The Drawing Ramsey Number $Dr(K_n)$

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Abstract Bounds are determined for the smallest $m = Dr(K_n)$ such that every drawing of K_m in the plane (two edges have at most one point in common) contains at least one drawing of K_n with the maximum number $\binom{n}{4}$ of crossings. For $n = 5$ these bounds are improved to $11 \leq Dr(K_5) \leq 113$.

A drawing $D(G)$ of a graph G is a special realization of G in the plane. The vertices are mapped into different points of the plane (also called vertices of $D(G)$), the edges are mapped into lines (also called edges of $D(G)$) connecting the corresponding vertices such that two edges have at most one point in common, which is either a common vertex or a crossing. Two drawings are said to be isomorphic if there exists an incidence-preserving one-to-one correspondence between vertices, crossings, edges, parts of edges and regions.

It is well known that every drawing of the complete graph K_4 has at most one crossing. Thus, the maximum number of crossings in a drawing $D(K_n)$ is at most $\binom{n}{4}$. Different nonisomorphic drawings $D(K_n)$ with $\binom{n}{4}$ crossings are discussed in [4]. In this note, we will show that for m sufficiently large every drawing of $D(K_m)$ must contain at least one drawing $D(K_n)$ with $\binom{n}{4}$ crossings. Moreover, bounds for the smallest such m, denoted by $Dr(K_n)$, will be deduced.

It can be observed that the question for a subdrawing $D(K_n)$ with maximum number of crossings is similar to the Esther Klein problem if lines are used instead of straight line segments and if convexity of n points is replaced by drawings $D(K_n)$ with $\binom{n}{4}$ crossings.

Theorem 1. For every positive integer n there exists a least integer $Dr(K_n)$ such that every drawing $D(K_m)$ with $m \geq Dr(K_n)$ contains a subdrawing $D(K_n)$ with $\binom{n}{4}$ crossings.

Proof. The existence of $Dr(K_n)$ will be deduced from Ramsey's theorem. Consider a drawing $D(K_m)$ with $m \geq r_4(5,n)$, where the Ramsey number $r_4(5,n)$ denotes the smallest l such that in every 2-coloring of the four-element subsets of an l-element set V, using colors green and red, there is a 5-element subset of V with all 4-element subsets green or an n-element subset of V with all 4-element subsets red. Color a 4-element subset of the vertex set V of $D(K_m)$ red if the four vertices determine a crossing and green otherwise. Among any five vertices there are four determining a crossing, since K_5 is nonplanar. Thus, there exists no 5-element subset of V with all 4-element subsets colored green, and there must be an n-element subset of V with all 4-element subsets red. These n vertices determine $\binom{n}{4}$ crossings and Theorem 1 is proved. ∎

The proof of Theorem 1 yields $Dr(K_n) \leq r_4(5,n)$. This bound might be very far from the truth, since none of the topological aspects of the problem besides the non-planarity of K_5 is taken into account. Moreover, in case $n \geq 5$ only rough upper bounds are available for $r_4(5,n)$ (see for example [3]). A lower bound for $Dr(K_n)$ can be deduced from the Esther Klein problem. In [5,6] it was shown that for $n \geq 2$ there are 2^{n-2} points in the plane no three of them collinear and no n of them determining a convex n-gon. Take 2^{n-2} such points as vertices of a drawing of a complete graph and draw all edges as straight line segments. Then no subdrawing $D(K_n)$ with $\binom{n}{4}$ crossings can occur, since among any n vertices there are four forming a non-convex 4-gon and hence having no crossing. Thus we obtain

Theorem 2. $2^{n-2}+1 \leq Dr(K_n) \leq r_4(5,n)$ for $n \geq 2$.

