Weakly $3 - DCI$ Abelian Groups

Mingyao Xu
(Inst. of Math., Peking Univ., Beijing 100871, China)
Jixiang Meng*
(Dept. of Math., Xinjiang University, Urumqi, Xinjiang 830046, China)

Let G be a group and S a subset of G not containing the identity element 1 of G. The Cayley digraph of G with respect to S, denoted $D(G,S)$, is a directed graph with vertex set G and for x and y in G, there is an arc from x to y if and only if $x^{-1}y \in S$. We call S a CI-subset of G, if for any isomorphism $D(G,S) \cong D(G,T)$ of Cayley digraphs, there is $f \in AutG$ such that $T = f(S)$. Let m be a positive integer. G is called a weakly $m - DCI-$ group if every generating subset S of G with $\mid S \mid \leq m$ is a $CI-$subset of G. Here we characterizes weakly $m - DCI$ abelian groups for $m \leq 3$.

§1. Introduction

Let G be a finite group and S a subset of G with $1 \notin S$. The Cayley digraph $D = D(G,S)$ of G with respect to S is defined by

$$\begin{aligned} V(D) = &\ G \\ E(D) = &\ \{(g, gs) : g \in G, s \in S\}. \end{aligned}$$

Clearly, Cayley digraphs are vertex transitive. If $S = S^{-1}$, then D is actually an undirected graph, called a Cayley graph.

Let G be a finite group. A subset S of G not containing the identity element 1 of G is called a $CI-$subset (CI stands for "Cayley isomorphism"), if for any isomorphism $D(G,S) \cong D(G,T)$, there is $f \in AutG$ such that $T = f(S)$. G is called a $DCI-$group if every subset of $G - \{1\}$ is a $CI-$subset. G is called a GCI-group or simply a $CI-$group if every inverse-closed subset of $G - \{1\}$ is a $CI-$subset. (DCI and GCI stands for "digraph Cayley isomorphism" and "Graph Cayley isomorphism" respectively).

Note that the Cayley digraphs of a group G corresponding to two CI-subsets are not necessarily isomorphic. For instance, let $G = Z_8$, $S = \{1,2\}$ and $T = \{1,5\}$. Then both S and T are $CI-$subsets of G. (See Theorem 3 in section 2.) But $D(G,S) \ncong D(G,T)$. For otherwise, since S is a $CI-$subset of G, there would exist an automorphism f of G mapping S into T. This is impossible since both 1 and 5 are generating elements of G, but 2 is not.

* supported by the Natural Science Foundation of China, Xinjiang Educational Committee and Xinjiang University.

In our terminology, the well-known Adam's conjecture [1] is equivalent to the statement that finite cyclic groups are $DCI-$groups. Although Elspas and Turner disproved this conjecture by showing that Z_8 is not a $DCI-$group and Z_{16} is not a $CI-$group [8], this led to further study of $CI-$ and $DCI-$groups. In [13], Godsil proved that $Z_p \oplus Z_p$ is a $CI-$group. In fact, it is a $DCI-$group. In [4], Babai and Frankl discussed the properties of Sylow subgroups of a $CI-$group of odd order. Considering that the determination of $CI-$ or $DCI-$groups is very difficult, Xu and Fang defined the so called $m-CI-$ and $m-DCI-$groups.

Definition 1. Let G be a finite group and m a positive integer. G is called an $m-CI-$group if every inverse-closed subset S of $G-\{1\}$ with $|S| \leq m$ is a CI-subset. G is called an $m-DCI-$ group if every subset of $G-\{1\}$ with $|S| \leq m$ is a $CI-$subset.

In [9-11], the authors determined all abelian $m-DCI-$groups $(m \leq 3)$. In [12], the authors characterized abelian $4-CI-$groups.

The main purpose of investigating $m-DCI-$ or $m-CI-$ groups is to classify Cayley digraphs or graphs of small degrees. But from the known results we can see that $m-DCI-$ or $m-CI-$groups are very exceptional. On the other hand, when considering the classifications of Cayley digraphs or graphs, we need only consider the strongly connected ones or connected ones. Based on these two points, the first author of this paper proposed the following definition.

