On the number of connecting lines determined by n points in the real plane

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ABSTRACT

We consider the problem of determining for which integers k and n can a minimum k-perfect covering on n points be embedded in the real projective plane.

1. Introduction

A pairwise balanced design (PBD) is a pair (X,B) where X is a set of points and B is a collection of subsets of X (called blocks) such that given any unordered pair x,y of distinct points there is exactly one block that contains them both. A PBD with n points in which $\max\{|b|:b\in B\} = k<n$ is called a k-perfect covering of n points. A k-perfect covering (X,B) on n points is called minimum if for any k-perfect covering (X,B′) we have $|B| \le |B'|$; the number $|B|$ is denoted by $g^{(k)}(1,2;n)$, or sometimes just $g^{(k)}(n)$.

A collection X of n points in the real plane determines (in the obvious way) a PBD: the members of B are the lines (called connecting lines) determined by all pairs of points from X. We will call such a system a plane PBD. A pairwise balanced design (X,B) will be called planar if there is a point-to-point incidence preserving bijection $\sigma:(X,B)\rightarrow(X',B')$ where (X′,B′) is a plane PBD. The problem with which we are herein

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concerned is the following: for which k and n does there exist a minimum k-perfect covering on n points which is planar?

The study of systems of connecting lines in the plane has a long history (see e.g. [6]). Early geometers were particularly interested in determining the number of ordinary lines, i.e. connecting lines containing exactly two points of the system, that must be present. In 1893 Sylvester conjectured that there must be at least one such line; it took half a century before Gallai finally settled the conjecture in the affirmative. The best result to date is due to Hansen (see Moser [6]):

Theorem 1.1 Given n points in the plane, not all collinear, where $n \neq 7,13$, there are at least $\frac{1}{2}n$ ordinary lines among the connecting lines determined by the points.

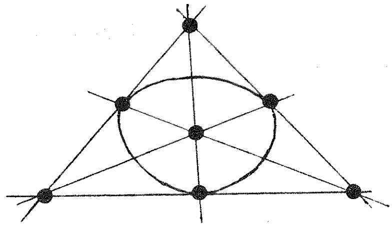
Indeed, Hansen's result is best possible when n is even, for Böröczky had produced configurations containing exactly $\frac{1}{2}n$ ordinary lines (see Crowe and McKee [3]).

Hansen's theorem is of fundamental importance to us here, for it implies:

Corollary 1.2 If (X,B) is a planar PBD on n points, $|B| > 1$ and $n \neq 7,13$, then there are at least $\frac{1}{2}n$ blocks each of which contains exactly two points of X.

A considerable amount of work has been done on determining the behavior of the function $g^{(k)}(n)$ (see e.g. [12], [10], [11]). We will make reference to specific results as the need arises.

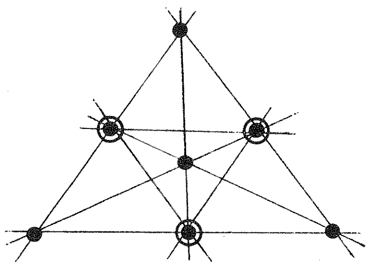
Before proceeding, we'll motivate our study with a simple example. It is known that $g^{(3)}(7) = 7$; the (unique) design achieving this bound is the plane of order 2:



Blocks: 1 2 4
2 3 5
3 4 6
4 5 7
5 6 1
6 7 2
7 1 3

Figure 1

Every block in this design contains three points, and so this design cannot be planar. If we dismantle the block 672 into its constituent pairs we do get a planar design:



Lines: 1 2 4
2 3 5
3 4 6
4 5 7
5 6 1
7 1 3
26,67,72

Figure 2

This raises an additional problem: let $l(k,n)$ denote the minimum number of connecting lines in a plane PBD on n points, where we require that k of the points be collinear but no more than k are collinear. If there is no planar minimum k-perfect covering on n points (i.e. if $l(k,n) \neq g^{(k)}(n)$) then how much larger than $g^{(k)}(n)$ is $l(k,n)$? The above example shows that $l(3,7) = 9$. (Among 7 points there are 21 pairs to be covered. A triple covers 3 pairs, and so the number of ordinary lines must be a multiple of 3.)

2. <u>Some cases where there exist planar minimum k-perfect coverings.</u>

<u>Theorem 2.1</u> If $k = 2$, or if $\frac{n}{2} \le k \le n-1$, then there is a planar minimum k-perfect covering of n points, i.e. $l(k,n) = g^{(k)}(n)$.

<u>Proof</u> The case $k = 2$ is trivial; $g^{(2)}(n) = \frac{n(n-1)}{2}$, and any 2-perfect covering is planar--one merely has to generate n points in the plane, no three of which are collinear.

