A CLASS OF EXTENDED TRIPLE SYSTEMS AND NUMBERS OF COMMON TRIPLES

Giovanni LO FARO[1]
Dipartimento di Matematica- Contrada Papardo- Salita Sperone, 31
98166 Sant'Agata , Messina - Italy
email: lofaro@imeuniv.unime.it

ABSTRACT. An extended triple system with no idempotent element (ETS) is a collection of non – ordered triples of type $\{x,y,z\}$ or $\{x,x,y\}$ chosen from a v – set in such a way that each pair (whether distinct or not) is contained in exactly one triple. (For example, in the block $\{x,x,y\}$, the pair $\{x,y\}$ is said to occur one time.) Such a design has $s_v = v(v+3)/6$ blocks and a necessary and sufficient condition for existence is that $v = 0 \pmod 3$. Let $J(v)$ denote the set of non – negative integers k such that there exist two ETS(v) with precisely k blocks in common. In this paper we determine $J(v)$ for all admissible v, in particular we show that $J(9) = I(9) - \{13\}$ and $J(v) = I(v)$, where $I(v) = \{0,1,\ldots, s_v - 3 , s_v\}$.

1. INTRODUCTION.

The concept of an extended triple system was introduced by D.M. Johnson and N.S. Mendelsohn [11]. An extended triple system is a pair (V,B), where V is a finite set and B is a collection of non – ordered triples from V , where each triple may have repeated elements, such that every pair of elements of V, not necessarily distinct, is contained in exactly one triple of B. The triple of B are of three types (1) $\{x,x,x\}$, (2) $\{y,y,z\}$ and (3) $\{a,b,c\}$, where the element x is called an idempotent and y a non – idempotent of the system (V,B). We shall denote by $\{v \, ; \, \alpha\}$ the class of all extended triple systems on v – elements containing exactly α idempotent elements.

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Necessary and sufficient conditions for the existence of an extended triple system$\{v\ ;\ \alpha\}$ with $0 \leq \alpha \leq v$ are :

(1) if $v = 0 \pmod 3$, then $\alpha = 0 \pmod 3$;

(2) if $v \neq 0 \pmod 3$, then $\alpha = 1 \pmod 3$;

(3) if $v = 0 \pmod 2$, then $\alpha \leq \frac{v}{2}$;

(4) if $\alpha = v - 1$, then $v = 2$.

D.M. Johnson and N.S, Mendelsohn [11] showed necessity, while in 1978, F.E. Bennet and N.S. Mendelsohn [1] showed the sufficiency of these conditions.

From now on we restrict our attention to extended triple systems with no idempotent element (i.e. without the triples of type $\{x,x,x\}$). We shall denote such a design, based on a v-set, by ETS(v). An ETS(v) has $\frac{v(v+3)}{6} = s_v$ blocks and a necessary and sufficient condition for existence is that $v = 0 \pmod 3$. Therefore in saying that a certain property concerning ETS(v) is true it is understood that $v = 0 \pmod 3$.

Various papers have dealt with the investigation of possible numbers of common blocks with two designs, with the same parameters, and based on the same v-set, may have in common. C.C. Lindner and A. Rosa [12] considered this problem for Steiner triple systems; M. Gionfriddo and C.C. Lindner [7], M. Gionfriddo and M.C Marino [9], A. Hartman and Z. Yehudai [10] , H.L. Fu [6] , G. Lo Faro [14] and others, for Steiner quadruple systems ; R.A.R. Butler and D.G. Hoffman [4] for group divisible triple systems ; E.J. Billington and D.G. Hoffman [2] for certain balanced ternary designs and E. J. Billington and E.S. Mahmoodian [3] for simple multi-set designs ; M. Gionfriddo, C.C. Lindner and C.A. Rodger for $K_4 - e$ designs.

Let $J(v)$ denote the set of non-negative integers k such that there exist two ETS(v) based on the same v-set, with k blocks in common and let

$$I(v) = \left\{0,1,\ldots, \frac{v(v+3)}{6} - 3 , \frac{v(v+3)}{6} = s_v\right\} .$$

It is seen instantly that $J(v) \subseteq I(v)$; in other words it is impossible to have two ETS(v) based on the same v-set which have all but one block the same, or all but two blocks the same.

The purpose of this paper is to prove the following result :

Main Theorem. $J(v) = I(v)$ **for** $v = 0$ **(mod 3) ,** $v \neq 9$ **and** $J(9) = I(9) - \{13\}$.

In the following section we give useful design constructions ; the subsequent section then deals with the adaption of the design constructions to obtain two ETS(v) with specified intersection. A later section deal with small v in order to start the recursive constructions.

From now on, where there is no confusion, we write blocks such as $\{x,y,z\}$ and $\{x,x,y\}$ as xyz and xxy for brevity ; it is not assumed that x , y and z are all distinct elements when using this notation.

2. AUXILIARY CONSTRUCTIONS OF ETS .

Let (V,B) an ETS(v) , where $V = \left\{a_i : i = 1 ,2 ,\ldots, v\right\}$.