Definition 2. A finite group G is called a weakly $m-DCI-$group if every generating subset S of G with $|S| \leq m$ is a $CI-$subset. G is called a weakly $CI-$group if every inverse-closed generating subset S of G with $|S| \leq m$ is a $CI-$subset.

From the results obtained in this paper we can see that the conditions imposed on weakly $m-DCI-$groups are weaker than those imposed on $m-DCI-$groups.

We list below some known results which will be used in our discussions. To this end, we introduce the so-called homocyclic groups.

We use mZ_n to denote the direct sum of m copies of the cyclic group Z_n and call it a homocyclic group.

Theorem 1.[9]. a). A finite abelian group G is $1-DCI$ if and only if every Sylow subgroup of G is homocyclic.

b). A finite abelian group G is 2−DCI if and only if G is $1-DCI$ and every Sylow 2-subgroup of G is cyclic or elementary abelian.

Theorem 2.[10-11]. a). A finite abelian group G of odd order is $3-DCI$ if and only if every Sylow subgroup of it is homocyclic and its Sylow 3-subgroup is cyclic or elementary abelian.

b). Let $G = H \oplus T$, where H is the Sylow 2-subgroup of G. Then G is $3 - DCI$ if and only if T is $3 - DCI$ and H is either cyclic of order at most 4 or elementary abelian.

Notations and definitions not defined here can be found in [5-7].

§2. The main results

The first author of this paper proposed at the Third China-USA Conference on Graph Theory and Combinatorics the following conjecture:

Conjecture 1. Every finite group is weakly $2 - DCI$.

The following theorem partially settles this conjecture.

Theorem 3. Every finite abelian group is weakly $2 - DCI$.

Proof. Since a cyclic group is $m - DCI$ if and only if it is weakly m-DCI (see [15]), the theorem thus follows by Theorem 1 for cyclic groups. Henceforth, we assume that G is a finite non-cyclic abelian group.

Let $S = \{a, b\}$ and $T = \{a', b'\}$ be two generating subsets of G with $D(G,S) \cong D(G,T)$. We consider two cases.

Case 1. $2a \neq 2b$.

Since Cayley digaphs are vertex transitive, we can find an isomorphism f from $D(G,S)$ to $D(G,T)$ such that $f(0) = 0$, where 0 is the zero element of G. Then $f(S) = T$. Since $\mid N^+(S) \mid= 3$, we have that $\mid N^+(T) \mid= 3$, therefore $2a' \neq 2b'$. Without loss of generality, we assume that $f(a) = a'$ and $f(b) = b'$. Since $a + b$ is the only common out-adjacency vertex of a and b, $f(a+b)$ must be the only common out-adjacency vertex of a' and b'. Thus $f(a+b) = a' + b'$, and so $f(2a) = 2a'$, $f(2b) = 2b'$. By using induction on $i + j$, it is not difficult to show that $f(ia + jb) = ia' + jb'$ for any non-negative integers i and j. Since S is a generating subset of G, each element of G can be expressed in the form $ia + jb$, thus $f \in AutG$.

Case 2. $2a = 2b$.

Since G is not cyclic, we have $G \cong \langle a \rangle \oplus \langle a - b \rangle \cong \langle a' \rangle \oplus \langle a' - b' \rangle$. Thus the following mapping

$$f : ia + j(a - b) \mapsto ia' + j(a' - b')$$

is an automorphism of G and clearly, $f(a) = a'$ and $f(b) = b'$. This completes the proof.

The following theorem characterizes weakly $3-DCI-$abelian groups.

Theorem 4. A finite abelian group is weakly $3-DCI$ if and only if one of the following conditions is satisfied:

i). The order of G is odd or the rank of G is at least 3.

ii).G is of even order with rank at most 2 and its Sylow 2-subgroup is a cyclic group of order at most 4 or a homocyclic group with rank 2.

To prove Theorem 4, we establish a sequence of lemmas.

