Single Machine Scheduling to Minimize Total Absolute Lateness for a Specified Number of Tardy Jobs

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Abstract

We consider the problem of assigning a common due-date to a set of simultaneously available jobs and sequencing them on a single machine, so that a specified number of the jobs are tardy. The objective is to determine the optimal combination of the common due-date and job sequence that minimizes the total absolute lateness. A simple algorithm is presented to achieve this. Effectively three independent procedures are developed: one to give the optimal sequence; a second to give the optimal due date and a third to give an explicit formula for the associated minimum penalty.

1. Introduction

A great deal of research has been directed towards single-machine scheduling where the objective is the minimizing of penalties associated with both tardiness and earliness. Surveys by Cheng and Gupta (1989) and Baker and Scudder (1990) attest to the volume of this research up to 1990. More recently, interest in the area shows no sign of diminishing. An important aspect of this research relates to the problem of completing all jobs in a batch as close as possible to a common due-date: due no doubt to the widespread adoption of the philosophy of 'just-in-time' (JIT) customer service in industry. The seminal work for the common due-date problem is due to Kanet (1981), and as outlined below his work has been developed and extended by a number of authors. It is the purpose of this paper to present and solve a new variation of the common due-date problem.

For a given number of tardy jobs, we give independent procedures for finding (a) a permutation sequence of the jobs which is optimal with respect to total absolute lateness in relation to a common due date; (b) the optimal common due date without finding the corresponding sequence, and (c) an explicit formula for the optimal total absolute lateness without finding the associated optimal permutation sequence or the optimal common due date.

In Kanet's original work, the number of tardy jobs and the number of early jobs are approximately equal and the optimal due date is located at about the centre of the makespan of the job set. Later studies provide for differential weighting of early and tardy jobs, and, as a result the optimal common due date can vary greatly. The actual number of tardy jobs in the optimal sequence then

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follows as a consequence of these weights. However, a due date for a batch of jobs may need to be set so that a specific number of the jobs are completed by that date. For example, this could occur if the client's or the manufacturer's storage facilities are limited, or, if there is a preference for a particular proportion of a batch being completed by a common due date. In this paper, we provide the scheduler with the opportunity to nominate any number of tardy jobs and in this context be provided with the corresponding optimal sequence, optimal due date or optimal penalty, or any combination of these three. This gives the scheduler flexibility in choosing prescribed numbers of tardy jobs and examining the consequences.

2. Previous Work

Kanet (1981) provides an O(n log n) algorithm for producing an optimal sequence for the total absolute lateness (early/tardy) problem and then computes the optimal common due date. The sequence is obtained before the optimal common date is calculated. Kanet's work has been extended in a variety of ways.

There are, for example, extensions to larger classes of penalty functions. Seidmann, Panwalker and Smith (1981) and Panwalker, Smith and Seidmann (1982) examine the problem when the penalty function is a linear combination of costs associated with the due date, the earliness and the tardiness. Lee, Danusaputro and Lin (1991) investigate the construction of a common due date where the penalty function is the weighted sum of the number of tardy jobs and earliness / tardiness penalties. Cheng (1992) considers Kanet's original problem and shows how the due date assignment problem can be separated from the job sequencing problem. He shows that for a given job sequence, the optimal due date is a simple function of the number of jobs. Cheng and Kahlbacher (1993) find an optimal combination of common due date and job sequence that minimizes a cost function based on the assigned due dates, job earliness and the number of tardy jobs. Kahlbacher (1993) considers the class of penalty functions that are monotonous with respect absolute lateness. In another direction, Chand and Chhajed (1992) using the same objective function as Panwalker, Smith and Seidmann, solve the problem of assigning p due dates to n jobs, where $1 \leq p \leq n$, and where specified numbers of jobs are to be assigned to each date; so that the total penalty is minimized.

An effect of these extensions of Kanet's original result is to provide an optimal common due date that is often not near the centre of the makespan of the job sequence. The reality is that the earliness of jobs and the tardiness of jobs are not always of equal importance to the scheduler, when a common due date for a batch of jobs is desired. In previous research, weighting factors are attached to early and tardy measures to take account of their relative importance. This action effectively results in the specification of a particular number of tardy (or early) jobs. In this paper, we approach the problem directly by allowing the scheduler to experiment with the specification of the number of tardy jobs and to find the consequent penalties.

A novelty in the results obtained here is the independent formulation of each factor of interest: the optimal schedule; the associated optimal due date; and, the associated optimal penalty. The scheduler can adopt a 'what if' approach and determine the results for each of a variety of numbers of tardy jobs. For example, if the scheduler applies Kanet's algorithm, then a common date is obtained which makes about half of the jobs tardy. What are the consequences of reducing the number of tardy jobs by one or two? How will this affect the size of the penalty, or the optimal due date, or the composition of the optimal schedule? Each of these questions can be pursued quickly and independently using the results of this paper.