1) v **to** $2v$ **,** v **even.**

Let $\mathcal{F} = \left\{F_i : i = 1,2,\ldots,v-1\right\}$ be a 1-factorization of K_v on $X = \left\{x_1,x_2,\ldots,x_v\right\}$, where $V \cap X = \emptyset$. Put $S = V \cup X$ and $T = B \cup C \cup D$ where $C = \left\{a_i xy : xy \in F_i, i = 1,2,\ldots, v-1\right\}$ and $D = \left\{a_v xx \text{ , for each } x \in X\right\}$. Then (S,T) is an ETS($2v$) that we denote by $\left((V \cup X) , (B,\mathcal{F})\right)$.

2) v to $2v$, v odd.

Put $X = \{x_1, x_2, \ldots, x_v, x_{v+1}\}$ with $V \cap X = \emptyset$. Let $\mathcal{F} = \{F_i : i = 1, 2, \ldots, v\}$ be a 1-factorization of K_{v+1} on X. Put $S = V \cup (X - \{x_{v+1}\})$ and $T = B \cup C \cup H$ where $C = \{a_i xy : xy \in F_i \text{ and } x_{v+1} \notin \{x, y\} ; i = 1, 2, \ldots, v\}$, and $H = \{a_i x_j x_j : x_j x_{v+1} \in F_i ; i = 1, 2, \ldots, v\}$. Then (S,T) is an ETS($2v$).

Let $\mathcal{F} = \{F_i : i = 1, 2, \ldots, 2n-1\}$ be a 1-factorization of a complete graph K_{2n} on $N = \{1, 2, \ldots, 2n\}$. If F_a , $F_b \in \mathcal{F}$, the notation $F_a \cdot F_b$ will denote the following set of blocks $\{11x_{i_2}, x_{i_2}x_{i_2}x_{i_3}, \ldots, x_{i_h}x_{i_h}1, x_{j_1}x_{j_1}x_{j_2}, x_{j_2}x_{j_2}x_{j_3}, \ldots, x_{j_s}x_{j_s}x_{j_1}, \ldots,$ $x_{t_1}x_{t_1}x_{t_2}, x_{t_2}x_{t_2}x_{t_3}, \ldots, x_{t_r}x_{t_r}x_{t_1}, x_{q_1}x_{q_1}x_{q_2}, x_{q_2}x_{q_2}x_{q_3}, \ldots, x_{q_m}x_{q_m}x_{q_1}\}$, where $x_{j_1} = \min\left(N - \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_h}\}\right), \ldots, x_{q_1} = \min\left(N - \{1, x_{i_2}, \ldots, x_{i_h}, x_{j_1}, x_{j_2}, \ldots, x_{j_s}, \ldots x_{t_1}, x_{t_2}, \ldots, x_{t_r}\}\right)$;

$F_a = \{1x_{i_2}, x_{i_3}x_{i_4}, \ldots, x_{i_{h-1}}x_{i_h}, x_{j_1}x_{j_2}, x_{j_3}x_{j_4}, \ldots, x_{j_{s-1}}x_{j_s}, \ldots, x_{t_1}x_{t_2}, x_{t_3}x_{t_4}, \ldots, x_{t_{r-1}}x_{t_r}, x_{q_1}x_{q_2}, x_{q_3}x_{q_4}, \ldots, x_{q_{m-1}}x_{q_m}\}$ and

$F_b = \{x_{i_2}x_{i_3}, x_{i_4}x_{i_5}, \ldots, x_{i_h}1, x_{j_2}x_{j_3}, x_{j_4}x_{j_5}, \ldots, x_{j_s}x_{j_1}, \ldots, x_{t_2}x_{t_3}, x_{t_4}x_{t_5}, \ldots, x_{t_r}x_{t_1}, x_{q_2}x_{q_3}, x_{q_4}x_{q_5}, \ldots, x_{q_m}x_{q_1}\}$.

Note that $(F_a \cdot F_b) \cap (F_b \cdot F_a) = \emptyset$.

We illustrate this when $2n = 12$; $N = \{1, 2, \ldots, 9, A, B, C\}$; $F_a = \{12, 34, 56, 78, 9A, BC\}$ and $F_b = \{15, 26, 39, 4A, 7B, 8C\}$. In this case $F_a \cdot F_b = \{112, 226, 665, 551, 334, 44A, AA9, 993, 778, 88C, CCB, BB7\}$.

3) v to $2v+3$, v odd.

Let $\mathcal{F} = \{F_i : i = 1, 2, \ldots, v+2\}$ be a 1-factorization of K_{v+3} on $X = \{x_1, x_2, \ldots, x_{v+3}\}$ with $V \cap X = \emptyset$. Put $S = V \cup X$ and $T = B \cup C \cup F_{v+1} \cdot F_{v+2}$ where $C = \{a_i xy : xy \in F_i , i = 1, 2, \ldots, v\}$. (S ,T) is an ETS($2v+3$).

