GENERALIZED CLIQUE COVERINGS OF CHORDAL GRAPHS

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Abstract. The generalized clique covering problem is defined. A polynomial algorithm is presented for the generalized clique covering problem for chordal graphs.

We assume that the reader is familiar with standard graph-theoretic ideas, and also with sorting algorithms (see for example [3]). A *clique* in a graph is a complete subgraph.

Two main types of clique coverings of graphs have been discussed. One is a clique covering of *vertices*, a set of cliques which between them contain every vertex at least once. The other, a clique covering of *edges*, is a set of cliques which between them contain every edge. So a clique covering of edges may be defined as a clique covering of vertices with the added restriction that the ends of each edge must together belong to at least one clique.

Suppose $\mathcal{F} = (S_1, S_2, \ldots, S_k)$ is a family of sets, where the elements of S_i may be vertices or edges of a graph G. A

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generalized clique covering (or GCC) of G for the family $\mathcal{F}$ is a family of cliques with the property that for each i there must be one clique which contains all members of S_i. The generalized clique covering problem is to find a generalized clique covering with the minimum number of cliques.

A graph is *chordal* if every cycle of length greater than 3 must have a chord. A necessary and sufficient condition for a graph to be chordal is that it has a perfect elimination ordering, a way of ordering the vertices as $v_1, v_2, \ldots, v_n$, so that

$$X_i = \{v_j : v_j \text{ is adjacent to } v_i,\ j > i\}$$

is a clique for each i. In the following we use $\sigma(v)$ to denote the index of vertex v in a perfect elimination ordering and $\overline{X}_i$ to denote the set $\{v_i\} \cup X_i$. Rose, Tarjan and Leuker [4] give a linear time algorithm to find a perfect elimination ordering of a chordal graph.

Linear time algorithms are presented in [1] and [2] which solve the clique covering problem on vertices and on edges in chordal graphs. The purpose of this note is to generalize those algorithms to solve the generalized clique covering problem.

A clique K contains all the elements of S_i if and only if it contains all the vertices of S_i and all the end-vertices of the edges in S_i. So without loss of generality we may assume that S_i contains only vertices. Let n and m be the number of vertices and edges of G, respectively; let s_i be the cardinality of S_i for each i, and write $L = s_1 + s_2 + \ldots + s_k$.

We now present an algorithm for the generalized clique covering problem.

ALGORITHM.

Input A chordal graph $G = (V,E)$ and a family $\mathcal{F} = \{S_1, S_2, \ldots, S_k\}$ of subsets of the vertex-set V.

Begin

Initially no S_i is marked.

1 Find a perfect elimination ordering $v_1, v_2, \ldots, v_n$ of G.

2 Sort the vertices of S_i as

$$u_{i,1}, u_{i,2}, \ldots, u_{i,s_i}$$

where

$$\sigma(u_{i,1}) < \sigma(u_{i,2}) < \ldots < \sigma(u_{i,s_i});$$

denote $\sigma(u_{i,1})$ by min (S_i) and denote the sequence $(\sigma(u_{i,1}), \sigma(u_{i,2}), \ldots, \sigma(u_{i,s_i}))$ by $\sigma(S_i)$.

3 Sort the family $\mathcal{F}$ as $(S_1, S_2, \ldots, S_k)$ where the sequence $\sigma(S_1), \sigma(S_2), \ldots, \sigma(S_k)$ is lexicographically non-decreasing.

4 For $j = 1, 2, \ldots, n$, process v_j as follows. If there is an unmarked set S_i with min $(S_i) = j$, then choose one set S_i which satisfies min $(S_i) = j$; call it a special set and call v_j a special vertex. Then mark all sets S in $\mathcal{F}$ which lie completely within $\bar{X}_j$.

5 Output the special sets and special vertices.

End.

Theorem: If there is an unmarked set S_i at the end of the algorithm, there is no GCC for $\mathcal{F}$. Otherwise, $\{\bar{X}_j : v_j$ is a special vertex$\}$ is a minimum generalized clique covering for $\mathcal{F}$.

Proof: The first part of the theorem is obvious. We prove the second part.

Let Σ be the set of special vertices produced in the algorithm. It is easy to see that $\{\bar{X}_j : v_j \in \Sigma\}$ is a GCC for $\mathcal{F}$. It uses $|\Sigma|$ cliques. The algorithm outputs the same number $|\Sigma|$ of special sets.

We claim that no two special sets are contained in a common clique of G. For assume the contrary; say special sets S^1 and S^2 are contained in a common clique of G. Without loss of generality suppose min (S^1) is less than min (S^2); write j = min (S^1). By the algorithm, the set S^1 was marked when we processed vertex v_j. Moreover S^2 is a subset of $\overline{X}_j$ since v_j and the vertices of S^2 are contained in a common clique. So S^2 was also marked when v_j was processed, and S^2 cannot be chosen as a special set in the algorithm.

So the claim is true. At least $|\Sigma|$ cliques must be used in any generalized clique covering for $\mathcal{F}$. Hence the algorithm produces a minimum generalized clique covering for $\mathcal{F}$. □

Now we estimate the complexity of the algorithm. The complexity of step 1 is $O(n+m)$ by [4]. By [3], the complexities of step 2 and step 3 are $\sum_{i=1}^{k} O(s_i \log s_i) = O(nk \log n)$ and $O(nk)$, respectively. It takes $O(|A| + |B|)$ time to test whether a sorted set A is a subset of another sorted set B. Hence the complexity of step 4 is $O(nL) = O(n^2k)$. The total complexity of our algorithm is $O(n+m) + O(nk \log n) + O(nk) + O(n^2k) = O(n^2k)$. It is polynomial in the input-length.

References

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