3. The Problem

Consider a set of n jobs with given integer processing times $t_1, t_2, \dots t_n$. For any job in position j in a particular sequence, we define its earliness E_j and its tardiness T_j by

$$E_j = \max\{d - C_j, 0\} \quad \text{and} \quad T_j = \max\{C_j - d, 0\}$$

respectively, where d is the common due date and C_j is the completion time of the job in position j in the particular sequence.

$$C_j = \sum_{i=1}^{j} t_i$$

The total absolute lateness of a set of jobs in a particular sequence is defined as

$$\sum_{j=1}^{n} abs\ (C_j - d) = \sum_{j=1}^{n} (E_j + T_j)$$

We remark that for any job in position j in a particular sequence, at most one of the measures E_j and T_j will be non-zero. In addition, at most one job j will exactly on time, that is, $E_j = T_j = 0$.

Let n_t be the given number of tardy jobs and define $m = \max\{n_t, n - n_t - 1\}$; and P as the total absolute lateness for the given number of tardy jobs with respect to an arbitrary sequence of jobs and common due date d. We note that m is the larger of the number of tardy jobs and the number of strictly early jobs.

The problem to be addressed can be stated as
GIVEN:

i) a set of n jobs with specified integer processing times to be processed on a single machine; and

ii) a prescribed number (n_t) of them to be tardy.

FIND:

a) A sequence of the jobs which will minimize the total absolute lateness of the set with respect to an optimal common due date, (which can always be found once the sequence is constructed).

b) The optimal common due date without necessarily first performing the procedure in part (a).

c) The optimal total absolute lateness without necessarily first performing the procedures in part (a) and/or part (b).

The problem will be solved by first proving a related simpler problem and then extending the result. This simpler problem will be referred to as "the restricted problem". Its statement is identical to the statement made above with an additional requirement in the "given" statement, namely:

"and iii) one job exactly on time".

4. The Results

4.1 Results for the restricted problem (where one job is exactly on time).

(a) Obtaining the sequence.

In this section we construct a sequence from the n jobs, which is optimal with respect to total absolute lateness; given n_t, the prescribed number of tardy jobs. The proof of optimality is given in Section 5. First, sequence the jobs in longest processing time (LPT) order. Assign the longest job to the first position in sequence the second longest job to the last position in sequence, the third longest job to second position in sequence, the fourth longest job to the second last position in sequence and so on, breaking ties where jobs are of equal duration, arbitrarily. This action continues until the conditions described below hold.

Consider two cases. First, where the number of tardy jobs is strictly greater than the number of early jobs. That is, suppose $m = n_t$ and $n \neq 2m + 1$. In this case, continue the above procedure until n - m jobs have been assigned to the beginning of the sequence and the same number of jobs has been assigned to the end of the sequence. Assign the remaining jobs, if any, in shortest processing time (SPT) order.

On the other hand, if $m \neq n_t$ or $n = 2m + 1$, then continue the procedure until n - m jobs have been assigned to the beginning of the sequence and n - m -1 jobs have been assigned to the end of the sequence. Assign the remaining jobs, if any, in LPT order.

In either case, the resulting sequence is optimal with respect to total absolute lateness in relation to the (optimal) common due date d, which is given independently in the next paragraph.

(b) The optimal due date.

We write down the optimal due date for the given set of jobs with respect to total absolute lateness given a prescribed number of tardy jobs. This result is independent of the previous section in the sense that the date can be calculated by the formula without first obtaining the optimal sequence. The proof is given in Section 5.

First, sequence the jobs in LPT order. Then

$$d = \sum_{r=1}^{n-n_t} t_{2r-1}$$

(c) The optimal penalty.

We state the optimal total absolute lateness P for the given set of jobs and the prescribed number of tardy jobs. The expression is independent of the previous two sections and is given by the following:

First sequence the jobs in LPT order. Then

$$P = \sum_{r=1}^{n-m} (r-1)t_{2r-1} + \sum_{r=1}^{n-m-1} rt_{2r} + A$$

where, $A = \sum_{r=0}^{2m-n} (m-r)t_{n-r}$, if $n \leq 2m$

$= 0,$ otherwise.

The proof is given in Section 5.

4.2 Results for the unrestricted problem.

In this section we consider the results where the due-date is not restricted to a job completion time.

- If $m \neq n_t$ or $n = 2n_t + 1$ or $n = 2n_t$, then

  a) the optimal sequence, and

  b) the optimal due date, and

  c) the optimal penalty are the same as for the restricted problem.

