Concerning Cyclic Group Divisible Designs with Block Size Three

Zhike Jiang
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1

Abstract

We determine a necessary and sufficient condition for the existence of a cyclic $\{3\}$-GDD with a uniform group size $6n$. This provides a fundamental class of ingredients for some recursive constructions which settle existence of k-rotational Steiner triple systems completely.

1 Preliminaries

A *group divisible design* (GDD) is an ordered triple $(V, \mathcal{G}, \mathcal{B})$, where V is a finite set, $\mathcal{G}$ a partition of V into *groups* and $\mathcal{B}$ a set of subsets of V, called *blocks*, such that each pair of elements from different groups appears in exactly one block and no block contains two elements from a common group. The *group type*, or simply the *type*, of a GDD $(V, \mathcal{G}, \mathcal{B})$ is denoted $g_1^{u_1} g_2^{u_2} \dots g_m^{u_m}$ when $\mathcal{G}$ contains exactly u_i $(1 \le i \le m)$ groups of size g_i. When the block sizes of a GDD all appear in an integer set K the GDD is a $K-GDD$. A $\{k\}-GDD$ is simply denoted $k-GDD$.

An *automorphism* of a GDD $(V, \mathcal{G}, \mathcal{B})$ is a permutation π on V with the property that $\pi(B) \in \mathcal{B}$ for each $B \in \mathcal{B}$. A GDD $(V, \mathcal{G}, \mathcal{B})$ is *cyclic* if it has an automorphism which permutes the elements in each group $G \in \mathcal{G}$ in a $|G|$-cycle. A cyclic $k-GDD$ is denoted $k-CGDD$.

Group divisible designs are essential ingredients in constructing many combinatorial designs. We use group divisible designs with certain automorphisms in constructing other designs with relevant automorphisms. A Steiner triple system of order v is *k-rotational* if it admits an automorphism consisting of one fixed point and k cycles of length $(v-1)/k$. In [3], recursive constructions for k-rotational Steiner triple systems are developed to settle existence completely. In this paper, we investigate the existence of $3-CGDD$s of type $(6n)^u$ for integers $u \ge 3$, $n \ge 1$. These $3-CGDD$s are then used as an ingredient for the recursive constructions appearing in [3].

Throughout this paper, Z_g denotes the residue class group modulo g with residue classes $\{0, 1, \ldots, g-1\}$. When causing no ambiguity, $[i, j]$ denotes the set of integers ℓ such that $i \leq \ell \leq j$, $O[i, j]$ (or $E[i, j]$) the set of odd (or even) integers in $[i, j]$, and x_i denotes an element (x, i) in the set $Z_g \times \{i\}$.

We first introduce the following basic constructions. The first one is trivial from the point of view of the difference methods. (For terminology of the difference methods the reader is referred to [2].)

Lemma 1.1 *Let $V = Z_g \times \{1, 2, \ldots, u\}$. Define $G_i = Z_g \times \{i\}$ for $i = 1, 2, \ldots, u$. Suppose $\mathcal{D}$ is a collection of 3-subsets of V, such that (i) no member of $\mathcal{D}$ contains two elements from the same G_i and (ii) $\mathcal{D}$ covers each possible mixed (i, j) difference x, or simply x_{ij}, exactly once, where $x \in Z_g$ and $1 \leq i \neq j \leq u$. Then developing all the members of $\mathcal{D}$ over Z_g yields the set of blocks of a cyclic $3-GDD$ of type g^u with point set V and groups $G_1, G_2, \ldots, G_u$. The members of $\mathcal{D}$ are called its base blocks (over Z_g).*

Proof. Let $\mathcal{B}$ be the set of blocks obtained by developing members of $\mathcal{D}$ over Z_g, $\mathcal{G} = \{G_1, G_2, \ldots, G_u\}$. Obviously, $(V, \mathcal{G}, \mathcal{B})$ has an automorphism

$$\pi = (0_1 1_1 \ldots (g-1)_1) \ldots (0_u 1_u \ldots (g-1)_u),$$

and therefore it is cyclic.

Let G_i, G_j be distinct groups, and let $x_i \in G_i$ and $y_j \in G_j$. Then there is exactly one $D \in \mathcal{D}$ which covers the mixed difference $(y-x)_{ji}$. Without loss of generality, we assume $D = \{d_i, e_j, f_k\}$, where $1 \leq i, j, k \leq u$ are distinct, $d, e, f \in Z_g$ and $e - d = y - x$. Then $\pi^{x-d}(D) \in \mathcal{B}$. But $\pi^{x-d}(D) = \{x_i, y_j, (f + x - d)_k\} \supseteq \{x_i, y_j\}$. Therefore $(V, \mathcal{G}, \mathcal{B})$ is a $3-GDD$ of type g^u. □

The following example (c.f. Lemma 5.1, [4]) is an easy application of this construction with $u = 4$.

Example 1.2

Let $n \geq 1$ be an integer. The following base blocks cover each of the mixed differences in $Z_{6n} \times \{1, 2, 3, 4\}$ exactly once and therefore developing them over Z_{6n} yields a $3-CGDD$ of type $(6n)^4$:

$$\begin{array}{ll} \{0_1, r_2, (2r)_3\} & r \in [0, 3n-1]; \\ \{0_2, r_3, (2r+1)_4\} & r \in [3n, 6n-1]; \\ \{0_3, r_4, (2r-1)_1\} & r \in [1, 3n]; \\ \{0_4, r_1, (2r)_2\} & r \in [3n, 6n-1]. \end{array}$$

Lemma 1.3 *Consider the residue class group Z_{gu}. Suppose there exists a set $\mathcal{D}$ of 3-subsets of Z_{gu} such that the members of $\mathcal{D}$ cover each possible non-zero difference $d \in Z_{gu}$ exactly once for $d \not\equiv 0 \pmod u$ and no member of $\mathcal{D}$ covers a difference d with $d \equiv 0 \pmod u$. Then there exists a $3-CGDD$ of type g^u.*

Proof. We construct a $3-CGDD$ on $V = Z_g \times \{0, 1, \ldots, u-1\}$ as follows.

First note that each integer x with $0 \le x \le gu-1$ can be uniquely expressed as $x = x_1u + x_2$, where x_1, x_2 are integers and $0 \le x_1 \le g-1, 0 \le x_2 \le u-1$. Therefore each element $x = x_1u + x_2 \in Z_{gu}$ corresponds to an element $(x_1, x_2) \in V$. Now for each $D = \{x_1u+x_2, y_1u+y_2, z_1u+z_2\} \in \mathcal{D}$, where $0 \le x_1, y_1, z_1 \le g-1, 0 \le x_2, y_2, z_2 \le u-1$, we construct u 3-sets

$$D_i = \{(x_1, x_2+i), (y_1, y_2+i), (z_1, z_2+i)\} \ (i = 0, 1, \ldots, u-1),$$

where the arithmetic is taken modulo u. Clearly, each $D_i \subseteq V$. We claim that

$$\mathcal{A} = \bigcup_{D \in \mathcal{D}} \{D_i | 0 \le i \le u-1\}$$

is the set of base blocks of a $3-CGDD$ of type g^u on V with groups $G_i = Z_g \times \{i\}$ $(i = 0, 1, \ldots, u-1)$.

Suppose $D_i = \{(x_1, x_2+i), (y_1, y_2+i), (z_1, z_2+i)\} \in \mathcal{A}$, and D_i contains two elements, say (x_1, x_2+i), (y_1, y_2+i) from the same group G_j, equivalently, $x_2+i \equiv y_2+i \equiv j \pmod u$. Then $(y_1u+y_2)-(x_1u+x_2) \equiv (y_1u+y_2+i)-(x_1u+x_2+i) \equiv 0 \pmod u$. So the member $D = \{x_1u+x_2, y_1u+y_2, z_1u+z_2\} \in \mathcal{D}$ covers the difference $(y_1u+y_2)-(x_1u+x_2) \equiv 0 \pmod u$. That is a contradiction.