Now suppose that $\frac{n}{2} \le k \le n-1$ where $k \ge 3$ (whence $n \ge 4$). It is known that $g^{(k)}(n) = 1 + \frac{1}{2}(n-k)(3k-n+1)$, and that any minimum covering consists of the one block of size k, and every other block has size 2 or 3 and intersects the block of size k (see [8] or [14]).

Let $n-k = t$. The cases where $t=1$ or 2 are quite simple; we illustrate solutions in Figure 3.

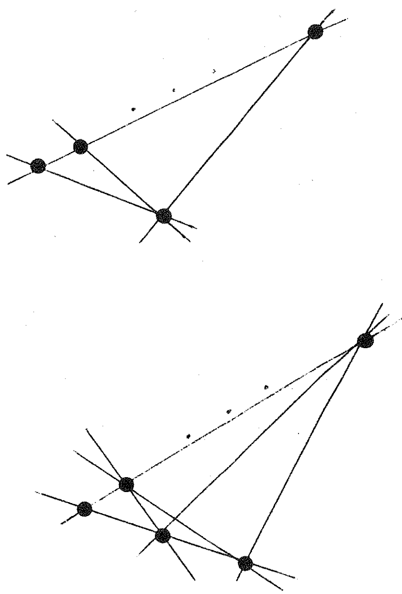


Figure 3

Now let $t \geq 3$, and take as our point set the vertices of a regular t-gon (in the affine plane) together with any k-subset of the projective line L_∞ which contains the t points determined by the directions of the t-gon. (In a plane PBD the directions are the slopes of its lines.) This can be done since $k \geq t$. It is readily verified that the number of connecting lines determined by these $t+k = n$ points is

$$t \cdot \left(\frac{t-1}{2} + 1\right) + (n-2t) \cdot t + 1 \quad \text{if } t \text{ is odd, or}$$

$$\frac{1}{2} t \cdot \frac{1}{2} t + \frac{1}{2} t \cdot \left(\frac{1}{2} t + 1\right) + (n-2t) \cdot t + 1 \quad \text{if } t \text{ is even.}$$

Recalling that $t = n-k$ we have in all $1 + \frac{1}{2}(n-k)(3k-n+1)$ lines, i.e. we have a (plane) minimum k-perfect covering.

The configurations constructed above contain $(n-k)(2k-n+1)$ ordinary lines; in particular when $n = 2k$ we get k ordinary lines. These are exactly the configurations of Böröczky referred to in the introduction.

Theorem 2.1 gives us a specific class of planar k-perfect coverings on n points for the range $\frac{n}{2} \leq k \leq n-1$; it is not true in general that any minimum covering with these parameters is planar, however. For example, there are four non-isomorphic covers for $g^{(6)}(10)$, corresponding to the four different ways of properly edge-coloring the complete graph K_4 with (at most) six colors:

a,b,c,d,e,f

a,1,2	b,1,3	c,1,4	d,1	e,1	f,1
a,3,4	b,2,4	c,2,3	d,2	e,2	f,2
			d,3	e,3	f,3
			d,4	e,4	f,4

Cover I

a,b,c,d,e,f

a,1,2	b,1,3	c,1,4	d,2,3	e,1	f,1
a,3,4	b,2,4	c,2	d,1	e,2	f,2
		c,3	d,4	e,3	f,3
				e,4	f,4

Cover II

a,b,c,d,e,f

a,1,2	b,1,3	c,2,4	d,1,4	e,2,3	f,1
a,3,4	b,2	c,1	d,2	e,1	f,2
	b,4	c,3	d,3	e,4	f,3
					f,4

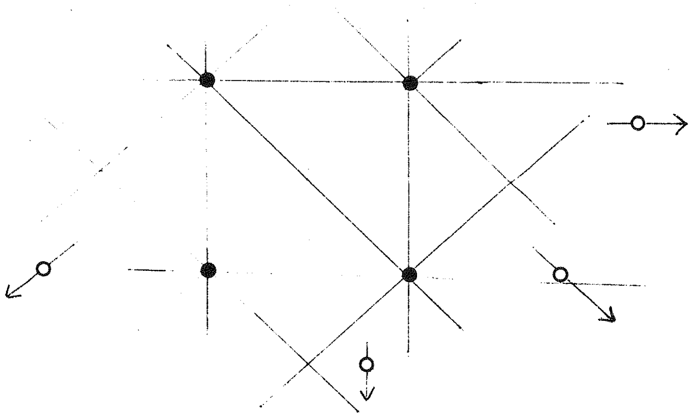
Cover III

a,b,c,d,e,f

a,1,2	b,3,4	c,1,3	d,2,4	e,1,4	f,2,3
a,3	b,1	c,2	d,1	e,2	f,1
a,4	b,2	c,4	d,3	e,3	f,4

Cover IV

Each of covers II, III and IV is planar (see figure 4). On the other hand, cover I is <u>not</u> planar since by deleting the points d, e and f from this cover we obtain the plane of order 2. In figure 4, and in all subsequent figures, we will use a filled circle (●) to denote an affine point and a hollow circle (○) to denote a point on the projective line L_∞.