- If $m = n_t$ and $n \neq 2n_t + 1$ and $n \neq 2n_t$ then

  a) the optimal sequence is the same as the optimal sequence for the restricted problem with n_t-1 tardy jobs.

  b) the optimal due date d is given by

$$d = \sum_{r=1}^{n-n_t+1} t_{2r-1} - 1$$

  That is, the optimal due date, is one unit of time less than the optimal due date for the restricted problem with n_t-1 jobs. This ensures there are n_t tardy jobs. One unit of time is subtracted as all processing times are integers.

  c) the optimal penalty is equal to the value of the optimal penalty for the restricted case with n_t-1 tardy jobs plus the number $2n_t$-n.

We note in addition, the time complexity for each procedure considered separately. For both the restricted and the unrestricted cases, the algorithms contained in parts (a) and (b) have polynomial-bound time complexity O(n log n) and the bound for the algorithm in part (c) is $O(n^2)$.

5. Proof of the Results

5.1 Proof of Results for the restricted problem.

We begin by considering the total absolute lateness of an arbitrary permutation sequence of n jobs with an arbitrary common due date d coinciding with the completion time of one of the jobs, so that as a result n_t jobs are tardy and one job is exactly on time. Later, in Section 5.2, we consider under which circumstances this date can be modified if necessary to become globally optimal for a given number of tardy jobs. The penalty for each job may be set out line by line as follows to facilitate vertical addition.

$d - t_1$

$d - t_1 - t_2$

....

$d - t_1 - t_2 - \ldots - t_{n-nt-1}$

$d - t_1 - t_2 - \ldots\ldots - t_{n-nt}\ (=0)$ (1)

$t_1 + t_2 + \ldots\ldots + t_{n-nt} + t_{n-nt+1} - d$

....

$t_1 + t_2 + \ldots\ldots\ldots\ldots\ldots\ldots + t_n - d$

(a) The optimal sequence.

We consider three cases. In case 1, $m = n_t$ and $n \neq 2m + 1$. Case 2 has $n = 2m + 1$ and in case 3, $m = n - n_t - 1$.

Case 1. (where $m = n_t$ and $n \neq 2m + 1$).

Adding vertically, we obtain

$$P = [(2n_t - n)t_1 + (2n_t - n + 1)t_2 + \dots + (n_t - 1)t_{n-nt}]$$
$$+ [n_t t_{n-nt+1} + (n_t - 1)t_{n-nt+2} + \dots + (n - n_t + 1)t_{nt}]$$
$$+ [(n - n_t)t_{nt+1} + \dots + 2\, t_{n-1} + t_n]$$
$$- (2n_t - n)d.$$

Now from (1), we can replace d in the last equation as a sum of processing times and on simplification, the penalty P can be re-written as

$$P = [0t_1 + 1t_2 + \dots + (n - n_t - 1)t_{n-nt}]$$
$$+ [n_t t_{n-nt+1} + (n_t - 1)t_{n-nt+2} + \dots + (n - n_t + 1\ )t_{nt}]$$
$$+ [(n - n_t)t_{nt+1} + \dots + 2\, t_{n-1} + t_n]. \qquad (2)$$

P is now expressed entirely in terms of processing times. The symbol 'd' is no longer in the expression. The expression for P consists of the sum of products of processing times and integer constants of various sizes. Such a sum is minimized by associating the longest processing time with the smallest constant, here 0; the second largest processing time with the next smallest constant, here 1; and so on. A proof is given in Hardy et al (1959). We will refer to this process as associating the constants and processing times in 'contra' order. Following these directions, and additionally re-assigning subscripts to the processing times in equation (2), so that t_1, t_2, t_3, ... are in LPT order we obtain:

$$P = [0t_1 + 1t_3 + \dots + (n - n_t - 1)t_{2n-2nt-1}]$$
$$+ [n_t t_n + (n_t - 1)t_{n-1} + \dots + (n - n_t + 1)t_{2n-2nt+1}]$$
$$+ [(n - n_t)t_{2n-2nt} + \dots + 2t_4 + t_2]. \qquad (3)$$

Remembering that $m = n_t$, we see that equation (3) indicates that n - m jobs have been assigned to the beginning of the sequence and n - m jobs have assigned to the end of the sequence and the remaining jobs, if any, have been placed in between in SPT order in the way described in part (a).

In the special case when $n = 2n_t$, the term $-(2n_t-n)d$ is zero and the second term in square brackets in equation (3) contains no elements. This case is covered in Kanet's original result.