Let K_{2n} be a complete graph on $2n$ vertices $(2n \geq 8)$. The edges of K_{2n} fall into n disjoint classes $P_1, P_2, \ldots, P_n$ where edge $\{i,k\}$ is in P_j if and only if $i-k=j$ (mod 2n).

R.G. Stanton and I.P. Goulden proved in [15] the following results :

(1) If $2x+1<n$ then $P_{2x} \cup P_{2x+1}$ splits into four one – factors ;

(2) If n is even , then P_n is a single one-factor . If n is odd , then $P_{n-1} \cup P_n$ can be split into three one – factors ;

(*) (3) The graph K_{2n} may be factored into a set of $2n$ triangles covering P_1, P_{2j}, P_{2j+1} and a set of $2n-7$ one factors covering the other P_i .

4) v to $2v+9$, v odd.

Factor the complete graph K_{v+9} on vertex set $X=\{x_i : i=1, 2, \ldots , v+9\}$, $V \cap X = \emptyset$, by (3) of (*) . Let $L=\{\{i, i+1, i+3\} : i=1, 2, \ldots, v+9\}$ be the set of triangles and $\mathcal{F}=\{F_i : i=1, 2, \ldots, v+2\}$ be the set of one – factors.

Put $S=V \cup X$ and $T=B \cup C \cup L \cup F_{v+1} \cdot F_{v+2}$ where $C=\{a_i xy : xy \in F_i$, $i=1, 2, \ldots, v\}$.

It is straightforward that (S,T) is an ETS$(2v+9)$.

5) v to $3v$.

Let (V,B_1) ; (V,B_2) ; (V,B_3) and (V,B) be ETS(v) . Put $S = V \times \{1,2,3\}$. We define a collection T of blocks on S as follows :

(1) $(x,i)(y,i)(z,i) \in T$ if and only if $xyz \in B_i$; $i=1, 2, 3$;

(2) $\{(x,1)(y,2)(z,3)$, $(x,1)(z,2)(y,3)$, $(y,1)(x,2)(z,3)$, $(y,1)(z,2)(x,3)$, $(z,1)(x,2)(y,3)$, $(z,1)(y,2)(x,3)\} \subseteq T$ if and only if $xyz \in B$ and

$|\{x, y, z\}| = 3$;

(3) $\{(x,1)(x,2)(y,3), (y,1)(x,2)(x,3), (x,1)(y,2)(x,3)\} \subseteq T$ if and only if $xxy \in B$.

It is straightforward to see that (S,T) is an ETS$(3v)$.

We close this section with two remarks:

REMARK 1. Let (W,R) be an STS(w) containing a parallel class $\pi \subseteq R$ (i.e. the blocks in π partition W). Obviously $w = 3 \pmod 6$. We can derive from (W,R) an ETS(w) (W,B) putting $B = (R - \pi) \cup L(\pi)$ where $\{xxy\ ,\ yyz\ ,\ zzx\} \subseteq L(\pi)$ if and only if $xyz \in \pi$.

REMARK 2. [1] Let $W = \{1, 2,..., w\}$, $w = 0 \pmod 3$. Put $E = H \cup Z$ where $H = \{xyz : x + y + z = 0 \pmod w\} - \{\frac{w}{3}\frac{w}{3}\frac{w}{3}, \frac{2w}{3}\frac{2w}{3}\frac{2w}{3}, www\}$ and $Z = \{\frac{w}{3}\frac{w}{3}\frac{2w}{3}, \frac{2w}{3}\frac{2w}{3}w, ww\frac{w}{3}\}$.

It a routine matter to see that (W, E) is an ETS(w) .

3. BASIC LEMMAS.

Take $N = \{1,2,...,2n\}$ and let $\mathcal{F}$ and $\mathcal{G}$ be two 1-factorizations of N where $\mathcal{F} = \{F_i : i = 1,2,...,2n-1\}$ and $\mathcal{G} = \{G_i : i = 1,2,...,2n-1\}$. We will say that $\mathcal{F}$ and $\mathcal{G}$ have k edges in common if $k = \sum_{i=1}^{2n-1} |F_i \cap G_i|$.

Let $U(2n)$ be the set of k such that a pair of 1-factorizations of order $2n$ having k edges in common exist. In [13] , C.C. Lindner and W.D. Wallis gave a complete solution to the intersection problem for 1-factorization by showing that $U(2) = \{1\}$; $U(4) = \{0, 2, 6\}$; $U(6) = \{0, 1, 2, 3, 5, 6, 7, 9, 15\}$ and $U(2n) = \{0, 1,..., u = n(2n-1)\} - \{u-5, u-3, u-2, u-1\}$, for all $n \geq 4$.

It is well known [5] that if n and m are even positive integers and $n \geq 2m$, then there exists a 1-factorization of order n containing a sub-1-factorization of order m .

LEMMA 1. For v even, if $(k,h) \in J(v) \times U(v)$ then $v+k+h \in J(2v)$.