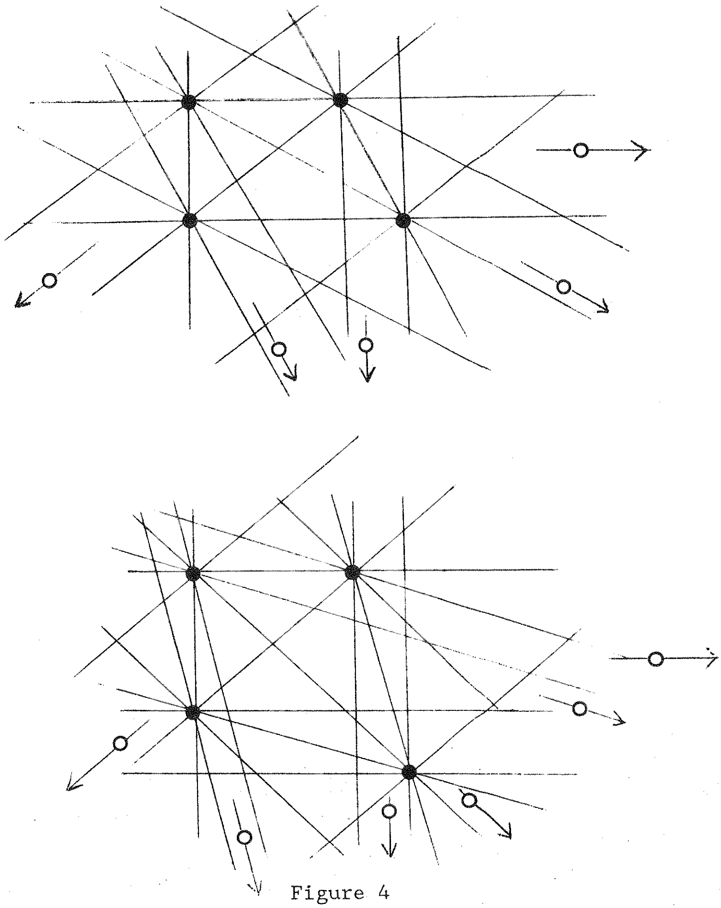


Figure 4

In contrast to Theorem 2.1, consider now the case $n = 2k + 1$, $k \geq 3$.

<u>Theorem 2.2</u> If $n = 2k + 1$, $k \geq 3$ then <u>no</u> minimum k-perfect covering is planar, i.e., $l(k,2k+1) > g^{(k)}(2k+1)$.

<u>Proof</u> If k is odd then $g^{(k)}(2k+1) = 1 + \frac{1}{2}(k+1)k$ (see [8] or [14]) and any minimum covering consists of the one block of size k, and every other block has size 3 and intersects the block of size k. (Such coverings admit to a simple description. Take a one-factorization of the complete

graph K_{k+1} and adjoin a new point to the pairs of each of the k one-factors; the block of size k is then taken to be the set of new points.) Since such a covering contains no blocks of size 2 it cannot be planar.

If k is even then $g^{(k)}(2k+1) = 1 + \frac{1}{2}(k+1)k + \lceil\frac{k}{4}\rceil$ (see [7]), and any minimum covering contains exactly $3\lceil\frac{k}{4}\rceil$ blocks of size 2. Now since $k \geq 4$ we have $2(3\lceil\frac{k}{4}\rceil) < 2k + 1$, so that by corollary 1.2 such a covering cannot be planar except possibly when k = 6. We consider the $g^{(6)}(13)$ covers. Any such cover must contain the block of size 6, six blocks of size 2, two blocks of size 4 and fifteen blocks of size 3; moreover, every block intersects the block of size 6. There are (at least) three non-isomorphic covers, which we give below:

a,b,c,d,e,f

a,0,1,2	b,0	c,0,3,4	d,0	e,2	f,3
a,3,6	b,2,3	c,2,5	d,1	e,5,0	f,6,0
a,4,5	b,1,4	c,1,6	d,4	e,1,3	f,1,5
	b,5,6		d,2,6	e,4,6	f,2,4
			d,3,5		