Case 2. (where $n = 2m + 1$)

In this situation, n is odd and $m = n_t = n - n_t - 1$, that is, the number of tardy jobs is equal to the number of strictly early jobs. This case is one covered by Kanet's original result. On assigning n - m jobs at the start and n - m - 1 jobs at the end of the sequence, all positions are filled. We again add vertically and note that $2n_t - n = -1$ and so the penalty P is given by:

$$P = [-1t_1 + 0t_2 + 1t_3 + \dots + (n_t - 1)t_{n-nt}]$$
$$+ [n_t t_{nt+2} + \dots + 2\, t_{n-1} + t_n] + d.$$

From (1) as $d = t_1 + t_2 + t_3 + \dots + t_{n-nt}$, we can re-write this as

$$P = [0t_1 + 1t_2 + 2t_3 + \dots + (n_t t_{n-nt}]$$
$$+ [n_t t_{nt+2} + \dots + 2t_{n-1} + t_n].$$

Following the minimizing procedure of ordering the processing times by LPT and assigning processing times to coefficients of contra size, we obtain

$$P = [0t_1 + 1t_3 + \dots + n_t t_{2n-2nt-1}]$$
$$+ [n_t t_{2nt} + \dots + 2t_4 + t_2] \qquad (4)$$

which again matches the requirements of part (a) of the results.

Case 3. (where $m \neq n_t$, that is, $m = n - n_t - 1 > n_t$)

Adding vertically as before we note that $n - m = n - n + n_t + 1 = n_t + 1$ and that the coefficient of t_{nt+1} is $2n_t - n + n - m - 1 = 3n_t - n$. On assigning n - m jobs at the start and n - m - 1 jobs at the end of the sequence we obtain

$$P = [(2n_t - n)t_1 + (2n_t - n + 1)t_2 + \dots + (3n_t - n)t_{nt+1}]$$
$$+ [(3n_t - n + 1)t_{nt+2} + (3n_t - n + 2)t_{nt+3} + \dots + (n_t - 1)t_{n-nt}]$$
$$+ [n_t t_{n-nt+1} + \dots + 2t_{n-1} + t_n]$$
$$- (2n_t - n)d.$$

We replace d as a sum of processing times using (1) and obtain

$$P = [0t_1 + 1t_2 + \dots + n_t t_{nt+1}]$$
$$+ [(n_t + 1)t_{nt+2} + (n_t + 2)t_{nt+3} + \dots + (n - n_t - 1)t_{n-nt}]$$
$$+ [n_t t_{n-nt+1} + \dots + 2t_{n-1} + t_n].$$

Next we again use the optimizing process of associating the coefficients and processing time in contra order and re-label these times in LPT order t_1, t_2, t_3... The expression for P is:

$$P = [0t_1 + 1t_3 + \dots + n_t t_{2nt+1}]$$
$$+ [(n_t + 1)t_{2nt+2} + (n_t + 2)t_{2nt+3} + \dots + (n - n_t - 1)t_n]$$

$+ [n_t t_{2n_t} + \dots + 2t_4 + t_2].$ (5)

Now in this case $m = n - n_t - 1$ and so $n_t = n - m - 1$. Hence

$$P = \sum_{r=1}^{n-m} (r-1)t_{2r-1} + \sum_{r=1}^{n-m-1} rt_{2r} + A$$

$$\text{where,} \quad A = \sum_{r=0}^{2m-n} (m-r)t_{n-r} \qquad \text{if } n \le 2m$$

$$= 0, \qquad \text{otherwise.} \tag{6}$$

Equation (6) gives the minimal total absolute lateness for a given number of tardy jobs, where the due date coincides with the completion of a job, in the three cases discussed. In equation (3), the substitution $m = n_t$ needs to be made and in equation (4), the substitution is also $m = n_t$.

(b) The optimal due date.

As the due date coincides with the completion of a job, the maximum number of tardy jobs is $n - 1$ and developments in section (a) show that if the jobs are sequenced in LPT order, optimal d is given by

$$d = \sum_{r=1}^{n-n_t} t_{2r-1} .$$

(c) The optimal penalty.

From part (a), the optimal total penalty for a given number of tardy jobs (where the common date is to coincide with a job completion time) is given by equation (6).

5.2 A Corollary to the Restricted Problem.

The absolute value of the difference between the optimal penalty for the case of n_t tardy jobs and the case of $n_t - 1$ tardy jobs is zero, if $n = 2n_t$ and

$$\sum_{r=0}^{|2n_t-n|-1} t_{n-r} \text{ , otherwise.}$$

Proof

Suppose $n = 2n_t$, Then from the remarks made at the end of case 1 of the proof of the restricted problem,

$$P = [0t_1 + 1t_3 + \dots + (n_t - 1)t_{n-1}] + [n_t\, t_n + \dots + 2t_4 + 1t_2].$$

Note that there is an equal number of (n_t) terms in each bracket. If we now reduce the number of tardy jobs by 1, then $m \neq n_t - 1$, and, on assigning n - m jobs at the start of the sequence and n - m - 1 jobs at the end of the sequence in the required way, the shortest job is left and is placed in between. Hence

$$P' = [0t_1 + 1t_3 + \ldots + (n_t - 1)t_{n-1}] + [n_t\, t_n] + [(n_t - 1)t_{n-2} + \ldots + 2t_4 + 1t_2].$$

We obtain $P - P' = 0$.