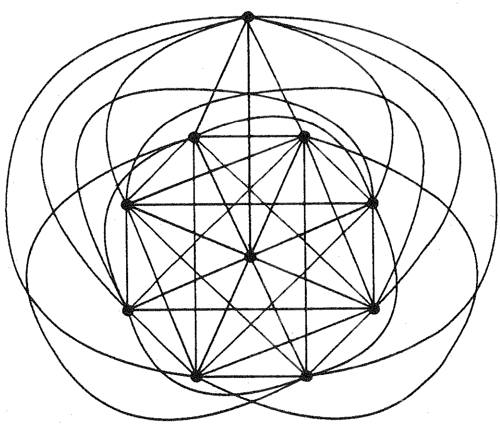

Figure 1. A $D(K_{10})$ containing no subdrawing $D(K_5)$ with five crossings

Trivially, $Dr(K_n) = n$ for $n \leq 3$, and Theorem 2 implies $Dr(K_4) = 5$. For $n \geq 5$, no exact values of $Dr(K_n)$ are known so far. The next theorem will improve the bounds given in Theorem 2 in case $n = 5$. For $n \geq 6$, no better bounds are known.

Theorem 3. $11 \leq Dr(K_5) \leq 113$.

Proof. The lower bound is given by the drawing $D(K_{10})$ in Figure 1. The proof of the upper bound is divided into four lemmas. The following Lemma 1 (due to P. Erdös) can also be found in [1] or [2].

Lemma 1. A sequence $a_1, a_2, ..., a_{st+1}$ of distinct real numbers either contains an increasing subsequence with $s + 1$ elements or a decreasing subsequence with $t + 1$ elements.

Proof. Assume there is no increasing subsequence with $s + 1$ elements. Give a_i label l where l is the length of the largest increasing subsequence starting at a_i. Clearly the possible labels are $1, 2, ..., s$. The sequence has $st + 1$ elements, so by the pigeonhole principle there are at least $t + 1$ with the same label. From the definition of the labelling these $t + 1$ (or more) elements with the same label form a decreasing subsequence. □

In the following lemmas some special notation will be used. Let G be a graph consisting of a triangle Δ with vertices v_1, v_2, v_3 and $n_1 + n_2 + n_3$ additional vertices of degree 1, n_i of them joined to v_i. A drawing $D(G)$ is denoted by $\Delta(n_1, n_2, n_3)$ if all $n_1 + n_2 + n_3$ vertices are placed outside (or inside) of Δ and if all edges from v_i to the n_i vertices intersect the edge of Δ not incident to v_i (see Figure 2). A $\Delta(n_1, n_2, n_3)$ with the vertices outside of Δ is isomorphic to one with the vertices inside; to see this, think of $\Delta(n_1, n_2, n_3)$ drawn on a sphere.

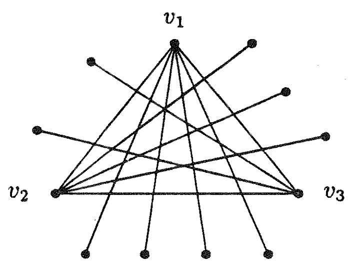

Figure 2. A drawing $\Delta(4, 3, 2)$

In $\Delta(n_1, n_2, n_3)$ the n_i vertices incident to v_i will always be labelled by $1^i, 2^i, ..., n_i^i$ in such a way that on edge (v_{i+1}, v_{i+2}) the point of intersection with (v_i, j^i) follows that with $(v_i, (j-1)^i)$ when (v_{i+1}, v_{i+2}) is oriented from v_{i+1} to v_{i+2} (all subscripts of the v_i are mod 3). Let $\Delta(n_1, n_2, n_3)$ be a subdrawing of $D(K_m)$. Consider l vertices $j_1^i, ..., j_l^i$ with $1 \leq j_1 < ... < j_l \leq n_i$. They are said to be of type $\mathrm{I}_{v_i, v_{i+k}}$ in $D(K_m)$,

$k = 1, 2$, if there is no point of intersection between the edges from the l vertices to v_i and to v_{i+k}. They are said to be of type $\text{II}_{v_i, v_{i+k}}$ if for $\lambda = 1, \ldots, l$ the edge (v_{i+k}, j^i_λ) intersects (in case $k = 1$) all the edges $(v_i, j^i_1), \ldots, (v_i, j^i_{\lambda-1})$ and (in case $k = 2$) all the edges $(v_i, j^i_{\lambda+1}), \ldots, (v_i, j^i_l)$ (see Figure 3).