Now suppose $\delta_{qp} = (\delta, q)-(0, p) = (\delta, q-p)$ is a mixed difference across $Z_g \times \{p\}, Z_g \times \{q\}$ with $0 \le p < q \le u-1, 0 \le \delta \le g-1$. Then $d = \delta u + (q-p)$ is an element of Z_{gu} and $d \not\equiv 0 \pmod u$. By the assumption on $\mathcal{D}$, there is exactly one $D \in \mathcal{D}$ which covers the difference d exactly once. Without loss of generality, we assume $D = \{x_1u+x_2, y_1u+y_2, z_1u+z_2\}$ and $(y_1u+y_2)-(x_1u+x_2) = d$. Then $y_1 - x_1 = \delta, y_2 - x_2 = q - p$ and so

$$\begin{aligned} D_{p-x_2} &= \{(x_1, x_2+(p-x_2)), (y_1, y_2+(p-x_2)), (z_1, z_2+(p-x_2))\} \\ &= \{(x_1, p), (y_1, q), (z_1, z_2+(p-x_2))\} \end{aligned}$$

covers δ_{qp} since $(y_1, q)-(x_1, p) = (\delta, q-p)$.

Applying Lemma 1.1 then establishes our claim. □

The guiding principle of the construction in Lemma 1.3 is that we identify each element $x = x_1u + x_2 \in Z_{gu}$ $(0 \le x_1 \le g-1)$ with an element $(x_1, x_2) \in Z_g \times \{x_2\}$, for a fixed x_2. Taking $G = \{x \in Z_{gu} | x \equiv 0 \pmod u\}$, a subgroup of Z_{gu} of order g, each $G_j = Z_g \times \{j\}$ is nothing but a copy of the coset $G + j$ and all the base blocks $D_i \in \mathcal{A}$ $(i = 0, 1, \ldots, u-1)$ are simply obtained by developing the corresponding $D \in \mathcal{D}$ over Z_u. Therefore, the $3-CGDD$ resulting from the construction can be viewed as on Z_{gu} with groups $G + j$ $(j = 0, 1, \ldots, u-1)$, $\mathcal{D}$ being its generating blocks, which we call its *base blocks* (over Z_{gu}).

2 Constructions from Skolem Sequences

In this section we investigate the applications of Lemma 1.3 to constructing $3-CGDD$s of type $(6n)^u$ with $u \ge 4$, where we assume $u = 4s, 4s+1, 4s+2, 4s+3$

according to the congruence class u falls in. The technique we use is an analogue to that appearing in [5], namely by establishing certain Skolem sequences. According to Lemma 1.3, to establish existence of a $3-CGDD$ of type $(6n)^u$ we may attempt to partition all the differences $d \in Z_{6nu}$ with $d \not\equiv 0 \pmod{u}$ into base blocks (over Z_{6nu}). Since if a base block covers a difference d then it also covers $-d$, we only need to consider the differences $1 \leq d \leq 3nu$ such that $d \not\equiv 0 \pmod{u}$. To be precise, we have the following Heffter-type problem:

HP(n,u): Partition $\{1,2,\ldots,3nu\} \setminus \{mu|1 \leq m \leq 3n\}$ into $n(u-1)$ triples $\{a_r, b_r, c_r\}$ for $r = 1,2,\ldots,n(u-1)$ such that $a_r + b_r = c_r$ or $a_r + b_r + c_r = (6n)u$.

Lemma 2.1 *If there is a solution to HP(n,u), then there is a $3-CGDD$ of type $(6n)^u$.*

Proof. Let $\{a_r, b_r, c_r\}$ for $r = 1,2,\ldots,n(u-1)$ be a solution to HP(n,u). Construct triples $\{0, a_r, c_r\}$ for r such that $a_r + b_r = c_r$, and $\{0, a_r, -c_r\}$ for r such that $a_r + b_r + c_r = (6n)u$. Then these triples cover each of the differences $d \in Z_{(6n)u}$ with $d \not\equiv 0 \pmod{u}$ exactly once. Applying Lemma 1.3, we get the result. □

A solution to HP(n,u) may be obtained by solving the following Skolem-type problem. Take

$$D(n,u) = \{1,2,\ldots,nu\} \setminus \{mu|1 \leq m \leq n\},$$

$$S(n,u) = \{1,2,\ldots,2nu\} \setminus \{mu|1 \leq m \leq 2n\},$$

and consider

SP(n,u): Partition $S(n,u)$ into $n(u-1)$ ordered pairs (a_r, b_r) for $r \in D(n,u)$ such that $b_r - a_r = r$ for each r.

Example 2.2 *Let $t \geq 1$ be an integer. There is a solution to SP$(2t,5)$.*

In this case, $D(2t,5) = \{1,2,\ldots,10t\} \setminus \{5,10,\ldots,10t\}$, $S(2t,5) = \{1,2,\ldots,20t\} \setminus \{5,10,\ldots,20t\}$. To produce differences $d \in D(2t,5)$, take the following pairs:

$$\begin{array}{llll}
d \equiv 1,9 \pmod{10}: & (10t-1, 20t-2) \text{ and} & & \\
 & (2+r, 10t-5-r) & r \in [1, 5t-4], r \equiv 1,2 & \pmod{5}; \\
d \equiv 2,8 \pmod{10}: & (10t-3+r, 20t-3-r) & r \in [1, 5t-1], r \equiv 1,4 & \pmod{5}; \\
d \equiv 3,7 \pmod{10}: & (10t+1+r, 20t-r) & r \in [1, 5t-2], r \equiv 1,3 & \pmod{5}; \\
d \equiv 4,6 \pmod{10}: & (r, 10t-2-r) & r \in [1, 5t-3], r \equiv 1,2 & \pmod{5}.
\end{array}$$

A variation of the problem SP(n,u) may be considered for obtaining solutions to HP(n,u). Taking $D(n,u)$ to be the same as above, modifying $S(n,u)$ to

$$S(n;u,x) = \{1,2,\ldots,2nu\} \setminus \{mu|1 \leq m \leq 2n\} \setminus \{2nu-x\} \bigcup \{2nu+x\},$$

where $1 \leq x \leq 2nu, x \not\equiv 0 \pmod{u}$, we obtain the following problem.

SP$(n;u,x)$: Partition $S(n;u,x)$ into $n(u-1)$ ordered pairs (a_r, b_r) for $r \in D(n,u)$ such that $b_r - a_r = r$ for each r.

Lemma 2.3 *If there is a solution to $SP(n,u)$ or $SP(n;u,x)$, then there is a solution to $HP(n,u)$.*

Proof. Suppose (a_r, b_r) $(r \in D(n,u))$ is a solution to SP$(n;u,x)$. Then there exists $r^* \in D(n,u)$ such that $b_{r^*} = 2nu + x$. Construct a triple $\{r, nu+a_r, nu+b_r\}$ for each $r \in D(n,u)$ with $r \neq r^*$ and, for $r = r^*$, construct the triple $\{r^*, nu + a_{r^*}, 3nu - x\}$. We have $r + (nu + a_r) = (nu + b_r)$ for $r \neq r^*$, and $r^* + (nu + a_{r^*}) + (3nu - x) = 6nu$ because $b_{r^*} - a_{r^*} = r^*$ implies $(3nu + x) - (nu + a_{r^*}) = r^*$. It is also easy to check that these triples form a partition to the set $\{1, 2, \dots, 3nu\} \setminus \{mu | 1 \leq m \leq 3n\}$ and therefore a solution to HP(n,u).

Now suppose (a_r, b_r) $(r \in D(n,u))$ is a solution to SP(n,u). Similarly, triples $\{r, nu + a_r, nu + b_r\}$ $(r \in D(n,u))$ form a solution to HP(n,u). □

From Lemma 2.1 and Lemma 2.3, we see that to construct a $3-CGDD$ of type $(6n)^u$, we may solve a problem SP(n,u) or SP$(n;u,x)$. Example 2.2 establishes a solution to SP(n,u) when $n = 2t$ $(t \geq 1)$ and $u = 5$ and therefore the existence of a $3-CGDD$ of type $(12t)^5$ for each $t \geq 1$. We now deal with the other cases for these $3-CGDD$s by considering a problem SP$(n;u,x)$ with a properly chosen x in each case.