Now suppose $m = n_t$ and $n \neq 2n_t$ and $n \neq 2n_t + 1$. The penalty P is given by equation (3). The corresponding penalty P′ for $n_t - 1$ tardy jobs is given by

$$\begin{aligned} P' = {} & [0t_1 + 1t_3 + 2t_5 + \ldots + (n - n_t - 1)t_{2n-2nt-1} + (n - n_t)t_{2n-2nt+1}) \\ & + [(n_t - 1)t_n + (n_t - 2)t_{n-1} + \ldots + (n - n_t + 3)t_{2n-2nt+4} \\ & + (n - n_t + 2)t_{2n-2nt+3}] \\ & + [(n - n_t + 1)t_{2n-2nt+2} + (n - n_t)t_{2n-2nt} + \ldots + 2t_4 + 1t_2]. \end{aligned}$$

Then $$P - P' = \sum_{r=0}^{2n_t-n-1} t_{n-r} > 0 .$$

The case where $n = 2n_t + 1$ can be proved in a way similar to the case where $n = 2m$ $(= 2n_t)$.

The case where $m \neq n_t$ can be established using a similar argument to the above and gives

$$P - P' = \sum_{r=0}^{n-2n_t-1} t_{n-r} > 0$$

Thus $|P - P'| = 0$ if $n = 2n_t$

and $$|P - P'| = \sum_{r=0}^{|2n_t-n|-1} t_{n-r} \text{ , otherwise.}$$

For a given set of jobs the optimal penalty P is monotone decreasing with respect to the number of tardy jobs for $0 \leq n_t \leq \lceil 0{\cdot}5n \rceil$ and monotone increasing for $\lceil 0{\cdot}5n \rceil \leq n_t \leq n$, where $\lceil x \rceil$ is defined as the smallest integer greater than x.

5.3 Proof of results for the unrestricted problem.

In this section, we consider how the results for the restricted problem need to be modified when the common due date is not limited to a job completion time. The case where $n = 2n_t + 1$ and where $n = 2n_t$ is covered by Kanet's work and so procedures (a), (b) and (c) for the restricted problem remain optimal for the problem addressed by Kanet.

We next employ an argument used by Cheng (1987) in relation to Kanet's work to show that the optimal due date for the unrestricted problem will coincide with or be within an arbitrarily small difference of a job completion time.

Let d' be any arbitrarily chosen common due date which does not coincide with any of the job completion times (i.e. $C_{[j-1]} < d' < C_{[j]}$, $j = 1, 2, \ldots, n$), where $C_{[j]}$ is the completion time of the job in position j.

Then d' expressed in the form of a Gantt chart will be as follows:

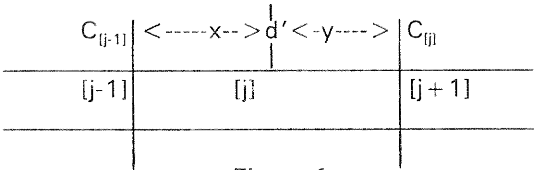


Figure 1

If we shift d' to the right so that it is equal to $C_{[j]}$, then the following change in penalty will arise

$$\Delta P_R = (j - 1)y - (n - j + 1)y = (2j - n - 2)y.$$

Similarly, if we shift d' to the left so that it equals $C_{[j-1]}$, then the following change in penalty occurs