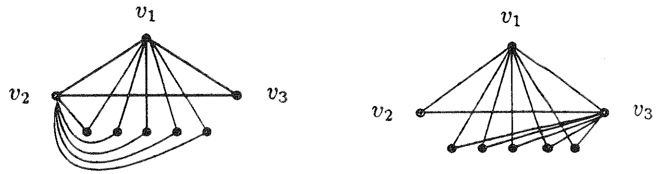

Figure 3. Five vertices of types I_{v_1,v_2} and II_{v_1,v_3}

Lemma 2. Let $\Delta(st + 1, 0, 0)$ be a subdrawing of $D(K_m)$. If $D(K_m)$ does not contain a subdrawing $D(K_5)$ with five crossings then the following assertions hold.

(i) For all j, k with $1 \leq j < k \leq st + 1$, the edges (v_2, j^1) and (v_1, k^1) have no common point of intersection in $D(K_m)$.

(ii) For $i = 2, 3$ there are either $s + 1$ vertices of type I_{v_1,v_i} or $t + 1$ vertices of type II_{v_1,v_i}.

Proof. (i) Assume that, for some j and k with $j < k$, (v_2, j^1) intersects (v_1, k^1). Then the missing edges between the vertices v_1, v_2, v_3, j^1 and k^1 can only be drawn in such a way that a subdrawing $D(K_5)$ with five crossings results.

(ii) We may assume that all $st + 1$ vertices of degree 1 in $\Delta(st + 1, 0, 0)$ are placed outside of Δ and that on Δ the vertex v_{i+1} follows v_i when taken in the counterclockwise direction. Set $e_0 = (v_1, v_2)$. Denote the edges from v_2 to the vertices $1^1, \ldots, (st + 1)^1$ by $e_1, \ldots, e_{st+1}$ such that in the counterclockwise direction (around v_2) e_j follows e_{j-1} for $j = 1, \ldots, st + 1$. Put $a_j = k$ if $e_j = (v_2, k^1)$. Apply Lemma 1 to the sequence $a_1, \ldots, a_{st+1}$. If an increasing subsequence of length $s + 1$ occurs, the corresponding vertices among $1^1, \ldots, (st + 1)^1$ are $s + 1$ vertices of type II_{v_1,v_2} by Lemma 2(i). Similarly a decreasing subsequence of length $t + 1$ leads to $t + 1$ vertices of type I_{v_1,v_2}. By symmetry, the corresponding result holds for v_3 instead of v_2. □

Lemma 3. If $D(K_m)$ for $m \geq 5$ contains no subdrawing $D(K_5)$ with five crossings a subdrawing $\Delta(n_1, n_2, n_3)$ with $n_1 + n_2 + n_3 \geq \lceil (m - 4)/4 \rceil$ must occur.

Proof. Consider a subdrawing $D(K_4)$ without crossings which must occur in $D(K_m)$. It divides the plane into four triangles $\Delta_1, \ldots, \Delta_4$. Let the vertices of $D(K_4)$ be u_1, u_2, u_3, u_4 such that u_j does not belong to the boundary of Δ_j. Add to $D(K_4)$ all those edges from $D(K_m)$ joining u_j to an inner vertex of Δ_j. Thus, we obtain four subdrawings $\Delta_j(n^j_1, n^j_2, n^j_3)$ where each of the $m - 4$ vertices of $D(K_m)$ different from $u_1, \ldots, u_4$ belongs to exactly one of them. This implies $\sum_{j=1}^4 (n^j_1 + n^j_2 + n^j_3) = m - 4$

and, for some j, $n_1^j + n_2^j + n_3^j \geq \lceil (m-4)/4 \rceil$. □

Lemma 4. A subdrawing $\Delta(n_1, n_2, n_3)$ in $D(K_m)$ with $n_1 + n_2 + n_3 \geq 28$ implies a subdrawing $D(K_5)$ with five crossings.

Proof. Assume that we have a $D(K_m)$ containing a subdrawing $\Delta(n_1, n_2, n_3)$ where $n_1 + n_2 + n_3 \geq 28$, and no subdrawing $D(K_5)$ with five crossings occurs. First we will show that this implies a subdrawing isomorphic to one of the drawings B_1 and B_2 in Figure 4. Note that $\Delta(n_1, n_2, n_3)$ must contain a subdrawing isomorphic to $A_1 = \Delta(7,7,1)$, $A_2 = \Delta(10,10,0)$, $A_3 = \Delta(13,1,1)$, $A_4 = \Delta(19,1,0)$ or $A_5 = \Delta(28,0,0)$.