Cover I (Mullin et al. [7])

a,b,c,d,e,f

a,0,1,2	b,0	c,0,3,4	d,2	e,0	f,1
a,3,5	b,1,4	c,1,6	d,3	e,1,3	f,2,4
a,4,6	b 2,3	c,2,5	d,4	e,2,6	f,3,6
	b 5,6		d,1,5	e,4,5	f,0,5
			d,0,6		

Cover II (Zhang [16])

a,b,c,d,e,f

a,0	b,0,1,4	c,4	d,4	e,2	f,3
a,1	b,2,6	c,0,2	d,0,6	e,4,5	f,4,6
a,2,3,4	b,3,5	c,1,5	d,1,3	e,0,3	f,0,5
a,5,6		c,3,6	d,2,5	e,1,6	f,1,2

Cover III

Each of Covers I and II contain exactly one point (namely 0) which is contained in exactly two blocks of size 2; Cover III contains two such points (a and 4). It is not difficult to show that a cover for $g^{(6)}(13)$ cannot contain more than two such points. On the other hand, a fundamental inequality from Kelly and Moser [5, Theorem 3.5] implies that in any 13-point configuration with six ordinary lines there must be at least three points, each of which is contained in exactly two ordinary lines. Therefore no minimum 6-perfect cover on 13 points can be planar.

■

We remark here that Crowe and McKee [3] have produced a 13-point configuration containing exactly six ordinary lines. We reproduce their design below, for we will refer to it in section 3.

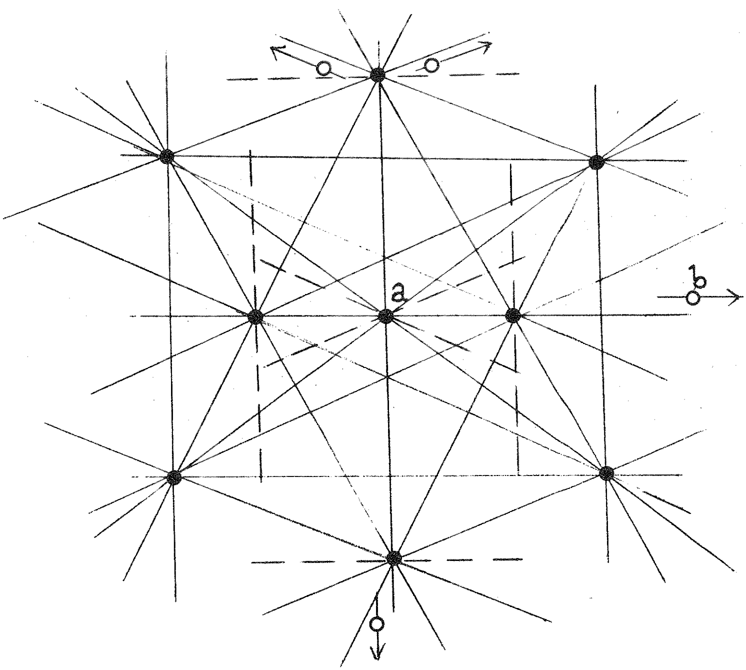


Figure 5

We close this section by remarking that a result analogous to theorem 2.2 can be proven for n = 2k+2, 2k+3 or 2k+4 and $k \geq 4$. The covering numbers $g^{(k)}(n)$ were determined for all such k and n by Rees in [9]. In all cases the minimum covers (necessarily) have insufficiently many blocks of size two to be planar.

3. <u>The cases k = 3 and 4</u>

The covering numbers $g^{(4)}(n)$ were determined for all $n \neq 17,18,19$ by Stanton and Stinson [15]. When $n \geq 11$ their covers contain at most one block of size two. Specifically, when $n \not\equiv 2,5,8$ or 11 (mod 12) there are no blocks of size two, while when $n \equiv 2,5,8$ or 11 (mod 12) there is one block of size two. It is <u>not</u> the case that <u>any</u> minimum covering has these properties however (for example when $n \equiv 5$ or 8 (mod 12) one can see from lemma 3.3 in [15]that minimum coverings with four blocks of size two can be constructed). If one considers the number of quadruples (blocks of size four) Q(n) in the covers one finds that in all cases $Q(n) \geq D(2,4,n) - \lfloor \frac{n}{4} \rfloor$, where D(2,4,n) denotes the maximum number of quadruples that can be constructed on n points so that no pair of points appears in more than one quadruple (see [1]).