Proof. Let (V,B_1) and (V,B_2) be two ETS(v) intersecting in k triples and let $\mathfrak{F}$ and $\mathfrak{G}$ be two 1-factorizations of K_v on X , where $|X| = v$ and $V \cap X = \emptyset$, such that $h = \sum_{i=1}^{v-1} |F_i \cap G_i|$. It is a routine matter to see that $((V \cup X)\ ,\ (B_1,\mathfrak{F}))$ and $((V \cup X)\ ,\ (B_2,\mathfrak{G}))$ are two ETS($2v$) with exactly $v+k+h$ blocks in common.

By construction 2 , the following can be shown in a similar fashion .

LEMMA 2. For v odd, if $(k,h) \in J(v) \times U(v+1)$ then $k+h \in J(2v)$.

By Lemmas 1 and 2 , we obtain the following

LEMMA 3. For $v \geq 9$, $J(v) = I(v)$ implies $J(2v) = I(2v)$.

Proof. If v is odd, it follows, without any undue difficulty, by Lemma 2 .

Suppose v even.By Lemma 1 , we obtain that $k \in J(2v)$ for $k \in \{v,\ v+1,\dots,\ s_{2v}-3,\ s_{2v}\}$. Put $V = \{a_i\ :\ i = 1,\ 2\ ,\dots,\ v\}$ and let (V, B_1) and (V, B_2) be two ETS(v) intersecting in r triples, $r \in \{0,\ 1,\dots,\ v-1\}$ and let $\mathfrak{F}$ and $\mathfrak{G}$ be two 1-factorizations on K_v on X $\left(|X| = v \text{ and } X \cap V = \emptyset\right)$ such that $\sum_{i=1}^{v-1} |F_i \cap G_i| = 0$, then $((V \cup X)\ ,\ (B_1,\mathfrak{F})) = (S\ ,T_1)$ and $((V \cup X)\ ,\ (B_2,\mathfrak{G})) = (S\ ,T_2)$ have $v+r$ blocks in common. If T_1^* is obtained from T_1 by removing the blocks $a_{v-1}xy$ $(xy \in F_{v-1})$ and $a_v xx$ $(x \in X)$ and replacing them with $a_v xy$ $(xy \in F_{v-1})$ and $a_{v-1}xx$ $(x \in X)$, we see that $T_1^* \cap T_2 = r$. This concludes the proof.

REMARK 3. We observe that the proof of Lemma 3 says also that for v

even $J(v) \subseteq J(2v)$.

LEMMA 4. ***Let*** v ***odd,*** $v \geq 9$. $J(v) = I(v)$ ***implies*** $J(2v+3) = I(2v+3)$.

Proof. Put $V = \{a_i : i = 1,2,\dots,v\}$. Let (V, B_1) and (V, B_2) be two ETS(v) intersecting in k triples and $\mathcal{F} = \{F_i : i = 1,2,\dots,v{+}2\}$ be a 1-factorization on X , where $|X| = v{+}3$ and $V \cap X = \emptyset$. Let α be any permutation of $\{1,2,\dots,v\}$ fixing exactly p elements ; obviously such an α exists for $\alpha = 0$, $1,\dots, v-2, v$.

Let now $C = \{a_i xy : xy \in F_i , i = 1,2,\dots,v\}$ and $C_\alpha = \{a_i xy : xy \in F_{\alpha(i)} , i = 1,2,\dots,v\}$.

C and C_α have exactly $p \cdot \frac{v+3}{2}$ triples in common .

Let $(S,T_1) = \big((V \cup X) , B_1 \cup C \cup F_{v+1} \cdot F_{v+2} \big)$ and $(S, T_2) = \big((V \cup X), B_2 \cup C_\alpha \cup$ $\cup F_{v+1} \cdot F_{v+2} \big)$ be as in costruction 3. Then the two ETS$(2v+3)$ (S,T_1) and (S,T_2) intersect in $v{+}3{+}k{+}p \cdot \frac{v+3}{2}$ triples. Taking into account that $s_v - 3 > \frac{v+3}{2}$ we obtain, by putting consecutively $p = 0, 1,\dots, v{-}2$, that $k \in J(2v{+}3)$ for $k \in \{v{+}3, v{+}4, \dots, s_{2v+3} - (v{+}6)\}$ $\big($ since $(s_v - 3 + (v-2) \cdot \frac{v+3}{2} + v + 3) = s_{2v+3} - (v+6) \big)$.

On the other hand when $p = v$ we have $k \in J(2v{+}3)$ for $k = s_{2v+3} - s_v$, $s_{2v+3} - (s_v - 1)$, $\dots$, $s_{2v+3} - 3$, s_{2v+3} and so $\{v{+}3 , v{+}4 ,\dots, s_{2v+3} - 3 , s_{2v+3}\} \subseteq J(2v{+}3)$.