$$\Delta P_L = (n - j + 1)x - (j - 1)x = (n - 2j + 2)x.$$

Since x, y and $n > 0$, it follows that

$$\Delta P_R \leq 0 \text{ if } j \leq n/2 + 1$$

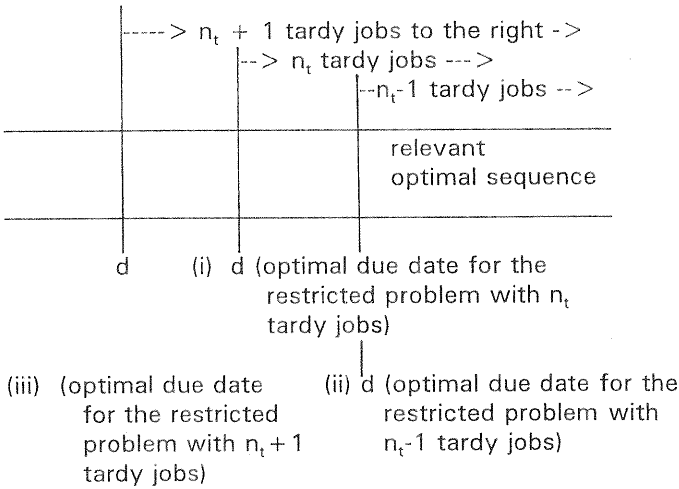
and

$$\Delta P_L \leq 0 \text{ if } j \geq n/2 + 1.$$

Thus for any given d' we can shift it to the left or to the right depending on its value so that a reduced or equally good penalty value can be achieved. Consequently the optimal due date must be equal to one of the job completion times; or be arbitrarily close to it to preserve the required number of tardy jobs.

This means that the optimal sequences developed for the restricted problem are also optimal for the unrestricted problem but it remains to associate a specified number of tardy jobs in the unrestricted problem with the appropriate optimal sequence from the restricted problem. That is, we must ascertain whether an optimal sequence for the restricted problem with n_t jobs is also an optimal sequence for the unrestricted problem with the same number of tardy jobs.

Suppose d is optimal, then $d = C_{[s]}$ where s is determined by the specified number of tardy jobs and by whether $m = n_t$. Suppose $m = n_t$, then s will have a value satisfying $s \leq n/2 + 1$ and so any arbitrary due dates which preserve the specified number of tardy jobs will shift to the right so as to coincide or nearly coincide with a job completion time. Three optimal sequences for the restricted problem each associated with different numbers of tardy jobs are the only candidates for the optimal sequence for the unrestricted problem with n_t tardy jobs. These sequences are illustrated in Figure 2. The sequence relevant to the unrestricted sequence with n_t tardy jobs is determined in what follows.



Optimal sequences for the restricted problem with (i) n_t, (ii) n_t-1 and (iii) n_t+1 tardy jobs.

Figure 2

The first sequence is illustrated in part (i) of Figure 2, and is the optimal sequence for the restricted problem with n_t tardy jobs (having penalty P). We shift its due date as far as possible to the right while still preserving the same number of tardy jobs, that is, shift it by an amount $t_n - 1$, as by equation (3) the first late job is the shortest job and has processing time t_n. The new penalty P_1 is given by

$$P_1 = P - (2n_t - n)(t_n - 1)$$

and as $m = n_t$, we have $2n_t - n \geq 0$ and so $P_1 \leq P$. The second sequence is illustrated in part (ii) of Figure 2. The specified number of tardy jobs can be obtained by taking the optimal sequence for the restricted version with $n_t - 1$ tardy jobs and an arbitrary due date to the left of the optimal date for this version. When such a date is moved as far as possible to the right (while still retaining n_t tardy jobs), it has a value one unit less than the optimal due date for

n_t - 1 tardy jobs. The resultant penalty P_2 is given by

$$P_2 = P - \sum_{r=0}^{2n_t-n-1} t_{n-r} + (2n_t - n)$$

where we have used the result of the corollary to the restricted problem to give the difference between the optimal penalties for sequences with n_t and n_t - 1 tardy jobs. We next compare the relative sizes of P_1 and P_2.

Upon simplification, we obtain

$$P_1 - P_2 = \sum_{r=0}^{2n_t-n-1} t_{n-r} - (2n_t - n)t_n$$

$$= \sum_{r=0}^{2n_t-n-1} (t_{n-r} - t_n) \geq 0$$

after noting that t_n is the smallest of the $2n_t$ - n numbers $t_n, t_{n-1}, \ldots, t_{2n-2nt+1}$ which lie in SPT order in the associated optimal sequence for the restricted problem. Note that when these numbers all have the same value, $P_1 = P_2$.

Thus $P_2 \leq P_1 \leq P$. The only other approach (illustrated in part (iii) of Figure 2) is to take the optimal sequence in the restricted problem for $n_t + 1$ tardy jobs and move the due date to the right until there are n_t tardy jobs. But the associated penalty then can be no less than P, which is the optimal penalty for the restricted problem.

Hence when $m = n_t$ in the unrestricted case, we choose to form the optimal sequence for n_t - 1 tardy jobs for the restricted problem and select a common date one unit less than the associated optimal due date for this restricted case. The associated optimal penalty for the unrestricted case is P_2, which is equal to the optimal penalty for the restricted case with n_t - 1 tardy jobs plus the number $2n_t$ - n.