The following lemmas [1-2] show that if the conditions in Theorem 4 do not hold for an abelian group, then it is not weakly $3-DCI$.

Lemma 1. Let $G = Z_n \oplus Z_m$, where $8 \mid n$, $(2, m) = 1$ and each prime divisor of m is also a prime divisor of n. Then G is not weakly $3-DCI$.

Proof. Set $S = \{a, b, c\}$, $T = \{a', b', c'\}$, where

$$\begin{cases} a = & (1,1) \\ b = & (n/2+1,1) \\ c = & (2,0) \end{cases}$$

and

$$\begin{cases} a' = & (1,1) \\ b' = & (n/2+1,1) \\ c' = & (n/2+2,0). \end{cases}$$

Then, since $8|n$, c' and c have the same order $\frac{n}{2}$ and it is easy to check that G has the following decompositions with respect to the subgroups $\langle c\rangle$ and $\langle c'\rangle$, respectively.

$$G = \bigcup_{i=0}^{2m-1} (\langle c\rangle + ia) = \bigcup_{i=0}^{2m-1} (\langle c'\rangle + ia').$$

Since m is odd and n is divisible by 8, we have the following equalities:

$$\begin{aligned} 2ma = & \ mc \\ b' = & \ a' + \tfrac{n}{4}c' \\ 2ma' = & \ (\tfrac{n}{4} + m)c'. \end{aligned}$$

With these equalities in mind, we can prove that the following mapping f:

$$kc + la \mapsto kc' + la' \ (0 \le k < n/2, \ 0 \le l < 2m)$$

is an isomorphism from $D(G, S)$ to $D(G, T)$. If there is $\phi \in AutG$ such that $\phi(S) = T$, then by considering the orders of elements in S and T we have that $\phi(c) = c'$. If $\phi(a) = a'$, then $\phi(b) = b'$. Since $2ma = mc$, we have that $2ma' = mc'$, and so $nm/2 \equiv 0 \ (mod\ n)$. This contradicts the condition that

$(2,m)=1$. When $\phi(a)=b'$, we can obtain a similar contradiction. Thus G is not a weakly $3-DCI$–group.

Lemma 2. Let $n=2^{n_1}q_1$, $m=2^{m_1}q_2$, where $(q_1,2)=(q_2,2)=1$, $n_1>m_1\geq 1$. Then $G=Z_n\oplus Z_m$ is not a weakly $3-DCI$–group.

Proof. Set $S=\{a,b,c\}$ and $T=\{a',b',c'\}$, where

$$\begin{cases} a= & (n/(m,n),1)\\ b= & (n/(m,n)+n/2,1)\\ c= & (1,0)\end{cases}$$

and

$$\begin{cases} a'= & (n/2(m,n),1)\\ b'= & (n/2(m,n)+n/2,1)\\ c'= & (1,0).\end{cases}$$

Then it is easy to see that G has the following decompositions with respect to the subgroups $\langle c\rangle$ and $\langle c'\rangle$.

$$G=\bigcup_{i=0}^{m-1}(\langle c\rangle+ia)=\bigcup_{i=0}^{m-1}(\langle c'\rangle+ia')$$

and the mapping f:

$$ka+lc\mapsto ka'+lc',\ (0\leq k\leq m-1,\ 0\leq l\leq n-1)$$

is an isomorphism from $D(G,S)$ to $D(G,T)$. Since the orders of a,b, and c are m,m, and n respectively, and the orders of a', b' and c' are $2m$, $2m$ and n respectively, there is no automorphism of G mapping S into T. Thus G is not weakly $3-DCI$.

The following lemmas [3-5] are presented in preparation for proving the sufficiency of Theorem 4.

Lemma 3. Suppose that $D(G,\{a,b,c\})\cong D(G,\{a',b',c'\})$, $D(G,\{a,b,c\})$ is strongly connected, and $2a$, $2b$, and $2c$ are distinct. Then there is an automorphism f of G such that $f\{a,b,c\}=\{a',b',c'\}$.