Suppose that we wish to increase the number of blocks of size two without increasing the total number of blocks in the cover. Then we clearly must also increase the number of blocks of size four, and so decrease the number of blocks of size three. The specific conditions involved are

$$3\Delta b_3 = \Delta b_2 + 6\Delta b_4$$

$$\Delta b_3 = \Delta b_2 + \Delta b_4$$

where Δb_i denotes the change in the number of blocks of size i (note that we are taking all Δb_i to be non-negative). From these equations we get

$$2\Delta b_2 = 3\Delta b_4 \tag{3.1}$$

whence admissible vectors $(\Delta b_2, \Delta b_4, \Delta b_3)$ look like (0,0,0), (3,2,5), (6,4,10), etc. But from the preceding discussion it must be that

$$\Delta b_4 \leq \lfloor \frac{n}{4} \rfloor$$

whence equation (3.1) gives

$$\Delta b_2 \leq \frac{3}{2} \lfloor \frac{n}{4} \rfloor \leq \frac{3}{8} n .$$

Therefore any minimum cover will contain at most $1 + \frac{3}{8} n < \frac{1}{2} n$ blocks of size two.

Values of $n \leq 9$ are handled in section 2. For $n = 10,13$ we have $g^{(4)}(10) = 12$ and $g^{(4)}(13) = 13$ and in each case the (unique) minimum cover has no blocks of size two (see [13]). With corollary 1.2 in hand we can now summarize the above:

<u>Theorem 3.1</u> Let $n \geq 5$, $n \neq 17,18,19$. There is a planar 4-perfect covering of n points if and only if n = 5,6,7 or 8; thus if $n \geq 9$ then $l(4,n) > g^{(4)}(n)$.

■

We now discuss the most challenging case, namely $k = 3$. The covering numbers $g^{(3)}(n)$ have been determined for all n by Stanton, Allston and Cowan [14]. We list these values in table 1. An STS(n) (<u>Steiner Triple System</u>) is a PBD on n points in which all blocks have size 3; it is well-known that there is a STS(n) if and only if $n \equiv 1$ or 3 (mod 6). By an FHS(n) we will mean a PBD on n points in which all but four blocks have size 3, where these four blocks each have size 2 and form a cycle, i.e. are of the form a,b b,c c,d d,a ; such systems were constructed for all $n \equiv 5$ (mod 6) by Fort and Hedlund [4].

n	$g^{(3)}(n)$	Characteristics of any minimum cover	# blocks of size two in any minimum cover
$n \equiv 1,3 \mod 6$	$\frac{n(n-1)}{6}$	STS(n)	0
$n \equiv 5 \mod 6$	$\frac{n(n-1)+16}{6}$	FHS(n)	4
$n \equiv 0,2 \mod 6$	$\frac{(n+1)n}{6}$	STS(n+1) with one point deleted	$\frac{1}{2}n$
$n \equiv 4 \mod 6$	$\frac{(n+1)n+4}{6}$	FHS(n+1); delete one of the points contained in two blocks of size two	$\frac{1}{2}n+1$

Table 1

Remark: In each case a minimum cover contains exactly D(2,3,n) triples, where D(2,3,n) is defined analogously to the quantity D(2,4,n) referred to in the k=4 case.

When n is odd, $n \geq 7$ it is clear that no minimum cover can be planar. When n is even, however, we can no longer use corollary 1.2 to rule out planarity.

A considerable amount of work has been done on a very closely related problem, namely the 'orchard', or 'three-in-a-row' problem. This asks for the maximum number t(n) of collinear triples possible in any arrangement of n points in the plane, a collinear triple meaning here a line containing <u>exactly</u> three points. For a comprehensive survey of this problem we refer the reader to Burr, Grünbaum and Sloane [2]. The best general upper bound on t(n) corresponds to the packing numbers D(2,3,n) when n is even, i.e.

$$t(n) \le \begin{cases} \dfrac{n(n-2)}{6} & \text{if } n \equiv 0,2 \bmod 6 \\ \dfrac{n(n-2)-2}{6} & \text{if } n \equiv 4 \bmod 6. \end{cases}$$

When n is odd, the best general upper bound on t(n) corresponds to subtracting from the packing numbers D(2,3,n) a quantity sufficient to allow for Hansen's theorem (Corollary 1.2):

$$t(n) \le \begin{cases} \dfrac{n(n-1)}{6} - \left\lceil \dfrac{n}{6} \right\rceil & \text{if } n \equiv 1,\ 3 \bmod 6 \\ & \quad (n > 3,\ n \ne 7,13) \\ \dfrac{n(n-1)-8}{6} - \dfrac{n-5}{6} & \text{if } n \equiv 5 \bmod 6 \end{cases}$$

Thus for all $n > 3$, $n \ne 7,13$, we have $t(n) \le \left\lfloor \frac{n(n-2)}{6} \right\rfloor$. On the other hand, one of the main results of Burr et al. was to prove that for all $n \ge 3$, we have $t(n) \ge 1 + \left\lfloor \frac{n(n-3)}{6} \right\rfloor$, a remarkably close approximation to the upper bound for t(n). Their constructions involved selecting special sets of points from a certain family of cubic curves. For our purposes it is important to notice that, in their constructions, no connecting line contains more than three points. The following result then is a direct consequence of the foregoing.