2.1 The Case $u \equiv 1 \pmod 4$

Assuming $u = 4s+1$ $(s \geq 2$ when $n \geq 2$ is even and $s \geq 1$ when $n \geq 1$ is odd$)$ and choosing $x = s$, we consider the problem SP$(n; 4s+1, s)$. We first partition $S(n; 4s+1, s)$ into $S_1(n; 4s+1, s)$ and $S_2(n; 4s+1, s)$, where

$$S_1(n; 4s+1, s) = \{1, 2, \dots, n(4s+1)\} \setminus \{m(4s+1) | 1 \leq m \leq n\},$$

$$S_2(n; 4s+1, s) = S(n; 4s+1, s) \setminus S_1(n; 4s+1, s).$$

Then we use the numbers in $S_1(n; 4s+1, s)$ to form ordered pairs which produce as differences all the odd (or even) numbers in $D(n, 4s+1)$ and those in $S_2(n; 4s+1, s)$ all the even (or odd) numbers in $D(n, 4s+1)$ when n is odd (or even). To construct these ordered pairs we first introduce some basic ingredients.

Lemma 2.4 *Assume $s \geq 1$. Let $E(s) = E[2, 4s]$ and $T(s) = ([1, 4s] \setminus \{3s+1\}) \bigcup \{5s+1\}$. Then there is a partition of $T(s)$ into $2s$ ordered pairs (c_i, d_i) $(i \in E(s))$ such that (i) $\bigcup_{i \in E(s)} c_i = [1, 2s]$, (ii) $\bigcup_{i \in E(s)} d_i = ([2s+1, 4s] \setminus \{3s+1\}) \bigcup \{5s+1\}$ and (iii) $d_i - c_i = i$ for $i \in E(s)$.*

Proof. Take the pairs as:

$$(1 + r, 4s + 1 - r) \quad r \in [1, s-1] \ (s \geq 2) \text{ and } r \in [s+1, 2s-1] \ (s \geq 2),$$

and $(1, 2s+1), (s+1, 5s+1)$. □

Lemma 2.5 *Assume $s \geq 1$. Let $F(s) = E[2, 8s]$ and $U(s) = [1, 8s+2] \setminus \{2s+1, 6s+2\}$. Then there is a partition of $U(s)$ into $4s$ ordered pairs (e_i, f_i) $(i \in F(s))$ such that (i) $\bigcup_{i \in F(s)} e_i = [1, 4s+1] \setminus \{2s+1\}$, (ii) $\bigcup_{i \in F(s)} f_i = [4s+2, 8s+2] \setminus \{6s+2\}$ and (iii) $f_i - e_i = i$ for $i \in F(s)$.*

Proof. Take the pairs as:

$$\begin{array}{ll} (1+r, 8s+3-r) & r \in [1, s-1] \ (s \geq 2) \text{ and } r \in [s+1, 2s-1] \ (s \geq 2), \\ (2s+1+r, 6s+1-r) & r \in [1, s-1] \ (s \geq 2) \text{ and } r \in [s+1, 2s-1] \ (s \geq 2), \end{array}$$

and $(4s+1, 6s+1), (s+1, 5s+1), (3s+1, 7s+3), (1, 6s+3)$. □

Lemma 2.6 *Assume $s \geq 2$. Let $G(s) = O[1, 4s-1]$ and $V(s) = [1, 4s+1] \setminus \{2s+1\}$. Then there is a partition of $V(s)$ into $2s$ ordered pairs (g_i, h_i) $(i \in G(s))$ such that $h_i - g_i = i$ for $i \in G(s)$.*

Proof. Take the pairs as:

$$\begin{array}{ll} (1+r, 4s+2-r) & r \in [1, s], \\ (s+1+r, 3s-r) & r \in [1, s-2] \ (s \geq 3), \end{array}$$

and $(3s, 3s+1), (1, 2s)$. □

The sequences of ordered pairs established in Lemma 2.4 - 2.6 can be used to build up a partition of $S_2(n; 4s+1, s)$ into $2ns$ ordered pairs which produce all the even numbers $d \in D(n, 4s+1)$ when n is odd or all the odd numbers $d \in D(n, 4s+1)$ when n is even (Assume $s \geq 2$ in the latter case). We illustrate our idea in the following examples, where (c_i, d_i) $(i \in E(s))$, (e_i, f_i) $(i \in F(s))$ and (g_i, h_i) $(i \in G(s))$ are defined, respectively, as in Lemma 2.4, Lemma 2.5 and Lemma 2.6.

Example 2.7

(For a $3-CGDD$ of type 6^{4s+1} with $s \geq 1$)

To partition $S_2(1; 4s+1, s)$ into $2s$ ordered pairs which produce all even numbers in $D(1, 4s+1)$, use the single "brick"

$$(4s+1+c_i, 4s+1+d_i) \quad i \in E(s).$$

(For a $3-CGDD$ of type 18^{4s+1} with $s \geq 1$)

To partition $S_2(3; 4s+1, s)$ into $6s$ ordered pairs which produce all even numbers in $D(3, 4s+1)$, pile the "brick"

$$(12s+3+c_i, 20s+5+d_i) \quad i \in E(s)$$

on top of the "brick"

$$(14s+3+e_i, 14s+3+f_i) \quad i \in F(s).$$

The top "brick" produces even differences $8s+4, 8s+6, \ldots, 12s+2$ and the bottom even differences $2, 4, \ldots, 8s$.

Example 2.8

(For a $3-CGDD$ of type 12^{4s+1} with $s \geq 2$)

To partition $S_2(2; 4s+1, s)$ into $4s$ ordered pairs which produce all odd numbers in $D(2, 4s+1)$, pile the "brick"

$$(8s+2+c_i, 12s+3+d_i) \quad i \in E(s)$$

on top of the "brick"

$$(10s+2+g_i, 10s+2+h_i) \quad i \in G(s).$$

The top "brick" produces odd differences $4s+3, 4s+5, \ldots, 8s+1$ and the bottom odd differences $1, 3, \ldots, 4s-1$.

(For a $3-CGDD$ of type 24^{4s+1} with $s \geq 2$)

To partition $S_2(4; 4s+1, s)$ into $8s$ ordered pairs which produce all odd numbers in $D(4, 4s+1)$, pile the following "bricks":

$$\begin{array}{ll} (16s+4+c_i, 28s+7+d_i) & i \in E(s), \\ (18s+4+e_i, 22s+5+f_i) & i \in F(s), \\ (22s+5+g_i, 22s+5+h_i) & i \in G(s). \end{array}$$

The differences produced by the these "bricks" are, respectively, $12s+5$, $12s+7, \ldots, 16s+3$, and $4s+3, 4s+5, \ldots, 12s+1$, and $1, 3, \ldots, 4s-1$.

Theorem 2.9 *There is a solution to $SP(n; 4s+1, s)$ for $s \geq 1$ when n is odd and $s \geq 2$ when n is even.*

Proof. Let (c_i, d_i) $(i \in E(s))$, (e_i, f_i) $(i \in F(s))$ and (g_i, h_i) $(i \in G(s))$ be, respectively, those partitions of $T(s)$, $U(s)$ and $V(s)$ obtained in Lemma 2.4, Lemma 2.5 and Lemma 2.6.

Suppose n is odd and $n = 2t+1$, $t \geq 0$. To produce all the odd numbers in $D(2t+1, 4s+1)$, take the following pairs:

$$(r, (2t+1)(4s+1)-r) \quad r \in [1, t(4s+1)+2s], \; r \not\equiv 0 \pmod{4s+1}.$$

To produce all the even numbers in $D(2t+1, 4s+1)$, take the following pairs:

$$((2t+1)(4s+1)+c_i, (4t+1)(4s+1)+d_i) \quad i \in E(s),$$

and, when $t \geq 1$, for each $j = 0, 1, 2, \ldots, t-1$,

$$((2t+1+j)(4s+1)+2s+e_i, (4t-j)(4s+1)-(2s+1)+f_i) \quad i \in F(s).$$

Suppose n is even and $n = 2t$, $t \geq 1$. To produce all the even numbers in $D(2t, 4s+1)$, take the following pairs:

$$(r, 2t(4s+1)-r) \quad r \in [1, t(4s+1)-1],\ r \not\equiv 0 \pmod{4s+1}.$$

To produce all the odd numbers in $D(2t, 4s+1)$, take the following pairs:

$$\begin{array}{ll} (2t(4s+1)+c_i, (4t-1)(4s+1)+d_i) & i \in E(s), \\ ((3t-1)(4s+1)+2s+g_i, (3t-1)(4s+1)+2s+h_i) & i \in G(s), \end{array}$$

and, when $t \geq 2$, for each $j = 0, 1, \ldots, t-2$,

$$((2t+j)(4s+1)+2s+e_i, (4t-2-j)(4s+1)-(2s+1)+f_i) \quad i \in F(s).$$

□

2.2 The Case $u \equiv 3 \pmod 4$

Assuming $u = 4s+3$ ($s \geq 1$) and choosing $x = s+1$, we consider the problem $\mathrm{SP}(n; 4s+3, s+1)$ when $n \geq 2$ is even. We first partition $S(n; 4s+3, s+1)$ into $S_1(n; 4s+3, s+1)$ and $S_2(n; 4s+3, s+1)$, where

$$S_1(n; 4s+3, s+1) = \{1, 2, \ldots, n(4s+3)\} \setminus \{m(4s+3) | 1 \leq m \leq n\},$$

$$S_2(n; 4s+3, s+1) = S(n; 4s+3, s+1) \setminus S_1(n; 4s+3, s+1).$$

Then we use the numbers in $S_1(n; 4s+3, s+1)$ to form ordered pairs which produce as differences all the even numbers in $D(n, 4s+3)$ and those in $S_2(n; 4s+3, s+1)$ all the odd numbers in $D(n, 4s+3)$. To construct these ordered pairs we first introduce some basic ingredients.