Proof. It suffices to consider the following case:

$$\begin{cases} 2a= & b+c\\ 2b= & a+c\\ 2c= & a+b\end{cases}$$

and

$$\begin{cases} 2a'= & c'+b'\\ 2b'= & a'+c'\\ 2c'= & a'+b'.\end{cases}$$

The other cases can be discussed in a similar way as in [8].

Set $x = b - a$ and $x' = b' - a'$. Then $ord(x) = ord(x') = 3$ and $G = \langle a, x \rangle = \langle a', x' \rangle$. If G is cyclic, then both a and a' are generating elements of G and so the mapping f: $ka \mapsto ka'$ is an automorphism of G satisfying the specified conditions of our lemma. If, otherwise, G is not cyclic, then $G \cong \langle a \rangle \oplus \langle x \rangle = \langle a' \rangle \oplus \langle x' \rangle$, and the mapping f: $ka + lc \mapsto ka' + lx'$ is the desired automorphism of G.

Lemma 4. Let G be an abelian group of even order, $G \not\cong Z_n \oplus Z_2 (4 \mid n)$, $G \not\cong Z_n (8 \mid n)$. Suppose that $S = \{a, b, c\}$ is a generating subset of G satisfying the following:

$$\begin{cases} 2a = & 2b \\ 2c = & a + b \end{cases}$$

and $D(G, S) \cong D(G, T)$. Then there exists some automorphism f of G such that $f(S) = T$.

Proof. Let f be an isomorphism from $D(G, S)$ to $D(G, T)$ satisfying $f(0) = 0$. Then $f(S) = T$. Suppose that $T = \{a', b', c'\}$. Without loss of generality, we assume that $f(a) = a'$, $f(b) = b'$ and $f(c) = c'$. It is easy to see that $2a' = 2b'$ and $2c' = a' + b'$. Thus $G = \langle a, a - c \rangle = \langle a', a' - c' \rangle$ and $ord(a - c) = ord(a' - c') = 4$. We consider two cases.

Case 1. $\langle a - c \rangle \cap \langle a \rangle = \{0\}$.

In this case, $G \cong \langle a \rangle \oplus \langle a - c \rangle$. If $\langle a' - c' \rangle \cap \langle a' \rangle \neq \{0\}$, and $a' - c' \in \langle a' \rangle$, then G is cyclic. By Theorem 2 we are done. If $a' - c' \in \langle a' \rangle$, then $2(a' - c') \in \langle a' \rangle$ and $2(a' - c') = ord(a')a'/2$. If, in addition, $ord(a')/2$ is odd, then $G \cong \langle a' - c' \rangle \oplus \langle 2a' \rangle$ is cyclic, by our condition and Theorem 2, we are done. If $ord(a')/2$ is even, set

$$b'' = (a' - c') - ord(a')/4$$

Then $2b'' = 0$, $G = \langle a', b'' \rangle$ and if $b'' \notin \langle a' \rangle$, then G is cyclic, and the theorem follows by Theorem 2. If $b'' \notin \langle a' \rangle$, then $G \cong \langle a' \rangle \oplus \langle b'' \rangle$, and $4 \mid ord(a')$, contradicting our condition. If $\langle a' - c' \rangle \cap \langle a' \rangle = \{0\}$, then $G \cong \langle a' - c' \rangle \oplus \langle a' \rangle$, and the theorem follows readily.

Case 2. $\langle a - c \rangle \cap \langle a \rangle = \{0\}$.

In this case, it is not difficult to check that $G \cong \langle 2a \rangle \oplus \langle a - c \rangle$ and $ord(a)/2$ is odd. Thus G is cyclic and the Sylow 2-subgroup of G is the cyclic group of order 4, and by Theorem 2, we are done.

Lemma 5. Let G be a finite abelian group the Sylow 2-subgroup of which is either homocyclic with rank 2 or cyclic of order at most 4. Let $S = \{a, b, c\}$ be a generating subset of G satisfying $2a = 2b$ and $2a$, $a + b$, $a + c$, $b + c$ and $2c$ are distinct. If $D(G, S) \cong D(G, T)$, then there exists an automorphism of G which maps S into T.