It remains to show that $\{0,1,\dots,v{+}2\} \subseteq J(2v{+}3)$. Let $p = 0$ then $(B_1 \cup C \cup F_{v+1} \cdot F_{v+2})$ and $(B_2 \cup C_\alpha \cup F_{v+2} \cdot F_{v+1})$ have exactly k blocks in common, consequently $J(2v{+}3) = I(2v{+}3)$.

LEMMA 5. ***Let*** v ***odd,*** $v \geq 15$. $J(v) = I(v)$ ***implies*** $J(2v+9) = I(2v+9)$.

Proof. Taking into account that $(F_{v+1} \cdot F_{v+2}) \cap (F_{v+2} \cdot F_{v+1}) = \emptyset$, we obtain from constuction 4) by a similar argument as Lemma 4 , but with more effort, that :

$$\{v{+}9 , v{+}10 ,\dots, s_{2v+9} - 3 , s_{2v+9}\} \subseteq J(2v{+}9) .$$

Let (V, B_1) and (V, B_2) be two ETS(v) intersecting in k triples , where $V = \{a_i : i = 1,2 ,\ldots,v\}$ and let X be a $(v+9)$ – set such that $X \cap V = \emptyset$. Let K_{v+9} be the the complete graph on vertex set X.

Put $L_1 = \{\{i, i+1, i+3\}\}$ and $L_2 = \{\{i, i+4, i+5\}\}$; $i = 1,2,\ldots, v+9$. From (1) of (*) , $P_2 \cup P_3$ splits into four 1-factors F_1, F_2 , F_3 , F_4 and $P_4 \cup P_5$ splits into four 1-factors G_1, G_2 , G_3 , G_4. From (*) (3), we have two sets of one-factors $\{F_i : i = 1,2 ,\ldots, v+2\}$ covering all P_j , $j = 4, 5,\ldots, \frac{v+9}{2}$ and $\{G_i : i = 1,2,\ldots,v+2\}$ covering all P_j , $j = 2, 3, 6, 7,\ldots, \frac{v+9}{2}$. We can assume that $F_i = G_i$, for $i = 5, 6,\ldots,v+2$.

Let α be a permutation of $\{1,2,\ldots,v\}$ fixing 0 element, $C = \{a_i xy : xy \in F_i, i = 1,2,\ldots,v\}$ and $C^*_\alpha = \{a_i xy : xy \in G_{\alpha(i)} , i = 1,2,\ldots,v\}$, then $(B_1 \cup C \cup L_1 \cup F_{v+1} \cdot F_{v+2})$ and $(B_2 \cup C^*_\alpha \cup L_2 \cup F_{v+2} \cdot F_{v+1})$ have exactly k blocks in common and so $\{0,1,\ldots,v+8\} \subseteq J(2v+9)$.

This completes the proof of the Lemma.

4. $J(v)$ FOR SMALL v.

$v = 3$.

There are precisely two ETS(3) ; call them designs A and B :

$A = \{112 , 223 , 331\}$; $B = \{113 , 221 , 332\}$. So we have $J(3) = \{0 , 3\}$.

$v = 6$.

Applying Lemma 2 to $J(3)$ we get $\{0, 2, 3, 5, 6, 9\} \subseteq J(6)$.

Take the following ETS(6) (V,T) based on the set $V = \{1 , 2 ,\ldots, 6\}$:

$T = \{112, 223, 331, 441, 553, 662, 156, 245, 346\}$. Consider the isomorphic designs got from T by permuting elements ; let $T_1 = (1,6)(2,3,4)T$, $T_2 = (3,4)(5,6)T$. Then it is easy to check that $|T \cap T_1| = 1$ and $|T \cap T_2| = 4$. So $J(6) = I(6)$

$v = 9$.

By a similar argument as Lemma 4, it is easy to see that $\{0,3,6,9,12,15,18\} \subseteq J(9)$.

Let D_1 , D_2 , D_3 be the following ETS(9) :

$D_1 = \{114, 221, 335, 442, 557, 669, 773, 886, 998, 136, 159, 178, 239, 258, 267, 438, 456, 479\}$;

$D_2 = \{112, 224, 336, 441, 559, 668, 775, 883, 997, 135, 167, 189, 237, 258, 269, 439, 456, 478\}$;

$D_3 = \{112, 223, 331, 445, 556, 664, 778, 889, 997, 147, 159, 168, 249, 258, 267, 438, 537, 639\}$.

Then $|D_1 \cap D_2| = 2$, $|D_1 \cap D_3| = 4$.

Now let D_4 come from D_3 by replacing 112, 445, 159, 249 by the blocks 115, 442, 129, 459 and let D_5 come from D_3 with 112, 445, 778 , 889 , 997, 249, 159 replaced by 115, 442, 779, 998, 887, 129, 459. We have $|D_3 \cap D_4| = 14$ and $|D_3 \cap D_5| = 11$.

Take the following ETS(9) :

$E_1 = \{112, 223, 331, 445, 556, 664, 778, 889, 997, 148, 157, 169, 247, 259, 268, 349, 358, 367\}$

and consider the isomorphic design got from E_1 by permuting elements; let $E_2 = (1,4)E_1$.