On the other hand, suppose $m \neq n_t$. We again consider the same sequences illustrated in Figure 2. In this case, n_t is relatively small and so s will have a value satisfying $s \geq n/2 + 1$. As a consequence, arbitrary due dates which preserve the required number of tardy jobs will shift to the left for optimality to coincide with the exact completion of n - n_t jobs. Thus $s = n - n_t$ and the relevant optimal sequence is that for the restricted problem with n_t tardy jobs (see part (i) of Figure 2). Note from part (ii) of Figure 2, that the optimal sequence for the restricted problem involving n_t - 1 tardy jobs can be made to have n_t tardy jobs by taking an arbitrary due date to the left of the optimal due date for the restricted version and shifting it further to the left to coincide with the nearest job completion time. However, this action does not result in any improvement in the penalty for the restricted version with n_t tardy jobs than that already achieved. An argument similar to that which established $P_2 \leq P_1$ earlier using the corollary to the restricted problem can be used to establish this fact. The third possibility involves the optimal sequence for the restricted problem with $n_t + 1$ tardy jobs and is illustrated in part (iii) of Figure 2. This can also be dismissed as the resultant penalty obtained by moving any due date to the left in this sequence (while preserving n_t tardy jobs) can be no better than that for n_t tardy jobs. Thus when $m \neq n_t$, the results for parts (a), (b) and (c) of the restricted problem are the same for the unrestricted problem. This completes the proof for the unrestricted version of the problem.

6. Discussion

In their review article, Baker and Scudder (1990) refer to the large variety of optimal due dates arising in Kanet's result from the way that pairs of jobs are assigned to the beginning and end of the optimal sequence. They mention the secondary criterion of minimizing the total processing time in the set of jobs scheduled before the common date. In many applications it is an advantage in terms of customer satisfaction to have the optimal common due date as early as possible, while the associated minimum penalty is preserved. In the current investigation, the number of tardy jobs is specified before sequencing and the manner of job assignment insures that the secondary criterion described above is achieved. A variety of optimal sequences still remain, when different jobs have the same processing times, but this variety does not change the optimal common date.

Bagchi, Sullivan and Chang (1986), in investigating the determination of an optimal sequence for unweighted total absolute lateness with respect to a given due date, refer to the V-shaped property of all optimal sequences. By this term they mean that the jobs preceding and succeeding the shortest job are in LPT and SPT orders, respectively. Krieger and Raghavachari (1992) prove that this property holds for optimal schedules with monotone penalty functions. In the current paper, the method of forming the optimal sequence, which is given in part (a) of the Results section; and its proof, show that the optimal sequences produced are also V-shaped.

As mentioned earlier, Cheng (1992) show that for a given job sequence, the optimal common due date k* is a simple function of the number of jobs. He shows

$$k^* = C_{[(n+1)/2]} \quad , \qquad n \text{ odd}$$
$$= C_{[n/2]} + ft_{[n/2+1]} \quad \text{for some } 0 \leq f \leq 1, \text{ if n is even, where } C_{[i]} \text{ is the}$$

completion time for the job in position i in sequence and $t_{[n/2+1]}$ is the processing time of the job in position $n/2 + 1$. In the notation of this paper this result translates to

$$k^* = C_{[n-m]} \quad , \qquad n = 2m + 1$$
$$= C_{[n-m]} + ft_{[n-nt+1]} \quad \text{for some } 0 \leq f \leq 1, \text{ if } n = 2n_t.$$

In the situation where we require the optimal due date for a given optimal job sequence, having a specified number of tardy jobs, the result is

$$k^* = C_{[n-m]}, \text{ for } m \neq n_t \text{ or } n = 2n_t \text{ or } n = 2m + 1$$
$$= C_{[n-m-1]} - 1 \text{ for } m = n_t \text{ and } n \neq 2m + 1 \text{ and } n \neq 2n_t. \quad (7)$$

In this paper, we have shown directly that the optimal common due date for the optimal sequence having a given number of tardy jobs is also a simple function of the number of jobs; and is given in part (b) of the Results section; and, equivalently by equation (7) once the optimal sequence is used.

The corollary to the restricted problem allows the generation of the penalty for any feasible number of tardy jobs once the penalty for a particular number of tardy jobs in a sequence is known. For example, if we begin by considering zero tardy jobs and follow the required procedures, the jobs are sequenced in LPT order and a common date equal to the sum of the processing times is assigned. The corresponding optimal penalty can also be calculated. The optimal penalty for successive numbers of tardy jobs decreases by the quantity given in the corollary until either $n = 2n_t$ or $n = 2n_t + 1$. It then increases in accordance with the requirements of both the corollary and the results of the unrestricted problem. This procedure is illustrated towards the end of the next section.

7. A Numerical example

We present a numerical example to illustrate the results. The example consists of twelve jobs.