Proof. Set $T = \{a', b', c'\}$. Let $G = G_1 \oplus G_2$, where G_1 is the Sylow 2-subgroup of G, and f be an isomorphism from $D(G, S)$ to $D(G, T)$ with $f(0) = 0$, $f(a) = a'$, $f(b) = b'$ and $f(c) = c'$. Then it is easy to see that $2a' = 2b'$, $a' + b'$, $a' + c'$, $b' + c'$ and $2c'$ are distinct.

By a similar proof as in the corresponding case in [11], we can deduce the following:

a). for any positive integer k, $f(kc) = kc'$.

b). $f(a - b) = a' - b'$.

c). if k is odd, then $f(ka) \in \{ka', kb'\}$, $f(kb) \in \{ka', kb'\}$.

d). if k is even, then $f(ka + b) \in \{(k+1)a', ka' + b'\}$.

e). if k is even, then $f(ka) \in \{ka', (k-1)a' + b'\}$.

f). if k is odd, then $f(ka + lc) \in \{ka' + lc', kb' + lc'\}$.

Let k_0 be the smallest positive integer such that $k_0 a \in \langle c \rangle$. Assume that

$$k_0 a = l_0 c.$$

In the following, we prove that $f(k_0 a) \in \{k_0 a', k_0 b'\}$. If k_0 is odd, this is the conclusion of c). We show that it holds for even k_0 by contradiction. For otherwise, we have by e) that $f(k_0 a) = (k_0 - 1)a' + b'$, and so

$$(k_0 - 1)a' + b' = l_0 c'$$

Set

$$\begin{cases} a = & (a_1, a_2) \\ b = & (b_1, b_2) \\ c = & (c_1, c_2) \end{cases}$$

and

$$\begin{cases} a' = & (a_1', a_2') \\ b' = & (b_1', b_2') \\ c' = & (c_1', c_2'). \end{cases}$$

where the first coordinate belongs to G_1 and the second coordinate belongs to G_2. Then $2a_1 = 2b_1$, $2a_1' = 2b_1'$, and $G_1 = \langle a_1, a_1 - b_1, c_1 \rangle = \langle a_1', a_1' - b_1', c_1' \rangle$.

If G_1 is homocyclic with rank 2 and $|G| > 4$, we claim that $G_1 \cong \langle a_1 \rangle \oplus \langle c_1 \rangle$. Suppose $a_1 - b_1 \notin \langle a_1, c_1 \rangle$. Since G_1 has rank 2, we know that $\langle a_1, c_1 \rangle$ is a cyclic group. Since $a_1 - b_1$ is of order 2, $G_1 = \langle a_1, c_1 \rangle \oplus \langle a_1 - b_1 \rangle$. Since G_1 is homocyclic, we must also have $|\langle a_1, c_1 \rangle| = 2$. But then $|G_1| = 4$, contrary to assumption. Thus $G_1 = \langle a_1, c_1 \rangle$, and by an elementary group-theoretical result (the Second Isomorphism Theorem applied to a homocyclic group of rank 2), $\langle a_1 \rangle \cap \langle c_1 \rangle$ is trivial, proving our claim.

Now set $ord(a_1) = ord(c_1) = ord(a_1') = ord(c_1') = 2^n$, then in view of the equalities $k_0 a = l_0 c$ and $G_1 = \langle a_1 \rangle \oplus \langle c_1 \rangle$, we have $k_0 a_1 = l_0 c_1 = 0$. Thus $ord(a_1) \mid k_0, ord(c_1) \mid l_0$. On the other hand, by the equality $(k_0 - 1)a' + b' = l_0 c'$, we have $(k_0 - 1)a_1' + b_1' = l_0 c_1'$, and so $a_1' = b_1'$, a contradiction.