Lemma 2.10 *Assume $s \geq 1$. Let $E(s) = O[3, 4s+3]$ and $T(s) = ([1, 4s+3] \setminus \{2s+3, 3s+3\}) \bigcup \{5s+5\}$. Then there is a partition of $T(s)$ into $2s+1$ ordered pairs (c_i, d_i) ($i \in E(s)$) such that (i) $\bigcup_{i \in E(s)} c_i = [1, 2s+1]$, (ii) $\bigcup_{i \in E(s)} d_i = ([2s+2, 4s+3] \setminus \{2s+3, 3s+3\}) \bigcup \{5s+5\}$ and (iii) $d_i - c_i = i$ for $i \in E(s)$.*

Proof. Take the pairs as:

$$\begin{array}{ll} (1+r, 4s+4-r) & r \in [1, s], \\ (s+2+r, 3s+3-r) & r \in [1, s-1]\ (s \geq 2), \end{array}$$

and $(1, 2s+2), (s+2, 5s+5)$. □

Lemma 2.11 *Assume $s \geq 1$. Let $F(s) = O[3, 8s+5]$ and $U(s) = [1, 8s+8] \setminus \{2s+2, 4s+5, 6s+6, 8s+7\}$. Then there is a partition of $U(s)$ into $4s+2$ ordered pairs (e_i, f_i) ($i \in F(s)$) such that (i) $\bigcup_{i \in F(s)} e_i = [1, 4s+3] \setminus \{2s+2\}$, (ii) $\bigcup_{i \in F(s)} f_i = [4s+4, 8s+8] \setminus \{4s+5, 6s+6, 8s+7\}$ and (iii) $f_i - e_i = i$ for $i \in F(s)$.*

Proof. Take the pairs as:

$$\begin{array}{ll} (r, 8s+7-r) & r \in [1, s-1]\ (s \geq 2), \\ (s+r, 7s+7-r) & r \in [1, s], \\ (2s+3+r, 6s+6-r) & r \in [1, s], \\ (3s+4+r, 5s+5-r) & r \in [1, s-1]\ (s \geq 2), \end{array}$$

and $(2s+3, 4s+4), (3s+4, 7s+7), (s, 5s+5), (2s+1, 8s+8)$. $\square$

Lemma 2.12 *Assume $s \geq 1$. Let $G(s) = O[1, 4s+1]$ and $V(s) = [1, 4s+4] \setminus \{2s+2, 4s+3\}$. Then there is a partition of $V(s)$ into $2s+1$ ordered pairs (g_i, h_i) ($i \in G(s)$) such that $h_i - g_i = i$ for $i \in G(s)$.*

Proof. Take the pairs as:

$$\begin{array}{ll} (r, 4s+3-r) & r \in [1, s], \\ (s+2+r, 3s+3-r) & r \in [1, s-1]\ (s \geq 2), \end{array}$$

and $(s+1, s+2), (2s+3, 4s+4)$. $\square$

Theorem 2.13 *Let $n \geq 2$ be even. Let $s \geq 1$. Then there is a solution to $SP(n; 4s+3, s+1)$.*

Proof. Assume $n = 2t$ with $t \geq 1$. The ordered pairs which produce all the even numbers in $D(n, 4s+3)$ can be taken as:

$$(r, 2t(4s+3)-r) \quad r \in [1, t(4s+3)-1],\ r \not\equiv 0 \pmod{4s+3}.$$

Now let (c_i, d_i) ($i \in E(s)$), (e_i, f_i) ($i \in F(s)$) and (g_i, h_i) ($i \in G(s)$) be those partitions of $T(s)$, $U(s)$ and $V(s)$ as defined in Lemma 2.10, Lemma 2.11 and Lemma 2.12. Then the following ordered pairs produce all the odd numbers in $D(n, 4s+3)$:

$$\begin{array}{ll} (2t(4s+3)+c_i, (4t-1)(4s+3)-1+d_i) & i \in E(s), \\ ((3t-1)(4s+3)+(2s+1)+g_i, (3t-1)(4s+3)+(2s+1)+h_i) & i \in G(s), \end{array}$$

and, when $t \geq 2$, for each $j = 0, 1, \ldots, t-2$,

$$((2t+j)(4s+3)+(2s+1)+e_i, (4t-2-j)(4s+3)-(2s+3)+f_i) \quad i \in F(s).$$

$\square$

2.3 The Case $u \equiv 2 \pmod 4$

Assuming $u = 4s+2$ ($s \geq 1$) and choosing $x = s+1$, we consider the problem SP$(n; 4s+2, s+1)$ when $n \geq 2$ is even.

Lemma 2.14 *Assume $s \geq 1$. Let $H(s) = (O[3, 4s-1] \setminus \{2s+1\}) \cup \{4\}$ and $W(s) = [1, 4s+1] \setminus \{s+1, 3s, 4s\}$. Then there is a partition of $W(s)$ into $2s-1$ ordered pairs (k_i, l_i) ($i \in H(s)$) such that (i) $\bigcup_{i \in H(s)} k_i = [1, 2s] \setminus \{s+1\}$, (ii) $\bigcup_{i \in H(s)} l_i = [2s+1, 4s+1] \setminus \{3s, 4s\}$ and (iii) $l_i - k_i = i$ for $i \in H(s)$.*

Proof. We construct the ordered pairs according to $s \equiv 0, 1, 2, 3 \pmod 4$ in the following.

Case (i): $s \equiv 0 \pmod 4$.

$s = 4$: (6, 9), (7, 11), (8, 13), (3, 10), (4, 15), (1, 14), (2, 17);

$s \geq 8$, $s = 4p$, $p \geq 2$:

$$\begin{array}{ll} (r, 16p - 1 - r) & r \in O[1, 4p - 3], \\ (1 + r, 16p + 2 - r) & r \in O[1, 4p - 1], \end{array}$$

which produce differences $d \in H(4p)$ such that $8p + 3 \leq d \leq 16p - 1$,

$$\begin{array}{ll} (4p + 11 + r, 12p - 4 - r) & r \in [1, 4p - 11], r \equiv 1 \pmod 4 \ (p \geq 3), \\ (4p + 4 + r, 12p - 5 - r) & r \in [1, 4p - 9], r \not\equiv 0 \pmod 4 \ (p \geq 3), \end{array}$$

which produce differences $d \in H(4p)$ such that $7 \leq d \leq 8p - 11$, and

$$\begin{array}{llll} (4p + 2, 12p + 1), & (4p - 1, 12p - 4), & (4p + 3, 12p - 2), & (4p + 4, 12p - 3), \\ (4p + 8, 12p - 1), & (8p - 2, 8p + 3), & (8p - 3, 8p + 1), & (8p - 1, 8p + 2), \end{array}$$

which produce differences $3, 4, 5, 8p - 9, 8p - 7, 8p - 5, 8p - 3, 8p - 1$.

Case (ii): $s \equiv 2 \pmod 4$.

