A Combined Approach to the Construction of Hadamard Matrices

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Abstract

We introduce a general approach which combines the strength of a method recently introduced by Kharaghani with that of orthogonal designs, orthogonal sets, and sequences, to give some very large classes of Hadamard matrices.

We construct Hadamard matrices of orders $4N(tw+h+k)$ and $4N(hw+kv)$, where v, w are the weights of 2-complementary $(0, \pm 1)$-sequences, and N, t, h and k can take on various values.

1 Introduction and Preliminaries

Let us review some ideas that have become central in the construction of Hadamard matrices.

A *weighing matrix*, $W = W(n, w)$ is a $(0, \pm 1)$-matrix of order n satisfying $WW^t = wI$. The number w is called the *weight* of W. Matrices A and B are *amicable* (respectively *antiamicable*) if $AB^t = BA^t$ (respectively $AB^t + BA^t = 0$), and *disjoint* if $A \cap B = 0$, where $\cap$ denotes the Hadamard, or entry-wise product of matrices.

An *Hadamard matrix*, $H(n)$, is the same as a $W(n, n)$.

An *orthogonal design* based on the commuting indeterminates $x_1, \ldots, x_u$ is a sum $D = x_1W_1 + \cdots + x_uW_u$, where $W_1, \ldots, W_u$ are pairwise disjoint, antiamicable weighing matrices. If $W_i = W(n, w_i)$, $i = 1, \ldots, u$, then we write $D = OD(n; w_1, \ldots, w_u)$. $w_1, \ldots, w_u$ are the *weights* of the orthogonal design. An equivalent formulation (the usual one) requires instead that $W_1, \ldots, W_n$ be disjoint $(0, \pm 1)$-matrices and that $DD^t = (w_1{x_1}^2 + \cdots + w_u{x_u}^2)I$.

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We shall concern ourselves mainly with the construction of Hadamard matrices, although our methods lend themselves readily to the construction of weighing matrices, orthogonal designs, and similar structures.

Orthogonal designs provide one of the best methods for the construction of Hadamard matrices, using the following simple (and well-known) idea.

Lemma 1 *Let* $D = x_1W_1 + \cdots + x_uW_u = OD(n; w_1, \ldots, w_u)$. *If there are pairwise amicable* $(0, \pm 1)$*-matrices* $A_1, \ldots, A_u$ *of order* m *such that* $w_1A_1{A_1}^t + \cdots + w_uA_u{A_u}^t = wI$, *then* $A_1 \otimes W_1 + \cdots + A_u \otimes W_u = W(mn, w)$.

The matrices A_i are called *suitable matrices* (for the orthogonal design D). Of course, if neither the orthogonal design nor the suitable matrices have any zeros, we shall have $w = mn$, and the resulting matrix is an Hadamard matrix.

A *circulant matrix* of order n is one whose (i, j) entry is determined solely by the value of $i - j$ mod n. Circulant matrices are normally specified by their first row; the circulant matrix A with first row $(a_1, \ldots, a_n)$ is denoted by $A = circ(a_1, \ldots, a_n)$. It is easy to show that the circulant matrices of any order n form a commutative ring.

Lemma 2 *Suppose* A *and* B *are commuting matrices satisfying* $AA^t + BB^t = wI$. *If* $W = \begin{pmatrix} A & -B \\ B^t & A^t \end{pmatrix}$, *then* $WW^t = wI$.

Here if A and B are $n \times n$ $(0, \pm 1)$-matrices, then $W = W(2n, w)$. On the other hand, if the nonzero entries of A and B are $\{\pm x_1, \ldots, \pm x_u\}$ and W is an appropriate quadratic form, then W is an orthogonal design.

Now if R is the matrix whose (i, j) entry is 1 if $i + j \equiv 1$ mod n, and 0 otherwise (R is the so-called *back-diagonal* matrix), then we may readily verify that AR is symmetric, where A is any circulant matrix. A matrix of this type is called *back-circulant*. It follows immediately that any circulant matrix is amicable to any back-circulant matrix.

Lemma 3 *Suppose* W, X, Y *and* Z *are circulant matrices satisfying* $WW^t + XX^t + YY^t + ZZ^t = wI$. *If* $H = \begin{pmatrix} W & XR & YR & ZR \\ -XR & W & Z^tR & -Y^tR \\ -YR & -Z^tR & W & X^tR \\ -ZR & Y^tR & -X^tR & W \end{pmatrix}$, *then* $HH^t = wI$.