If G_1 is cyclic of order 4, by the statement of a) and the fact that $f(0) = 0$, we have $ord(c) = ord(c')$, and so $ord(c_1) = ord(c_1')$. In what follows, we show that $ord(a_1) = ord(b_1) = 4$. Assume, without less of generality that $ord(a_1) \geq ord(b_1)$. Then $a_1 \neq 0$. If $ord(a_1) = 2$, then since $G_1 = \langle a_1, b_1, c_1 \rangle$ and $ord(G_1) = 4$, we have $ord(c_1) = ord(c_1') = 4$. Since k_0 is even, $k_0 a_1 = l_0 c_1 = 0$, thus $4 \mid l_0$, and $k_0 a_1' + (b_1' - a_1') = l_0 c_1' = 0$, that is, $k_0 a_1' = b_1' - a_1'$. As $b_1' - a_1' \neq 0$ and k_0 is even, we have $4 \mid k_0$. We will obtain the contradiction by showing that $\frac{k_0}{2} a \in \langle c \rangle$. In fact, since $k_0 a = l_0 c$ and the order of G_2 is odd, we have $\frac{k_0}{2} a_2 = \frac{l_0}{2} c_2$. If $8 \mid l_0$, then, since $a_1 = 2c_1$, we have

$$\begin{aligned}
\tfrac{l_0}{2} c = &\ (\tfrac{l_0}{2} c_1, \tfrac{l_0}{2} c_2) \\
= &\ (\tfrac{l_0}{4} a_1, \tfrac{k_0}{2} a_2) \\
= &\ (a_1, \tfrac{k_0}{2} a_2) \\
= &\ \tfrac{k_0}{2} a.
\end{aligned}$$

If $8 \mid l_0$, then, since $ord(c_2)$ is odd, we have

$$\begin{aligned}
(\tfrac{l_0}{2} + 2ord(c_2))c = &\ (2ord(c_2)c_1, \tfrac{l_0}{2} c_2) \\
= &\ (2c_1, \tfrac{k_0}{2} a_2) \\
= &\ (a_1, \tfrac{k_0}{2} a_2) \\
= &\ \tfrac{k_0}{2} a.
\end{aligned}$$

Thus, $\frac{k_0}{2} a \in \langle c \rangle$, a contradiction to the choice of k_0. Thus, $ord(a_1) = 4$. As $2b_1 = 2a_1 \neq 0$, we have $ord(b_1) = 4$. Next, we show by contradiction that $ord(a_1') = ord(b_1') = 4$. If, otherwise, we assume that $ord(a_1') > ord(b_1')$, then $ord(a_1') = 2$, $b_1' = 0$. Since $\{a_1', b_1', c_1'\}$ generates G_1, we have $ord(c_1') = 4$. Now, since $l_0 c_1' = k_0 a_1' + b_1' - a_1' = b_1' - a_1' \neq 0$ and $2 l_0 c_1' = 0$, we have $2 \mid l_0$ but $4 \mid l_0$. On the other hand, since $k_0 a_1 = l_0 c_1 \neq 0$ $(ord(c_1) = ord(c_1') = 4)$, we have $4 \mid k_0$. We then deduce that

$$\frac{k_0}{2} a = \begin{cases} (a_1, \frac{k_0}{2} a_2) & k_0 \equiv 2 \ mod\ 8, \\ (-a_1, \frac{k_0}{2} a_2) & k_0 \equiv 6 \ mod\ 8. \end{cases}$$

$$\frac{l_0}{2} c = \begin{cases} (c_1, \frac{l_0}{2} c_2) & l_0 \equiv 2 \ mod\ 8, \\ (-c_1, \frac{l_0}{2} c_2) & l_0 \equiv 6 \ mod\ 8. \end{cases}$$

and

$$(\frac{l_0}{2} + 2ord(c_2))c = \begin{cases} (-c_1, \frac{l_0}{2} c_2) & l_0 \equiv 2 \ mod\ 8, \\ (c_1, \frac{l_0}{2} c_2) & l_0 \equiv 6 \ mod\ 8. \end{cases}$$