Data		Summary of Results			Sample (n_t = 8)			
j	t_j	n_t	d	P	j	t_j	C_j	P
1	112	0	1254	6031	6	171	171	417
2	101	1	1140	5062	1	112	283	305
4	103	3	924	3784	10	109	392	196
5	71	4	823	3456	4	103	495	93
6	171	5	734	3311	9	94	589	1
7	89	6	663	3311	5	71	660	72
8	114	7	662	3313	11	74	734	146
9	94	8	588	3460	7	89	823	235
10	109	9	494	3790	2	101	924	336
11	74	10	391	4324	3	105	1029	441
12	111	11	282	5072	12	111	1140	552
		12	170	6043	8	114	1254	666

In the table above, the Data column gives the job numbers and processing times for the twelve jobs. The Summary of Results column gives for each possible number of tardy jobs (from 0 to 12), the optimal due date and the associated optimal penalty. The Sample column gives for $n_t = 8$, the optimal sequence in terms of the original jobs numbers; and, for each job in this sequence, its completion time C_j and the penalty it attracts, due to its position relative to the optimal due date of 588. We next verify the contents of the Sample column using the results developed in this paper.

First, place the jobs in LPT order and re-label them. We note that $n = 12$, $n_t = 8$ and that $n - n_t - 1 = 3$. Thus $m = 8$ and as $m = n_t$, we consider the restricted problem with $n_t - 1 = 7 = m$ tardy jobs and assign the first $n - m = 5$ odd numbered jobs at the beginning of the sequence and the first $n - m = 5$ even numbered jobs at the end of the sequence. The remaining two jobs are placed in between in SPT order. This can be verified in the Sample column where the original job numbers have been retained. The sequence is the optimal sequence.

Next we independently determine the optimal due date given by

$$d = \sum_{r=1}^{n-n_t+1=5} t_{2r-1} - 1$$

$$= 589 - 1$$

$$= 588$$

where the jobs have first been placed in LPT order.

The optimal penalty is given independently by

$$P = \sum_{r=1}^{5} (r-1)\, t_{2r-1} + \sum_{r=1}^{4} r t_{2r} + \sum_{r=0}^{2} (7-r) t_{12-r} + (2n_t - n)$$

$$= 1015 + 1055 + 1386 + 4$$

$$= 3460.$$

We now illustrate how the optimum penalty for each feasible number of tardy jobs can be generated from a particular optimal penalty. The optimal penalty for zero tardy jobs is obtained by sequencing the jobs in LPT order and assigning a common date of 1254, which is the makespan of the job set. The optimal penalty is 6031. To calculate the penalty for one tardy job we proceed as in the corollary and calculate

$$\sum_{r=0}^{n-2n_t-1} t_{n-r}\ , \quad \text{where } n_t = 1.$$

In this case, this sum is 969. The optimal penalty for one tardy job is then 6031 - 969 = 5062 and we can proceed in a similar way for increasing numbers of tardy jobs. In summary:

n_t	$\sum_{r=0}^{n-2n_t-1} t_{n-r}$	Optimal Penalty
2	746	5062 - 746 = 4316
3	532	4316 - 532 = 3784
4	328	3784 - 328 = 3456
5	145	3456 - 145 = 3311
6	0 (as $n = 2n_t$)	3311 - 0 = 3311

When $n_t > 6$, $m = n_t$ and we follow the results for the unrestricted problem. Thus for $n_t = 7$ we use the penalty for 6 tardy jobs calculated above and add $2n_t - n = 2$ to obtain the optimal penalty of 3311 + 2 = 3313. For $n_t > 7$ we calculate the penalty for the restricted version first. These penalties can also be obtained conveniently by symmetry with the optimal penalties for $n_t \leq 6$.

n_t	Optimal penalty for n_t - 1 jobs	$2n_t - n$	Optimal penalty
8	3456	4	3460
9	3784	6	3790
10	4316	8	4324
11	5062	10	5072
12	6031	12	6043

These results match those summarised in the previous table.

8. Concluding statement

In this paper we consider the problem of optimal sequencing of a set of jobs on a single machine to minimize total absolute lateness, where a prescribed number of jobs is tardy. Three independent procedures are presented. The first determines the optimal sequence and the second determines the optimal due date. Each of these procedures runs in O(n log n) time. The third determines the optimal penalty and runs in $O(n^2)$ time. The theoretical treatment involves a consideration of an arbitrary job sequence and an arbitrary common due date, which coincides with a job completion time. This date is then replaced with the corresponding sum of processing times to give a total penalty in terms of job processing times. A standard optimizing procedure follows and an adjustment is made to the resultant due date to make the results globally optimal. A numerical example is presented to illustrate the application of the procedures.

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