Now let E_3 come from E_1 by replacing 778, 889, 997 by the blocks 779, 998, 887 . Next let E_4 come from E_1 by replacing 112, 223, 331, 445, 556, 664, 778, 889, 997 by the blocks 113, 332, 221, 446, 665, 554, 779, 998, 887 and let E_5 come from E_1 with 112, 223, 331, 445, 556, 664 replaced by 113, 332, 221, 446, 665, 554 . It is seen that $|E_1 \cap E_2| = 10$, $|E_2 \cap E_3| = 7$, $|E_2 \cap E_4| = 5$, $|E_2 \cap E_5| = 8$ without any undue difficulty .

Finally, if D have the following blocks $\{118, 221, 335, 443, 554, 667, 779, 882,$

996, 139, 147, 156, 237, 246, 259, 368, 489, 578} then $1 \in J(9)$ because $|D \cap D_3| = 1$.

Thus we have $I(9) - \{13\} \subseteq J(9) \subseteq I(9)$.

Let (V,B) an ETS(9) , it is straightforward to show that each element has to occur singly in four blocks and twice in one block.Using graph theoretic terminology we will say that each element x of V has degree $d(x) = 6$.

For every $H \subseteq V$, $|H| = h$, put :

$$T_H = \{b \in B : b \subseteq H\} \text{ and } I_H = \{b \in T_H : |b| = 2\}.$$

From Inclusion – Exclusion Principle, we have

$$|T_H| + |T_{V\text{-}H}| = 18 - 6 \cdot h + \frac{h(h+1)}{2} + |I_H| =$$
$$= 18 - 6(9-h) + \frac{(9-h)(10-h)}{2} + |I_{V\text{-}H}|$$

and so

$$|T_H| \leq 18 - 6 \cdot (9-h) + \frac{(9-h)\cdot(10-h)}{2} = 9 - \frac{h \cdot (7-h)}{2} .$$

Suppose (V,B_1) and (V,B_2) are two ETS(9) with $|B_1 \cap B_2| = 13$. This means that the triples not in common to the two ETS(9) , namely $B_1 - B_2$ and $B_2 - B_1$, are disjoint sets, each containing 5 triples which are *mutually balanced* .That is , the 5 triples in $Q_1 = B_1 - B_2$ covering precisely the same pairs of elements , *not necessarily distinct* , as $Q_2 = B_2 - B_1$. Let the triples of Q_1 and Q_2 involve h elements , so necessarily $6 \leq h \leq 7$.

Elementary considerations show that there is not possible to find Q_1 and Q_2 . Thus $13 \notin J(9)$ and then $J(9) = I(9) - \{13\}$.

$v = 12$.

Applying Lemma 1 to $J(6)$ and $U(6)$ we get $k \in J(12)$ for all $k \in I(12)$ except for $k = 0, 1, \ldots, 5$. By Remark 3, since $\{0, 1, \ldots, 5\} \subseteq J(6)$ we have $J(12) = I(12)$.

$v = 15$.

Let (V,B) be an STS(7) where $V = \{a_i : i = 1,2,\ldots,7\}$ and $a_3a_4a_7 \in B$.

let $\mathcal{F} = \{F_i : i = 1,2,l \ldots,7\}$ be the following 1-factorization of K_8 with the vertex – set $X = \{1, 2, \ldots, 8\}$:

$F_1 = \{12, 34, 56, 78\}$; $F_2 = \{13, 24, 57, 68\}$; $F_3 = \{14, 23, 58, 67\}$; $F_4 = \{15, 26, 37, 48\}$; $F_5 = \{16, 25, 38, 47\}$; $F_6 = \{17, 28, 35, 46\}$; $F_7 = \{18, 27, 36, 45\}$.

Put $V \cup X = S$ and $C = \{a_i xy : xy \in F_i, \ i = 1, 2,l \ldots, 7\}$ .Then $(S, B \cup C)$ is an STS(15).

$\pi = \{a_1 56, a_2 13, a_5 47, a_6 28, a_3a_4a_7\}$ is a parallel class of $(S, B \cup C)$ and so by Remark 1, we can construct an ETS(15) (S,T) , with $T = ((B \cup C) - \pi) \cup L(\pi)$.