Since $ord(c_1) = ord(a_1) = 4$, we have $a_1 = \pm c_1$. Clearly, $\frac{k_0}{2}a_2 = \frac{l_0}{2}c_2$. Thus $\frac{k_0}{2}a \in \langle c \rangle$, a contradiction to the choice of k_0. Thus $ord(a_1') = ord(b_1') = 4$. Since k_0 is even, we have $k_0 a_1 = k_0 a_1'$. On the other hand, if $ord(c_1) = 2$, then $ord(c_1') = 2$ and so $c_1 = c_1'$. If $ord(c_1) = 4$, then from the equality $k_0 a_0 = l_0 c_1$, we know that l_0 is even and so $l_0 c_1 = l_0 c_1'$. But then $k_0 a_1 = l_0 c_1 = l_0 c_1' = k_0 a_1' + (b_1' - a_1') = k_0 a_1 + b_1' - a_1'$, thus $a_1' = b_1'$, and so $a' = b'$, a contradiction.

If G_1 is homocyclic of rank 2 and $|G_1| = 4$ or G_1 is cyclic of order 2, then $2a_1 = 2b_1 = 2c_1 = 2a_1' = 2b_1' = 2c_1' = 0$. As pointed out before, $ord(c) = ord(c')$. Thus $ord(c_1) = ord(c_1')$. Since k_0 is even, we have $k_0 a_1 = l_0 c_1 = 0$. Thus $l_0 c_1' = 0$. On the other hand, from the equality $(k_0 - 1)a_1' + b_1' = l_0 c_1'$, we have $b_1' - a_1' = (k_0 - 1)a_1' + b_1' = l_0 c_1' = 0$, thus $a_1' = b_1'$, this is imposible.

We thus conclude that

$$f(k_0 a) = k_0 a'.$$

Thus

$$l_0 c' \in \{k_0 a', k_0 b'\}.$$

If $l_0 a' = k_0 c'$, consider the following correspondence:

$$f_0 : ka + lc \mapsto ka' + lc'.$$

If $ka = lc$ for some non-negative integers k and l, then by the choice of k_0, we have $k_0 \mid k$. Let $k = k_0 q$. Then

$$k_0 qa = q(k_0 a) = ql_0 c = lc$$

and so $l = ql_0 + q' ord(c)$ for some q'. Thus $lc' = (ql_0 + q' ord(c))c' = ka'$, and so f_0 is well-defined. f_0 is clearly an automorphism of G and $f_0(a) = a'$, $f_0(c) = c'$. On the other hand, if the Sylow 2-subgroup of G is cyclic, then $a - b = a' - b'$ and $f_0(a - b) = a' - b'$, therefore, $f_0(b) = b'$. If the Sylow 2-subgroup of G is homocyclic, assume that $b = k_1 a + l_1 c$, then $f(b) = b' = k_1 a' + l_1 c'$ (the proof is similar to the above). Thus $f_0(b) = k_1 f_0(a) + f_0(c) = k_1 a' + l_1 c' = b'$.

If $l_0 b' = k_0 c'$, we define the mapping f_0 as follows

$$f_0 : ka + lc \mapsto kb' + lc'.$$

Then by a similar discussion, we can deduce that $f_0 \in AutG$ and $f_0(S) = T$. This completes the proof.

Now we are in the position to prove Theorem 4.

Proof of Theorem 4:

$\Rightarrow$. Let G be a weakly $3-DCI$–finite abelian group that does not satisfy i); we prove that ii) holds for G. Now G is of even order and can be generated by some two elements of G. Suppose that $G \cong Z_n \oplus Z_m$. Then mn is even. Assume, without loss of generality that n is even. If m is odd, we can assume that each prime divisor of m is a prime divisor of n. By Lemma 1 , we know that the Sylow 2-subgroup of G is a cyclic group of order at most 4. If m is even, by Lemma 2, we conclude that the Sylow 2-subgroup of G is homocyclic.

$\Leftarrow$. By Theorem 3, it suffices to check that any any 3-element generating subset of G is a CI-subset. This is clear if G is of odd order (Lemma 3). If G satisfies ii), then by Lemmas [3-5], we know that G is weakly $3-DCI$. In what follows, we assume that the rank of G is at least 3. Suppose that $D(G, \{a, b, c\}) \cong D(G, \{a', b', c'\})$ and $\{a, b, c\}$ is a generating subset of G. Let f be an isomorphism from the first digraph to the second satisfying that $f(0) = 0$. We consider three cases.