$s = 2$: (4, 7), (1, 5), (2, 9);

$s \geq 6$, $s = 4p + 2$, $p \geq 1$:

$$\begin{array}{ll} (r, 16p + 7 - r) & r \in O[1, 4p - 1], \\ (1 + r, 16p + 10 - r) & r \in O[1, 4p + 1], \end{array}$$

which produce differences $d \in H(4p + 2)$ such that $8p + 7 \leq d \leq 16p + 7$,

$$\begin{array}{ll} (4p + 11 + r, 12p + 4 - r) & r \in [1, 4p - 7], r \equiv 1 \pmod 4 \ (p \geq 2), \\ (4p + 4 + r, 12p + 3 - r) & r \in [1, 4p - 5], r \not\equiv 0 \pmod 4 \ (p \geq 2), \end{array}$$

which produce differences $d \in H(4p + 2)$ such that $7 \leq d \leq 8p - 3$, and

$$\begin{array}{lll} (4p + 1, 12p + 4), & (4p + 4, 12p + 5), & (4p + 8, 12p + 7), \\ (8p + 2, 8p + 7), & (8p + 1, 8p + 5), & (8p + 3, 8p + 6), \end{array}$$

which produce differences $3, 4, 5, 8p - 1, 8p + 1, 8p + 3$.

Case (iii): $s \equiv 1 \pmod 4$.

$s = 1$: (1, 5);

$s \geq 5$, $s = 4p + 1$, $p \geq 1$:

$$\begin{array}{ll} (r, 16p + 3 - r) & r \in O[1, 4p - 1], \\ (1 + r, 16p + 6 - r) & r \in O[1, 4p - 1], \end{array}$$

which produce differences $d \in H(4p + 1)$ such that $8p + 5 \leq d \leq 16p + 3$,

$$\begin{array}{ll} (4p + 2 + r, 12p - 3 - r) & r \in [1, 4p - 7], r \equiv 1 \pmod 4 \ (p \geq 2), \\ (4p + 7 + r, 12p - r) & r \in [1, 4p - 5], r \not\equiv 0 \pmod 4 \ (p \geq 2), \end{array}$$

which produce differences $d \in H(4p+1)$ such that $3 \le d \le 8p-7$, and

$$\begin{array}{lll} (4p+1, 12p+2), & (4p+6, 12p+5), & (4p+4, 12p+1), \\ (4p+5, 12p), & (8p-1, 8p+3), & \end{array}$$

which produce differences $4, 8p-5, 8p-3, 8p-1, 8p+1$.

Case (iv): $s \equiv 3 \pmod 4$.

$s \ge 3$, $s = 4p+3$, $p \ge 0$:

$$\begin{array}{ll} (r, 16p+11-r) & r \in O[1, 4p+1], \\ (1+r, 16p+14-r) & r \in O[1, 4p+1], \end{array}$$

which produce differences $d \in H(4p+3)$ such that $8p+9 \le d \le 16p+11$,

$$\begin{array}{ll} (4p+2+r, 12p+5-r) & r \in [1, 4p-3], r \equiv 1 \pmod 4\ (p \ge 1), \\ (4p+7+r, 12p+8-r) & r \in [1, 4p-1], r \not\equiv 0 \pmod 4\ (p \ge 1), \end{array}$$

which produce differences $d \in H(4p+3)$ such that $3 \le d \le 8p+1$, and

$$(4p+6, 12p+11), \quad (4p+5, 12p+8), \quad (8p+3, 8p+7),$$

which produce differences $4, 8p+3, 8p+5$. □

We now construct the following ingredients.

Lemma 2.15 *Assume $s \ge 1$. Let $E'(s) = E[4, 4s+2]$ and $T'(s) = [1, 4s+3] \setminus \{2s+1, 2s+2, 2s+3\}$. Then there is a partition of $T'(s)$ into $2s$ ordered pairs (c_i', d_i') ($i \in E'(s)$) such that (i) $\bigcup_{i \in E'(s)} c_i' = [1, 2s]$, (ii) $\bigcup_{i \in E'(s)} d_i' = [2s+4, 4s+3]$ and (iii) $d_i' - c_i' = i$ for $i \in E'(s)$.*

Proof. Simply take the following pairs:

$$(r, 4s+4-r) \quad r \in [1, 2s].$$

□

Lemma 2.16 *Assume $s \ge 1$.*

1. *Let $F(s) = (O[3, 4s+3] \bigcup \{4s+9\}) \bigcup (E[4s+8, 8s+4] \setminus \{6s+6\} \bigcup \{4s+2\})$ and $U(s) = [1, 8s+6] \setminus \{2s+1, 4s+4, 6s+5, 8s+5\}$. Then there is a partition of $U(s)$ into $4s+1$ ordered pairs (e_i, f_i) ($i \in F(s)$) such that (i) $\bigcup_{i \in F(s)} e_i = [1, 4s+2] \setminus \{2s+1\}$, (ii) $\bigcup_{i \in F(s)} f_i = [4s+3, 8s+6] \setminus \{4s+4, 6s+5, 8s+5\}$ and (iii) $f_i - e_i = i$ for $i \in F(s)$.*

2. *Let $F'(s) = (E[4, 4s] \bigcup \{4s+6, 6s+6\}) \bigcup (O[4s+5, 8s+5] \setminus \{4s+9\})$ and $U'(s) = [1, 8s+7] \setminus \{2s+2, 4s+3, 4s+4, 4s+5, 6s+6\}$. Then there is a partition of $U'(s)$ into $4s+1$ ordered pairs (e_i', f_i') ($i \in F'(s)$) such that (i) $\bigcup_{i \in F'(s)} e_i' = [1, 4s+2] \setminus \{2s+2\}$, (ii) $\bigcup_{i \in F'(s)} f_i' = [4s+6, 8s+7] \setminus \{6s+6\}$ and (iii) $f_i' - e_i' = i$ for $i \in F'(s)$.*

Proof. Take (k_i, l_i) $(i \in H(s))$ to be the partiton of $W(s)$ obtained in Lemma 2.14. Then the $4s+1$ ordered pairs (e_i, f_i) $(i \in F(s))$ can be taken as in the following:

$$\begin{array}{ll} (k_i, 4s+5+l_i) & i \in H(s), \\ (2s+2+r, 6s+5-r) & r \in [1, s], \\ (3s+3+r, 5s+4-r) & r \in [1, s-1] \ (s \geq 2), \end{array}$$

and $(2s+2, 4s+3), (3s+3, 7s+5), (s+1, 5s+4)$.

The $4s+1$ ordered pairs $(e_i^{'}, f_i^{'})$ $(i \in F^{'}(s))$ can be taken as in the following:

$$\begin{array}{ll} (1+r, 8s+8-r) & r \in [1, 2s-2] \ (s \geq 2), \\ (2s+3+r, 6s+5-r) & r \in [1, 2s-1], \end{array}$$

and $(2s, 6s+5), (2s+3, 6s+9), (2s+1, 6s+8), (1, 6s+7)$. □

Lemma 2.17 *Assume $s \geq 1$.*

1. *Let $G(s) = (E[2, 4s] \setminus \{2s+2\}) \bigcup \{1\}$ and $V(s) = [1, 4s+2] \setminus \{2s+1, 4s+1\}$. Then there is a partition of $V(s)$ into $2s$ ordered pairs (g_i, h_i) $(i \in G(s))$ such that $h_i - g_i = i$ for $i \in G(s)$.*

2. *Let $G^{'}(s) = O[3, 4s+1] \bigcup \{2s+2\}$ and $V^{'}(s) = [1, 4s+3] \setminus \{2s+2\}$. Then there is a partition of $V^{'}(s)$ into $2s+1$ ordered pairs $(g_i^{'}, h_i^{'})$ $(i \in G^{'}(s))$ such that $h_i^{'} - g_i^{'} = i$ for $i \in G^{'}(s)$.*

Proof. The $2s+1$ ordered pairs $(g_i^{'}, h_i^{'})$ $(i \in G^{'}(s))$ can be taken as:

$$(1+r, 4s+4-r) \quad r \in [1, 2s],$$

and $(1, 2s+3)$.

Now we construct (g_i, h_i) $(i \in G(s))$ by distinguishing s into two cases.

Suppose $s \geq 2$ is even. Then take the pairs as:

$$\begin{array}{ll} (r, 4s-r) & r \in O[1, s-3] \ (s \geq 4), \\ (1+r, 4s+3-r) & r \in O[1, s-1] \text{ and } r \in [s+1, 2s-1], \end{array}$$

and $(2s+2, 2s+3), (s-1, s+1)$.