<u>Theorem 3.2</u> For each $n > 3$, $n \ne 7, 13$,

$$\binom{n}{2} - 2\left\lfloor \frac{n(n-2)}{6} \right\rfloor \le l(3,n) \le \binom{n}{2} - 2\left(1 + \left\lfloor \frac{n(n-3)}{6} \right\rfloor\right).$$

In particular, if n is even the lower bound for $l(3,n)$ coincides with $g^{(3)}(n)$ from Table 1. <u>It remains a challenging open problem to determine for which even values of n it is true that $l(3,n) = g^{(3)}(n)$</u>. Exact answers to this question are known only up to n=16.

Table 2 summarizes the behavior of $g^{(3)}(n)$ and $l(3,n)$ for $7 \le n \le 20$. The lower and upper bounds on $l(3,n)$ are from theorem 3.2, and values for t(n) are taken from Burr et al. [2]. Values for $g^{(3)}(n)$ are as in table 1.

Then, in figures 6 through 11, we illustrate configurations on n points having $l(3,n)$ lines for n = 8,9,10,11,12 and 16 (figure 2 illustrates the unique such configuration for n=7). The ordinary lines are indicated by broken lines. The first configuration in each figure is taken from Burr et al. (see their Figs. 1 and 6). Note that in each case (except for n=12) we have configurations whose lines can be considered as being obtained from the blocks of a minimum 3-perfect covering on n points by 'dismantling' some triples (in the sense illustrated by figures 1 and 2). The 16-point configuration is of particular interest since it shows $l(3,16) = g^{(3)}(16)$.

n	$g^{(3)}(n)$	lower bound on $l(3,n)$	upper bound on $l(3,n)$	t(n)	$l(3,n)$
7	7	-	-	6	9
8	12	12	14	7	14
9	12	16	16	10	16
10	19	19	21	12	21
11	21	23	25	16	23
12	26	26	28	19	28
13	26	-	-	22-24	30-34
14	35	35	39	26-27	37-39
15	35	41	43	31-32	41-43
16	46	46	50	37	46
17	48	52	56	40-42	52-56
18	57	57	61	46-48	57-61
19	57	65	69	52-53	65-67
20	70	70	76	57-60	70-76

Table 2

Remark: For values of $n > 20$ the best known estimates on $l(3,n)$ are given by theorem 3.2. It is worth noting that Burr, Grünbaum and Sloane have conjectured that for $n \geq 20$ their constructions on cubic curves actually maximize the number of collinear triples possible on n points, i.e. that $t(n) = 1 + \lfloor \frac{n(n-3)}{6} \rfloor$. If true, this would imply that $l(3,n)$ is given by the upper bound in theorem 3.2.