So, now :

i) if T_1 is obtained from T by removing the blocks $a_4 15$, $a_4 26$, $a_5 16$, $a_5 25$ and replacing them with $a_5 15$, $a_5 26$, $a_4 16$, $a_4 25$,

ii) if T_2 is obtained from T by removing the blocks $a_3 14$, $a_3 23$, $a_3 58$, $a_3 67$, $a_4 15$, $a_4 26$, $a_4 37$, $a_4 48$ and replacing them with $a_4 14$, $a_4 23$, $a_4 58$, $a_4 67$, $a_3 15$, $a_3 26$, $a_3 37$, $a_3 48$,

iii) if T_3 is obtained from T by removing the blocks $a_3 14$, $a_3 23$, $a_3 58$, $a_3 67$, $a_4 15$, $a_4 26$, $a_4 37$, $a_4 48$, $a_7 18$, $a_7 27$, $a_7 36$, $a_7 45$ and replacing them with $a_4 14$, $a_4 23$, $a_4 58$, $a_4 67$, $a_7 15$, $a_7 26$, $a_7 37$, $a_7 48$, $a_3 18$, $a_3 27$, $a_3 36$, $a_3 45$,

iv) if $L_1(\pi)$ and $L_2(\pi)$ have precisely $3r$ blocks in common, $r = 0, 1, \ldots, 5$,

v) noting that we can find two STS(7) (S, B_1) and (S, B_2) such that $a_3a_4a_7 \in B_1 \cap B_2$ with $|B_1 \cap B_2| = k$, $k \in \{1, 3, 7\}$, it is easy to check that :

$$45 - (7 - k + 15 - 3r + q) = (23 + k + 3r - q) \in J(15) ,$$

$k = 1,3,7$; $r = 0,1,\ldots,5$ and $q = 0,4,8,12$. So $\{12, 14, 15, \ldots, 39, 41,42,45\} \subseteq J(15)$.

Let $\alpha = (1, 2)(3, 4, 7)(5, 6)$ be a permutation on $\{1, 2, \dots, 7\}$, and $C_\alpha = \{a_i xy : xy \in F_{\alpha(i)}\}$, then $(S, B \cup C_\alpha)$ is an STS(15) containing the parallel class $\pi^* = \{a_1 13, a_2 56, a_5 28, a_6 47, a_3 a_4 a_7\}$. Put $T' = ((B \cup C_\alpha) - \pi^*) \cup L(\pi^*)$. Then (S, T') is an ETS(15).

Noting that we can find $L(\pi)$ and $L(\pi^*)$ with precisely 0,1,...,7 blocks in common, it is not difficult to see that $i + k - 1 \in J(15)$; $i = 0, 1, \dots, 7$, $k = 1, 3, 7$. So $\{0, 1, \dots, 13\} \subseteq J(15)$.

It remains to show that $40 = (s_{15} - 5) \in J(15)$.

Let (W, E) be the ETS(15) construct in Remark 2. If E^* is obtained from E removing the blocks $\{\{14,14,2\};\{2,2,11\};\{11,11,8\};\{5,11,14\};\{5,2,8\}\}$ and replacing them with $\{\{14,14,11\};\{11,11,2\};\{2,2,8\};\{5,11,8\};\{5,2,14\}\}$, we see that $|E \cap E^*| = 40$. Hence $J(15) = I(15)$.

$v = 18$.

Applying Lemma 2 to $J(9)$ and $U(10)$ we get $k \in J(18)$ for all $k \in I(18) - \{58\}$ By Construction 5), since $s_6 - 5 = 4 \in J(6)$, it is readily verified that $s_{18} - 5 = 58 \in J(18)$ and so $J(18) = I(18)$.

$v = 21$.

By a similar argument as Lemma 4, it is easy to see that $J(21) \supseteq I(21) - \{79\}$.

Let (V,B) be an ETS(9) where $V = \{a_i : i = 1,2,\dots,9\}$. Let $\mathcal{F} = \{F_i : i = 1,2,\dots,11\}$ and $\mathcal{G} = \{G_j : j = 1, 2, 3\}$ be two 1-factorizations of K_{12} and K_4 respectively with the vertex-set $X = \{1, 2, \dots, 12\}$ and $X' = \{1, 2, 3, 4\}$, such that $G_1 \subseteq F_1$; $G_2 \subseteq F_{10}$ and $G_3 \subseteq F_{11}$. Suppose $G_1 = \{14, 23\}$; $G_2 = \{12, 34\}$;

$G_3 = \{13, 24\}$. Put $S = V \cup X$ and $T = B \cup C \cup F_{10} \cdot F_{11}$ where $C = \{a_i xy : xy \in F_i, i = 1, 2, \dots, 9\}$. (S, T) is an ETS(21).

If T^* is obtained from T removing the blocks 112, 224, 443, $a_1 14$, $a_1 23$ and replacing them with 114, 223, 442, $a_1 34$, $a_1 12$, we see that $|T \cap T^*| = 79$. Hence $J(21) = I(21)$.

$v = 24$.

Since $J(12) = I(12)$, applying Lemma 3 we obtain $J(24) = I(24)$.

$v = 27$.

By a similar argument as Lemma 5, it is a routine matter to check $J(27) \supseteq I(27) - \{112, 115, 116, 130\}$. By Construction 5), it is readily verified that if $h_i \in s_9$, $i = 1,2,3$ then $s_{27} - ((18 - h_1) + (18 - h_2) + (18 - h_3)) = 81 + h_1 + h_2 + h_3 \in$ $\in J(27)$ and so $\{112, 115, 116\} \subseteq J(27)$.