Suppose $s \geq 1$ is odd. Then take the pairs as:

$$\begin{array}{ll} (1+r, 4s+3-r) & r \in O[1, s-2] \ (s \geq 3), \\ (r, 4s-r) & r \in O[1, s-2] \ (s \geq 3) \text{ and } r \in [s, 2s-2] \ (s \geq 3), \end{array}$$

and $(2s-1, 2s), (3s+1, 3s+3)$. □

Theorem 2.18 *Let $n \geq 2$ be even. Let $s \geq 1$. Then there is a solution to $SP(n; 4s+2, s+1)$.*

Proof. Let (c_i, d_i) $(i \in E(s))$ be the partition of $T(s)$ obtained in Lemma 2.10, (c_i', d_i') $(i \in E'(s))$ be the partition of $T'(s)$ obtained in Lemma 2.15, (e_i, f_i) $(i \in F(s))$, (e_i', f_i') $(i \in F'(s))$ be the partitions of $U(s)$, $U'(s)$ obtained in Lemma 2.16 and (g_i, h_i) $(i \in G(s))$, (g_i', h_i') $(i \in G'(s))$ be the partitions of $V(s)$, $V'(s)$ obtained in Lemma 2.17.

Suppose $n = 2t$ with $t \geq 1$. The following pairs form a solution to $\mathrm{SP}(n; 4s + 2, s + 1)$:

$$\begin{array}{ll} (c_i', (2t-1)(4s+2) - 2 + d_i') & i \in E'(s), \\ (2t(4s+2) + c_i, (4t-1)(4s+2) - 2 + d_i) & i \in E(s), \end{array}$$

which produce the numbers $d \in D(n, 4s+2)$ with $(2t-1)(4s+2) + 1 \leq d \leq 2t(4s+2) - 1$,

$$\begin{array}{ll} ((t-1)(4s+2) + 2s + g_i', (t-1)(4s+2) + 2s + h_i') & i \in G'(s), \\ ((3t-1)(4s+2) + (2s+1) + g_i, (3t-1)(4s+2) + (2s+1) + h_i) & i \in G(s), \end{array}$$

which produce the numbers $d \in D(n, 4s+2)$ with $1 \leq d \leq 4s+1$, and, when $t \geq 2$, for each $j = 0, 1, \ldots, t-2$,

$$\begin{array}{ll} (j(4s+2) + 2s + e_i', (2t-2-j)(4s+2) - (2s+4) + f_i') & i \in F'(s), \\ ((2t+j)(4s+2) + (2s+1) + e_i, (4t-2-j)(4s+2) - (2s+3) + f_i) & i \in F(s), \end{array}$$

which produce the numbers $d \in D(n, 4s+2)$ with $(2t-3-2j)(4s+2) + 1 \leq d \leq (2t-1-2j)(4s+2) - 1$. $\square$

2.4 The Case $u \equiv 0 \pmod 4$

Assuming $u = 4s$ $(s \geq 2)$, we consider the problem $\mathrm{SP}(n; 4s, x)$, where $x = s$ when $n \geq 2$ is even and $x = s - 1$ when $n \geq 1$ is odd. We first deal with the case where n is even.

Lemma 2.19 *Assume $s \geq 2$.*

1. *Let $F(s) = (O[4s+1, 12s-1] \setminus \{8s+1, 10s-1\}) \bigcup \{10s\}$ and $U(s) = [1, 4s-1] \bigcup ([8s+1, 12s-1] \setminus \{9s, 11s\} \bigcup \{7s, 13s\})$. Then there is a partition of $U(s)$ into $4s-1$ ordered pairs (e_i, f_i) $(i \in F(s))$ such that (i) $\bigcup_{i \in F(s)} e_i = [1, 4s-1]$, (ii) $\bigcup_{i \in F(s)} f_i = [8s+1, 12s-1] \setminus \{9s, 11s\} \bigcup \{7s, 13s\}$ and (iii) $f_i - e_i = i$ for $i \in F(s)$.*

2. *Let $F'(s) = (E[4s+2, 12s-2] \setminus \{8s, 10s\}) \bigcup \{8s+1, 10s-1\}$ and $U'(s) = [1, 4s-1] \bigcup [8s+1, 12s-1]$. Then there is a partition of $U'(s)$ into $4s-1$ ordered pairs (e_i', f_i') $(i \in F'(s))$ such that (i) $\bigcup_{i \in F'(s)} e_i' = [1, 4s-1]$, (ii) $\bigcup_{i \in F'(s)} f_i' = [8s+1, 12s-1]$ and (iii) $f_i' - e_i' = i$ for $i \in F'(s)$.*

Proof. The $4s-1$ ordered pairs (e_i, f_i) $(i \in F(s))$ can be taken as:

$$\begin{array}{ll}
(1+r, 12s-r) & r \in [1, s-1], \\
(s+1+r, 11s-r) & r \in [1, s-2] \ (s \geq 3), \\
(2s-1+r, 10s-r) & r \in [1, s-1], \\
(3s-1+r, 9s-r) & r \in [1, s-1],
\end{array}$$

and $(3s-1, 7s), (4s-1, 10s), (1, 10s+1), (s+1, 13s)$.

The $4s-1$ ordered pairs $(e_i^{'}, f_i^{'})$ $(i \in F^{'}(s))$ can be taken as:

$$\begin{array}{ll}
(r, 12s-r) & r \in [1, s-1], \\
(s+r, 11s-2-r) & r \equiv 1 \pmod 2 \\
 & \text{with } r \in [1, s-2] \text{ if } s \text{ is odd, or} \\
 & \text{with } r \in [1, s-3] \text{ if } s \text{ is even } (s \geq 4), \\
(s+1+r, 11s+1-r) & r \equiv 1 \pmod 2 \\
 & \text{with } r \in [1, s-2] \text{ if } s \text{ is odd, or} \\
 & \text{with } r \in [1, s-1] \text{ if } s \text{ is even}, \\
(2s+r, 10s-r) & r \in [1, 2s-1],
\end{array}$$

and $(s, 11s-1), (2s, 10s+1)$ if s is odd, or $(s, 11s-1), (2s-1, 10s)$ if s is even. □

Lemma 2.20 *Assume $s \geq 2$.*

1. *Let $G(s) = E[2, 4s-2] \bigcup O[4s+1, 8s-1\}$, $V(s) = ([1, 8s-1] \setminus \{4s, 7s\}) \bigcup \{9s\}$. Then there is a partition of $V(s)$ into $4s-1$ ordered pairs (g_i, h_i) $(i \in G(s))$ such that $h_i - g_i = i$ for $i \in G(s)$.*

2. *Let $G^{'}(s) = O[1, 4s-1] \bigcup E[4s+2, 8s-2]$ and $V^{'}(s) = [1, 8s-1] \setminus \{4s\}$. Then there is a partition of $V^{'}(s)$ into $4s-1$ ordered pairs $(g_i^{'}, h_i^{'})$ $(i \in G^{'}(s))$ such that $h_i^{'} - g_i^{'} = i$ for $i \in G^{'}(s)$.*

Proof. The $4s-1$ ordered pairs (g_i, h_i) $(i \in G(s))$ can be taken as:

$$\begin{array}{ll}
(1+r, 8s-r) & r \in [1, s-1], \\
(s+1+r, 7s-r) & r \in [1, s-1], \\
(2s+r, 6s-r) & r \in [1, 2s-1],
\end{array}$$

and $(1, 6s), (s+1, 9s)$.

The $4s-1$ ordered pairs $(g_i^{'}, h_i^{'})$ $(i \in G^{'}(s))$ can be taken as:

$$\begin{array}{ll}
(r, 8s-r) & r \in [1, 2s-1], \\
(2s-1+r, 6s-r) & r \in [1, s], \\
(3s+1+r, 5s-r) & r \in [1, s-2] \ (s \geq 3),
\end{array}$$

and $(3s, 3s+1), (4s+1, 6s)$. □

Theorem 2.21 *Let $n \geq 2$ be even and $s \geq 2$. Then there is a solution to $SP(n; 4s, s)$.*

Proof. Let (e_i, f_i) $(i \in F(s))$, (e_i', f_i') $(i \in F'(s))$ be the partitions of $U(s)$, $U'(s)$ obtained in Lemma 2.19 and (g_i, h_i) $(i \in G(s))$, (g_i', h_i') $(i \in G'(s))$ those of $V(s)$, $V'(s)$ obtained in Lemma 2.20.