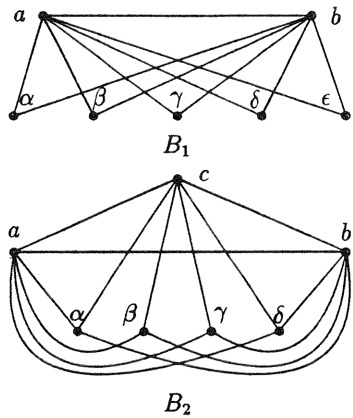

Figure 4

Case 1. A_2, A_4 or A_5 occurs. First suppose that in one of these three drawings there are ten vertices of type I_{v_1,v_2}. If among these ten vertices there are four of type I_{v_1,v_3}, a subdrawing isomorphic to B_2 occurs. Otherwise, by Lemma 2(ii), there are four vertices of type II_{v_1,v_3} yielding a subdrawing isomorphic to B_1 together with v_1, v_2 and v_3. For the remaining case, that there are no ten vertices of type I_{v_1,v_2}, we will deduce from Lemma 2(ii) the existence of a subdrawing isomorphic to B_1. If A_2 occurs, we may assume by symmetry that there are also no ten vertices of type I_{v_2,v_1}. Then, by Lemma 2(ii), there must be two vertices of type II_{v_1,v_2} and two of type II_{v_2,v_1} yielding the desired subdrawing B_1 together with v_1, v_2 and v_3. If A_4 occurs, then there are three vertices of type II_{v_1,v_2}. These yield a subdrawing isomorphic to B_1 together with v_1, v_2, v_3 and the neighbor of degree one of v_2 in A_4. If A_5 occurs, there must be four vertices of type II_{v_1,v_2} which yield the desired subdrawing B_1 together with v_1, v_2 and v_3.

Case 2. A_1 or A_3 occurs. Suppose there are seven vertices of type I_{v_1,v_2}. If among these seven vertices there are four of type I_{v_1,v_3}, a subdrawing isomorphic to B_2 occurs. Otherwise, by Lemma 2(ii), there are three vertices of type II_{v_1,v_3} which together with v_1, v_2, v_3, and one of the n_3 neighbors of v_3 yield a subdrawing isomorphic to B_1.

By Lemma 2(ii) it remains for A_3 that there are three vertices of type II_{v_1,v_2} which together with v_1, v_2, v_3, and one of the n_2 neighbors of v_2 determine a subdrawing isomorphic to B_1. By symmetry and Lemma 2(ii) it remains for A_1 that there are two vertices of type II_{v_1,v_2} and two vertices of type II_{v_2,v_1} which together with v_1, v_2 and v_3 yield a subdrawing isomorphic to B_1.

To complete the proof of Lemma 4 we now show that a subdrawing isomorphic to B_1 or B_2 implies a subdrawing $D(K_5)$ with five vertices. If among the five vertices $\alpha, \beta, \gamma, \delta, \epsilon$ from B_1, or among the four vertices $\alpha, \beta, \gamma, \delta$ from B_2, there are three vertices u, v, w such that in $D(K_m)$ the edge (u, v) intersects an edge from w to a or b, then five crossings are determined by u, v, w, a, b. Otherwise we obtain five crossings determined by $\alpha, \beta, \gamma, \delta, \epsilon$ from B_1 and five crossings determined by $c, \alpha, \beta, \gamma, \delta$ from B_2. □

It follows from Lemmas 3 and 4 that every drawing $D(K_{113})$ contains a subdrawing $D(K_5)$ with five crossings. This gives $Dr(K_5) \leq 113$ and the proof of Theorem 3 is complete. ■

Finally, we note that there exist only two nonisomorphic drawings $D_1(K_5)$ and $D_2(K_5)$ which have the maximum number of five crossings. In [4], nonisomorphic drawings $D_1(K_m)$ and $D_2(K_m)$ were constructed such that every subdrawing $D(K_5)$ of $D_i(K_m)$ is isomorphic to $D_i(K_5)$. Moreover, for every $n \leq m$ all subdrawings $D(K_n)$ of $D_i(K_m)$ are pairwise isomorphic. Thus Ramsey like numbers for any single drawing $D(K_n)$ do not exist for $n \geq 5$.

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