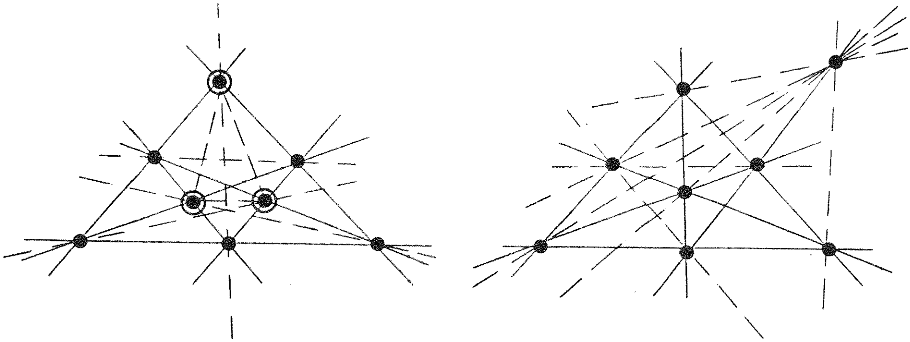


Figure 6

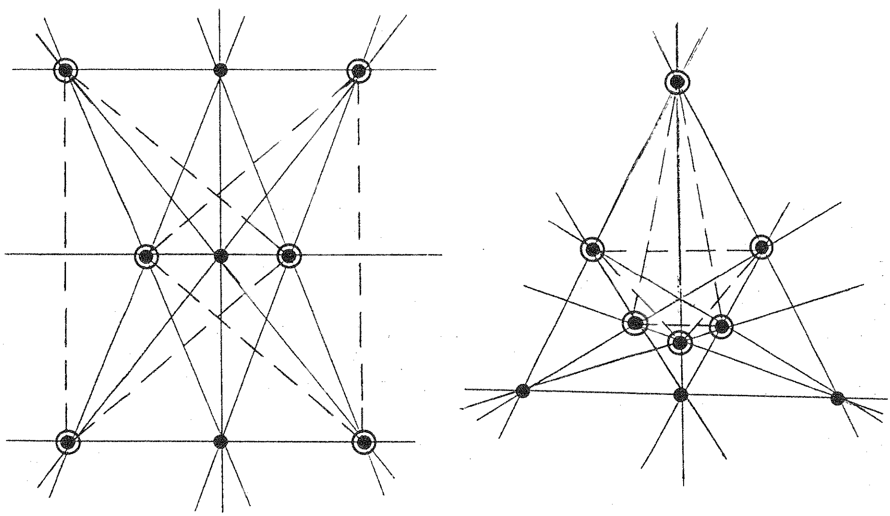


Figure 7

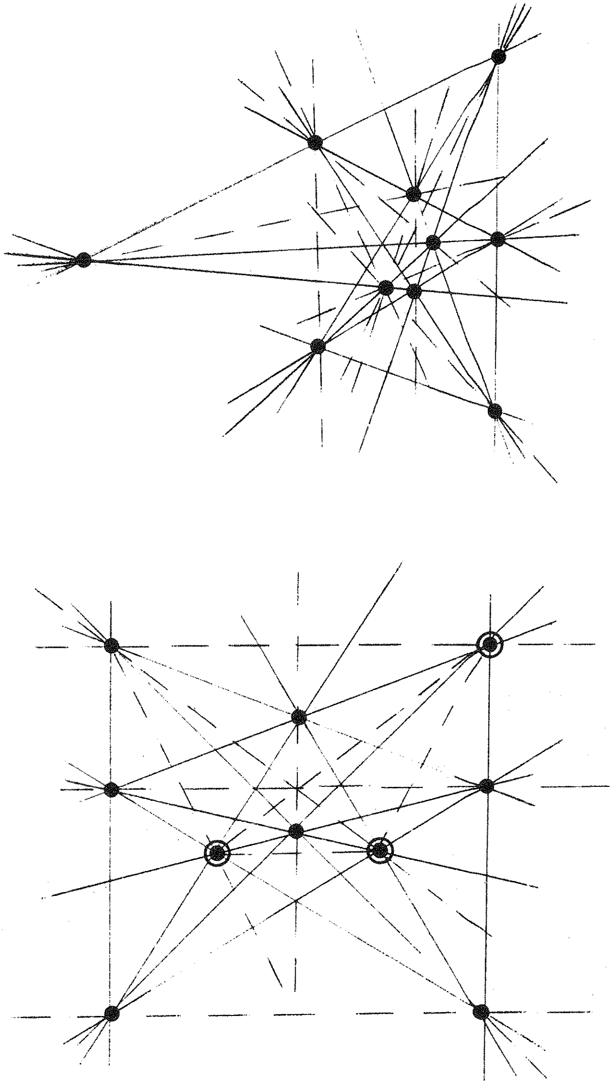
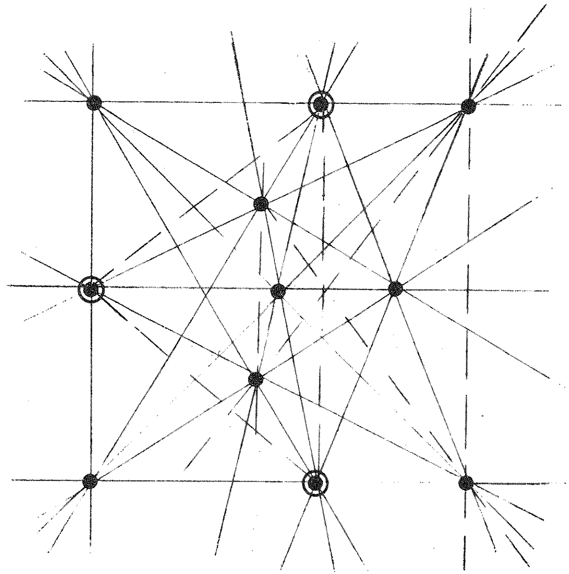


Figure 8



-this configuration can be obtained by deleting points a and b from the Crowe and McKee configuration (see figure 5).