Let (V,B) be an ETS(9) where $V = \{a_i : i = 1,2,\dots,9\}$. By (3) of $(*)$, we can factor the complete graph K_{18} on vertex set $X = \{1, 2, \dots, 18\}$ into a set of 18 triangles covering P_1, P_2, P_3 and a set of 11 one factors covering the other P_j $(j = 4, 5, \dots, 9)$.

Let $L = \{\{i, i+1, i+3\} : i = 1, 2, \dots, 18\}$ be the set of triangles and $\mathcal{F} = \{F_i : i = 1, 2, \dots, 11\}$ be the set of one-factors, where :

$F_1 = \{\{1,5\},\{2,11\},\{3,12\},\{4,13\},\{6,16\},\{7,14\},\{8,15\},\{9,18\},\{10,17\}\}$;

$F_2 = \{\{1,6\},\{2,7\},\{3,8\},\{4,9\},\{5,10\},\{11,15\},\{12,16\},\{13,17\},\{14,18\}\}$;

$F_3 = \{\{1,7\},\{2,6\},\{3,9\},\{4,18\},\{5,14\},\{8,13\},\{10,15\},\{11,16\},\{12,17\}\}$;

$F_4 = \{\{1,8\},\{2,9\},\{3,10\},\{4,12\},\{5,16\},\{6,14\},\{7,15\},\{11,17\},\{13,18\}\}$;

$F_5 = \{\{1,9\},\{2,14\},\{3,13\},\{4,8\},\{5,15\},\{6,17\},\{7,12\},\{10,16\},\{11,18\}\}$;

$F_6 = \Big\{\{1,10\},\{2, 16\}, \{3,7\},\{4,11\},\{5,17\},\{6,13\},\{8,14\},\{9,15\},\{12,18\}\Big\}$;

$F_7 = \Big\{\{1,12\},\{2, 13\}, \{3,17\},\{4,15\},\{5,9\},\{6,11\},\{7,16\},\{8,18\},\{10,14\}\Big\}$;

$F_8 = \Big\{\{1,13\},\{2, 8\}, \{3,15\},\{4,14\},\{5,11\},\{6,12\},\{7,17\},\{9,16\},\{10,18\}\Big\}$;

$F_9 = \Big\{\{1,15\},\{2, 12\}, \{3,14\},\{4,10\},\{5,13\},\{6,18\},\{7,11\},\{8,16\},\{9,17\}\Big\}$;

$F_{10} = \Big\{\{1,11\},\{2, 15\}, \{3,16\},\{4,17\},\{5,18\},\{6,10\},\{7,13\},\{8,12\},\{9,14\}\Big\}$;

$F_{11} = \Big\{\{1,14\},\{2, 10\}, \{3,11\},\{4,16\},\{5,12\},\{6,15\},\{7,18\},\{8,17\},\{9,13\}\Big\}$;

Put $S = V \cup X$ and $T = B \cup C \cup L \cup F_{10} \cdot F_{11}$ where $C = \{a_i xy \ : \ xy \in F_i \ , \ i = 1, 2,\dots,9\}$.

If T^* is obtained from T removing the blocks $\{2,2,15\}$, $\{15,15,6\}$, $\{6,6,10\}$, $\{a_3,2,6\}$, $\{a_3,10,15\}$ and replacing them with $\{2,2,6\}$,$\{15,15,10\}$,$\{6,6,15\}$ $\{a_3,2,15\}$,$\{a_3,6,10\}$, we see that $|T \cap T^*| = 130$. Hence $J(27) = I(27)$.

5. CONCLUSION.

We now have our required result :

MAIN THEOREM. $J(v) = I(v)$ **for** $v \equiv 0 \pmod 3$, $v \neq 9$ **and** $J(9) = I(9) - \{13\}$.

Proof. For $v = 3 \cdot t$, $t = 1, 2, \dots, 9$ our statement follows from Section 4 .

Assume therefore $v \geq 30$, and assume that for all $w < v$ $(w \geq 15)$, $J(w) = I(w)$.

If $v \equiv 0$ or $6 \pmod{12}$ then $\frac{v}{2} \equiv 0$ or $3 \pmod 6$ and $\frac{v}{2} \geq 15$. Therefore $J\left(\frac{v}{2}\right) = I\left(\frac{v}{2}\right)$ and by Lemma 3 , $J(v) = I(v)$ as well.

If $v \equiv 3 \pmod{12}$ then $\frac{v-9}{2} \equiv 3 \pmod 6$ and $\frac{v-9}{2} \geq 15$. Therefore $J\left(\frac{v-9}{2}\right) = I\left(\frac{v-9}{2}\right)$ and by Lemma 5 , $J(v) = I(v)$ as well.

If $v \equiv 9 \pmod{12}$ then $\frac{v-3}{2} \equiv 3 \pmod 6$ and $\frac{v-3}{2} \geq 15$. Therefore $J\left(\frac{v-3}{2}\right) = I\left(\frac{v-3}{2}\right)$ and by Lemma 4 , $J(v) = I(v)$ as well.

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