Case 1. $2a$, $2b$ and $2c$ are distinct.

In this case, the theorem follows readily from Lemma 3.

Case 2. $2a = 2b = 2c$.

In this case, we have that $2a' = 2b' = 2c'$. Since G can not be generated by two elements, we deduce that $G \cong \langle a\rangle \oplus \langle a-b\rangle \oplus \langle a-c\rangle \cong \langle a'\rangle \oplus \langle a'-b'\rangle \oplus \langle a'-c'\rangle$. From this, the theorem follows readily.

Case 3. $2a = 2b \neq 2c$.

Clearly, $2c \neq a + b$. Assume, without loss of generality that $f(a) = a'$, $f(b) = b'$ and $f(c) = c'$. Then it is easy to see that $2a' = 2b'$, $2a'$, $a' + b'$, $a' + c'$, $b' + c'$ and $2c'$ are distinct, and $G = \langle a, c, a - b\rangle = \langle a', c', a' - b'\rangle$.

If there exist non-negative integers k and l such that

$$ka = lc \tag{1}$$

then by the proof of Lemma 5 (the statements c) and e)), we know that $f(ka) \in \{ka', kb', (k-1)a' + b'\}$ and $f(lc) = lc'$. If $f(ka) = (k-1)a' + b'$, then by (1), we have the following

$$(k-1)a' + b' = lc'$$

and so G can be generated by a' and c', a contradiction. Thus $f(ka) = ka'$ or kb'.

Now suppose that k_0 is the smallest positive integer such that $k_0a \in \langle c\rangle$ and suppose

$$k_0a = l_0c \tag{2}$$

Then by the above conclusion, we have that $f(k_0a) \in \{k_0a', k_0b'\}$.

If $f(k_0 a) = k_0 a'$, we establish a correspondence f_0 in G as follows:

$$f_0 : ka + lc + m(a-b) \mapsto ka' + lc' + m(a'-b') \ (0 \leq m \leq 1).$$

In the following, we prove that $f_0 \in AutG$.

If $k_1 a + l_1 c + m_1(a-b) = k_2 a + l_2 c + m_2(a-b)$, then $m_1 = m_2$, since otherwise, G can be generated by a and b, a contradiction. Now suppose, without loss of generality that $k_1 \geq k_2$. Then $(k_1 - k_2)a = (l_2 - l_1)c$. By letting f act on both sides of the above equality, we obtain the following

$$f((k_1 - k_2)a) = (l_2 - l_1)c'.$$

By the former proved statement, we know that $f((k_1-k_2)a) \in \{(k_1-k_2)a', (k_1-k_2)b'\}$. If $f((k_1 - k_2)a) = (k_1 - k_2)b' \neq (k_1 - k_2)a'$, then $k_1 - k_2$ is odd and

$$\begin{aligned}(l_2 - l_1)c' &= (k_1 - k_2)b' \\ &= (k_1 - k_2 - 1)b' + b' \\ &= (k_1 - k_2 - 1)a' + b'\end{aligned}$$

Thus G can be generated by a' and c', a contradiction. Therefore $f((k_1-k_2)a) = (k_1 - k_2)a'$, and so

$$(k_1 - k_2)a' = (l_2 - l_1)c'$$

This implies that f_0 is well-defined, similarly, f_0 is a bijection. Thus $f_0 \in AutG$. Clearly, $f_0\{a, b, c\} = \{a', b', c'\}$.

If $f(k_0 a) = k_0 b'$, define f_0 as follows:

$$f_0 : ka + lc + m(a-b) \mapsto kb' + lc' + m(b'-a')$$

By a similar argument, we can show that $f_0 \in$ AutG and $f_0\{a, b, c\} = \{a', b', c'\}$. This completes the proof.

Acknowledgment

The authors are grateful to the referee for his valuable comments.

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(Received 16/8/94; revised 14/7/95)