Figure 9

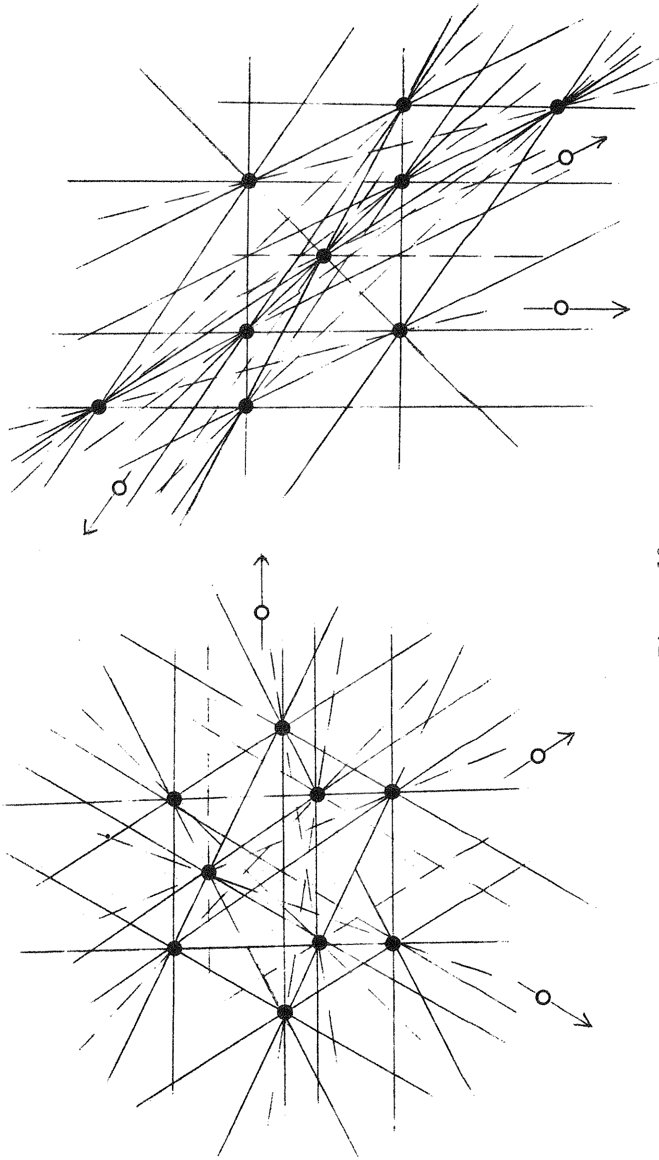


Figure 10

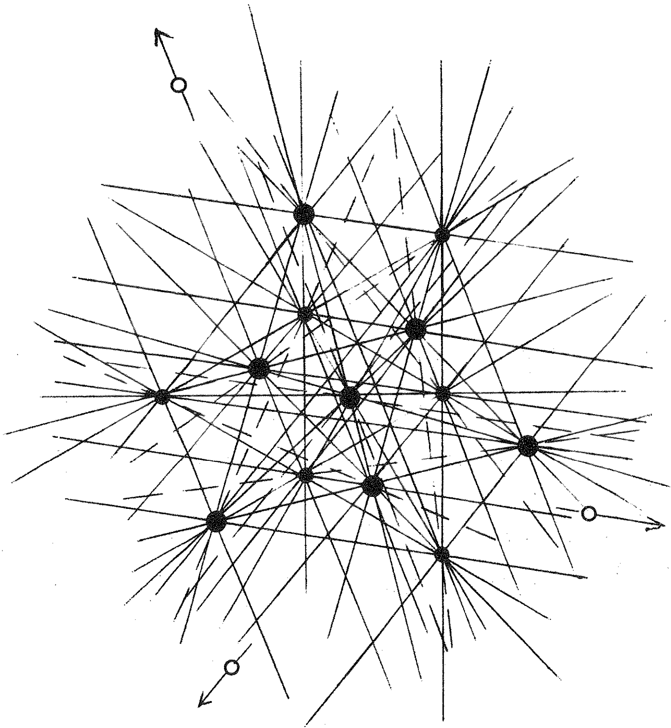


Figure 11

4. Conclusion.

Our observations here suggest that planar minimum k-perfect coverings on n points exist only if k is very large (i.e. $k \geq \frac{n}{2}$) or very small (i.e. k=2 and some cases with k=3). Determining for which n it is true that $l(3,n) = g^{(3)}(n)$ remains an interesting open problem.

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