Suppose $n = 2t$, $t \geq 1$. Then the following pairs form a solution to SP$(2t; 4s, s)$:

$$\begin{array}{ll} ((t-1)(4s) + g_i', (t-1)(4s) + h_i') & i \in G'(s), \\ ((3t-1)(4s) + g_i, (3t-1)(4s) + h_i) & i \in G(s), \end{array}$$

which produce the numbers $d \in D(2t, 4s)$ with $1 \leq d \leq 2(4s) - 1$, and, when $t \geq 2$, for each $j = 0, 1, \ldots, t-2$,

$$\begin{array}{ll} (j(4s) + e_i', (2t-3-j)(4s) + f_i') & i \in F'(s), \\ ((2t+j)(4s) + e_i, (4t-3-j)(4s) + f_i) & i \in F(s), \end{array}$$

which produce the numbers $d \in D(2t, 4s)$ with $(2t-2-2j)(4s) + 1 \leq d \leq (2t-2j)(4s) - 1$. □

Now we deal with the case when n is odd.

Lemma 2.22 *Assume $s \geq 2$. Let $H(s) = \{4\} \bigcup O[5, 2s+1]$ and $W(s) = [1, 2s+3] \setminus \{s, s+2, 2s+1\}$. Then there is a partition of $W(s)$ into s ordered pairs (k_i, l_i) $(i \in H(s))$ such that (i) $\bigcup_{i \in H(s)} k_i = [1, s+1] \setminus \{s\}$, (ii) $\bigcup_{i \in H(s)} l_i = [s+3, 2s+3] \setminus \{2s+1\}$ and (iii) $l_i - k_i = i$ for $i \in H(s)$.*

Proof. Case (i): $s \equiv 1 \pmod 4$, $s = 4p+1$, $p \geq 1$.

$$\begin{array}{lll} (r, 8p+5-r) & r \in [1, 4p-3],\ r \equiv 1 & \pmod 2, \\ (1+r, 8p-r) & r \in [1, 4p-7],\ r \equiv 1 & \pmod 4\ (p \geq 2), \\ (3+r, 8p+6-r) & r \in [1, 4p-3],\ r \equiv 1 & \pmod 4, \end{array}$$

and $(4p+2, 4p+6), (4p-1, 4p+4), (4p-2, 4p+5)$.

Case (ii): $s \equiv 2 \pmod 4$, $s = 4p+2$, $p \geq 0$.

$$\begin{array}{lll} (r, 8p+7-r) & r \in [1, 4p+1],\ r \equiv 1 & \pmod 2, \\ (1+r, 8p+2-r) & r \in [1, 4p-3],\ r \equiv 1 & \pmod 4\ (p \geq 1), \\ (3+r, 8p+8-r) & r \in [1, 4p-3],\ r \equiv 1 & \pmod 4\ (p \geq 1), \end{array}$$

and $(4p+3, 4p+7)$.

Case (iii): $s \equiv 3 \pmod 4$, $s = 4p+3$, $p \geq 0$.

$$\begin{array}{lll} (r, 8p+9-r) & r \in [1, 4p+1],\ r \equiv 1 & \pmod 2, \\ (1+r, 8p+4-r) & r \in [1, 4p-3],\ r \equiv 1 & \pmod 4\ (p \geq 1), \\ (3+r, 8p+10-r) & r \in [1, 4p+1],\ r \equiv 1 & \pmod 4, \end{array}$$

and $(4p+2, 4p+6)$.

Case (iv): $s \equiv 0 \pmod 4$, $s = 4p$, $p \geq 1$.

$$\begin{array}{ll}
(3+r,8p-r) & r\in[1,4p-5],\ r\equiv 1 \pmod 2\ (p\geq 2),\\
(2+r,8p-3-r) & r\in[1,4p-7],\ r\equiv 1 \pmod 4\ (p\geq 2),\\
(4+r,8p+3-r) & r\in[1,4p-3],\ r\equiv 1 \pmod 4,
\end{array}$$

and $(4p-1,4p+3),(1,8p),(2,8p+3)$. □

Lemma 2.23 *Assume $s \geq 2$.*

1. *Let $F(s) = (O[4s+1,12s-1] \setminus \{6s+1,10s+1\}) \bigcup \{4s+2\}$ and $U(s) = ([2,4s+1] \setminus \{4s\}) \bigcup ([8s+1,12s-1] \setminus \{9s-1,11s+1\} \bigcup \{7s+1,13s-1\})$. Then there is a partition of $U(s)$ into $4s-1$ ordered pairs (e_i,f_i) $(i \in F(s))$ such that (i) $\bigcup_{i\in F(s)} e_i = [2,4s+1] \setminus \{4s\}$, (ii) $\bigcup_{i\in F(s)} f_i = ([8s+1,12s-1] \setminus \{9s-1,11s+1\} \bigcup \{7s+1,13s-1\}$ and (iii) $f_i - e_i = i$ for $i \in F(s)$.*

2. *Let $F'(s) = (E[4s+2,12s-2] \setminus \{4s+2,8s\}) \bigcup \{6s+1,10s+1\}$ and $U'(s) = [1,4s-1] \bigcup ([8s+2,12s+1] \setminus \{12s\})$. Then there is a partition of $U'(s)$ into $4s-1$ ordered pairs (e'_i,f'_i) $(i \in F'(s))$ such that (i) $\bigcup_{i\in F'(s)} e'_i = [1,4s-1]$, (ii) $\bigcup_{i\in F'(s)} f'_i = [8s+2,12s+1] \setminus \{12s\}$ and (iii) $f'_i - e'_i = i$ for $i \in F'(s)$.*

Proof. Let (k_i,l_i) $(i \in H(s))$ be the partition of $W(s)$ defined in Lemma 2.22. Then the $4s-1$ ordered pairs (e_i,f_i) $(i \in F(s))$ can be taken as:

$$\begin{array}{ll}
(1+r,12s-r) & r\in[1,s-2]\ (s\geq 3),\\
(s+r,11s+1-r) & r\in[1,2s-1],\\
(3s+k_i,7s-2+l_i) & i\in H(s),
\end{array}$$

and $(3s,7s+1),(s,13s-1)$.

The $4s-1$ ordered pairs (e'_i,f'_i) $(i \in F'(s))$ can be taken as:

$$\begin{array}{ll}
(r,12s-r) & r\in[1,2s-1],\\
(2s+1+r,10s+1-r) & r\in[1,2s-2],
\end{array}$$

and $(2s+1,8s+2),(2s,12s+1)$.

Lemma 2.24 *Assume $s \geq 2$.*

1. *Let $G(s) = O[1,4s-1] \setminus \{2s+1\}$ and $V(s) = ([2,4s-1] \setminus \{3s+1\}) \bigcup \{5s-1\}$. Then there is a partition of $V(s)$ into $2s-1$ ordered pairs (g_i,h_i) $(i \in G(s))$ such that $h_i - g_i = i$ for $i \in G(s)$.*

2. *Let $G'(s) = E[2,4s-2] \bigcup \{2s+1\}$ and $V'(s) = [1,4s-1] \bigcup \{4s+1\}$. Then there is a partition of $V'(s)$ into $2s$ ordered pairs (g'_i,h'_i) $(i \in G'(s))$ such that $h'_i - g'_i = i$ for $i \in G'(s)$.*

Proof. The $2s-1$ ordered pairs (g_i, h_i) $(i \in G(s))$ can be taken as:

$$\begin{array}{ll} (1+r, 4s-r) & r \in [1, s-2]\ (s \geq 3), \\ (s+r, 3s+1-r) & r \in [1, s], \end{array}$$

and $(s, 5s-1)$.

The $2s$ ordered pairs $(g_i^{'}, h_i^{'})$ $(i \in G^{'}(s))$ can be taken as:

$$(r, 4s-r) \quad r \in [1, 2s-1],$$

and $(2s, 4s+1)$. □

Theorem 2.25 *Let $n \geq 1$ be odd and $s \geq 2$. Then there is a solution to $SP(n; 4s, s-1)$.*

Proof. Let (e_i, f_i) $(i \in F(s))$, $(e_i^{'}, f_i^{'})$ $(i \in F^{'}(s))$ be the partitions of $U(s)$, $U^{'}(s)$ obtained in Lemma 2.23 and (g_i, h_i) $(i \in G(s))$, $(g_i^{'}, h_i^{'})$ $(i \in G^{'}(s))$ be those of $V(s)$, $V^{'}(s)$ obtained in Lemma 2.24.

Suppose $n = 2t+1$, $t \geq 0$. Then the following pairs form a solution to $\mathrm{SP}(2t+1; 4s, s-1)$:

$$\begin{array}{ll} (t(4s)+g_i^{'}, t(4s)+h_i^{'}) & i \in G^{'}(s), \\ ((3t+1)(4s)+g_i, (3t+1)(4s)+h_i) & i \in G(s), \end{array}$$

which produce the numbers $d \in D(2t+1, 4s)$ such that $1 \leq d \leq 4s-1$, and, when $t \geq 1$, for each $j = 0, 1, \ldots, t-1$,

$$\begin{array}{ll} (j(4s)+e_i^{'}, (2t-2-j)(4s)+f_i^{'}) & i \in F^{'}(s), \\ ((2t+1+j)(4s)+e_i, (4t-1-j)(4s)+f_i) & i \in F(s), \end{array}$$

which produce the numbers $d \in D(2t+1, 4s)$ such that $(2t-1-2j)(4s)+1 \leq d \leq (2t+1-2j)(4s)-1$. □

3 Conclusions

We now consider the existence of a $3-CGDD$ of type $(6n)^u$ when $u \equiv 2, 3 \pmod 4$ and n is odd. A $3-CGDD$ of this type can not exist.

Lemma 3.1 *There is no $3-CGDD$ of type $(6n)^u$ whenever $u \equiv 2, 3 \pmod 4$ and n is odd.*

Proof. Assume there is a $3-CGDD$ $(V, \mathcal{G}, \mathcal{B})$ of type $(6n)^u$. Without loss of generality, we may assume that $V = Z_{6n} \times \{1, 2, \ldots, u\}$, $\mathcal{G} = \{Z_{6n} \times \{i\} | i = 1, 2, \ldots, u\}$ and the cyclic automorphism is

$$\pi = (0_1 1_1 \ldots (6n-1)_1) \ldots (0_u 1_u \ldots (6n-1)_u).$$

The automorphism π yields a partition of $\mathcal{B}$ into equivalence classes, namely the orbits of π, such that $B_1, B_2 \in \mathcal{B}$ are in the same class if and only if $\pi^{\alpha}(B_1) = B_2$ for some integer α. Let $\mathcal{D}$ be a set of representatives of the orbits of π. We claim that $\mathcal{D}$ covers each of the mixed differences in $V = Z_{6n} \times \{1, 2, \ldots, u\}$ exactly once.

Since each pair $\{0_i, x_j\}$ ($x \in Z_{6n}$) is contained in one block B in $\mathcal{B}$, each of the pairs $\{y_i, (x+y)_j\}$ is contained in a block which is in the orbit containing B. Therefore the representative block in $\mathcal{D}$ of the orbit containing B covers the mixed difference x_{ji}. Furthermore, each mixed difference in $V = Z_{6n} \times \{1, 2, \ldots, u\}$ is covered by at most one block in $\mathcal{D}$. For otherwise, suppose $B_1, B_2 \in \mathcal{D}$ both cover a difference x_{ji}. Without loss of generality, we assume that $B_1 = \{a_i, b_j, c_k\}$, $B_2 = \{d_i, e_j, f_\ell\}$. Then $b - a = e - d = x$. Consequently,

$$\begin{aligned} \pi^{d-a}(B_1) &= \{(a+d-a)_i, (b+d-a)_j, (c+d-a)_k\} \\ &= \{d_i, e_j, (c+d-a)_k\} \end{aligned}$$

contains $\{d_i, e_j\}$. Since $\pi^{d-a}(B_1)$ and B_2 are in different orbits they are distinct. Then the pair $\{d_i, e_j\}$ is contained in two distinct blocks in $\mathcal{B}$, which contradicts that $(V, \mathcal{G}, \mathcal{B})$ is a GDD.

Now consider a block $B = \{a_i, b_j, c_k\}$ in $\mathcal{D}$. Note that the mixed difference x_{ij} is equal to $(-x)_{ji}$. B covers six differences $(\pm(a-b))_{ij}, (\pm(b-c))_{jk}, (\pm(a-c))_{ik}$. Obviously, $(a-b) + (b-c) = (a-c)$. Since $\pm(a-b)$ (respectively, $\pm(b-c)$, $\pm(a-c)$) are both even or both odd in Z_{6n}, the number of odd differences covered by B is either 0 or 4. Consequently, the total number of odd differences covered by blocks in $\mathcal{D}$ is a multiple of 4. Since $\mathcal{D}$ covers each of the mixed differences in $V = Z_{6n} \times \{1, 2, \ldots, u\}$ exactly once, we conclude that the total number of odd mixed differences in V is divisible by 4. An easy calculation shows that the total number of odd mixed differences in V is $2\binom{u}{2}3n = 3nu(u-1)$ and this number is not divisible by 4 when $u \equiv 2, 3 \pmod 4$ and n is odd. Therefore there is no $3-CGDD$ of type $(6n)^u$ whenever $u \equiv 2, 3 \pmod 4$ and n is odd. □

Even if n is even, a $3-CGDD$ of type $(6n)^3$ does not exist. This is an immediate consequence of the following result, which appeared in a preliminary version of [1] in terms of Latin squares.

Lemma 3.2 *There is no $3-CGDD$ of type g^3 whenever g is even.*

Proof. Assume $(V, \mathcal{G}, \mathcal{B})$ is a $3-CGDD$ of type g^3. As in Lemma 3.1, we assume that $V = Z_g \times \{1, 2, 3\}$, $\mathcal{G} = \{Z_g \times \{i\} | i = 1, 2, 3\}$ and the cyclic automorphism is

$$\pi = (0_1 1_1 \ldots (g-1)_1) \ldots (0_3 1_3 \ldots (g-1)_3).$$

Let $\{0_1, r_2, (x_r)_3\}$, where $x_r \in Z_g$, be the block in $\mathcal{B}$ containing the pair $\{0_1, r_2\}$ for each $r \in Z_g$. Define

$$\mathcal{D} = \{\{0_1, r_2, (x_r)_3\} | r \in Z_g\}.$$

Clearly, each block in $\mathcal{D}$ is in a distinct orbit of π. Since each orbit of π has cardinality g and $|\mathcal{B}| = g^2$, π has g orbits. Therefore $\mathcal{D}$ is a set of representatives of the orbits of π. Similar to Lemma 3.1, we see that $\mathcal{D}$ covers each of the mixed differences in $V = Z_g \times \{1,2,3\}$ exactly once. This implies that $x_r \neq x_s$, $x_r - r \neq x_s - s$ for each $r, s \in Z_g$ with $r \neq s$. In particular, $x_r \neq x_0$, $x_r - r \neq x_0$ for each $r \in Z_g$ with $r \neq 0$. Therefore (with arithmetic taken in the group Z_g) $\sum_{r \in Z_g \setminus \{0\}} x_r = \sum_{r \in Z_g \setminus \{0\}} (x_r - r)$, and consequently we get $\sum_{r \in Z_g \setminus \{0\}} r = 0$. But this is true only when the order g of Z_g is odd. This establishes the result. □

Lemma 3.1, Lemma 3.2 and results in the previous sections can be used to completely determine the spectrum of a $3 - CGDD$ of type $(6n)^u$.

Theorem 3.3 *There is a* $3 - CGDD$ *of type* $(6n)^u$ *if and only if (i)* $u \geq 4$ *and (ii)* $u \not\equiv 2, 3 \pmod 4$ *when* n *is odd.*

Proof. The necessity follows from Lemma 3.1 and Lemma 3.2.

For the sufficiency, Example 1.2 establishes a $3 - CGDD$ of type $(6n)^u$ for $u = 4$ and any integer $n \geq 1$; For the remaining cases of n and u, a solution to SP(n, u) or to SP$(n; u, x)$, for some x, is established in Example 2.2, Theorem 2.9, Theorem 2.13, Theorem 2.18, Theorem 2.21 and Theorem 2.25 and therefore successively applying Lemma 2.3 and Lemma 2.1 establishes the existence of a $3 - CGDD$ of type $(6n)^u$ in these cases. □

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