The array used in this Lemma is known as the *Goethals-Seidel array*; as with the previous result, W, X, Y and Z may have commuting indeterminates and their negations as entries, etc.

Generally, wherever they appear, circulant and back-circulant matrices may be replaced by *type I* and *type II* matrices over any abelian group, though we shall not use this amount of generality, and such matrices are not inferred as naturally as circulant ones from the sequence constructions we employ. For further information on these topics, see [7].

Now with each sequence $a = (a_1, \ldots, a_n)$ of length n, we associate the polynomial $f_a(x) = \sum_{a=1}^{n} a_i x^i$. We define an involution, $f^*(x) = f(x^{-1})$, in the ring of Laurent polynomials. If $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $c = (c_1, \ldots, c_n), \ldots$ are k $(0, \pm 1)$-sequences (of *length* n) such that

$$f_a(x)f_a{}^*(x) + f_b(x)f_b{}^*(x) + f_c(x)f_c{}^*(x) + \cdots = w, \tag{1}$$

then we say that they are *k-complementary sequences* with *weight* w (the definition is often given in terms of having zero *non-periodic autocorrelation function* [15]). We also use the briefer appellation, *complementary pair*, for 2-complementary $(0, \pm 1)$-sequences. It is easy to see that, if $A = circ(a)$, $B = circ(b)$, $C = circ(c), \ldots$, then

$$AA^t + BB^t + CC^t + \cdots = wI. \tag{2}$$

This is also true if (1) holds in a weaker sense—namely modulo $x^n - 1$—in which case the sequences are said to have zero *periodic* autocorrelation. The advantage of the stronger condition (which we shall use exclusively in this article) is that it gives circulant matrices satisfying (2) in *all* orders $\geq n$ (these are obtained by appending an appropriate number of zeros to each sequence), while the weaker condition does so only for order n (actually, any multiple of n). *Golay sequences* are 2-complementary (± 1)-sequences. These are known to exist with lengths $2^a 10^b 26^c$, $a, b, c \geq 0$ — so-called *Golay numbers.*

When sequences are displayed, commas indicate concatenation, and "scalar multiplication" is as with n-tuples. The *direct product* of sequences $A = (a_1, \ldots, a_n)$ and B is $A \times B = (a_1 B, \ldots, a_n B)$. The *conjugate* of A is $A^* = (a_n, \ldots, a_1)$. As with matrices, we say that sequences are *disjoint* if their Hadamard (entry-wise) product is zero. If A, B are $(0, \pm 1)$-sequences having zeros in all the same positions (such sequences are sometimes called *disjointable*), then clearly $U = \frac{1}{2}(A + B)$, $V = \frac{1}{2}(A - B)$ are disjoint $(0, \pm 1)$-sequences and $f_U f_U{}^* + f_V f_V{}^* = \frac{1}{2}(f_A f_A{}^* + f_B f_B{}^*)$. In particular, A, B are complementary if and only if U, V are. We shall call a $(0, \pm 1)$-sequence with exactly one nonzero entry a *monomial sequence*. We shall refer to $2k$ mutually disjoint sequences $U_1, V_1, \ldots, U_k, V_k$ as k disjoint complementary pairs if, for each i, (U_i, V_i) is a complementary pair (weights given explicitly, not necessarily all the same).

The *Cooper-Wallis construction* [1] uses four disjoint circulant $n \times n$ $(0, \pm 1)$-matrices X_1, X_2, X_3, X_4 satisfying (2), with $w = n$ (such matrices are called *T-matrices*). From these, circulant matrices are formed, based on the variables x_1, x_2, $x3, x_4$:

$$W = x_1 X_1 + x_2 X_2 + x_3 X_3 + x_4 X_4,$$

$$X = -x_2 X_1 + x_1 X_2 + x_4 X_3 - x_3 X_4,$$

$$Y = -x_3 X_1 - x_4 X_2 + x_1 X_3 + x_2 X_4$$

and

$$Z = -x_4 X_1 + x_3 X_2 - x_2 X_3 + x_1 X_4.$$

W, X, Y and Z may now be used in Lemma 3 to obtain $OD(4n; n, n, n, n)$. If we do not require $w = n$, we get $OD(4n; w, w, w, w)$. Later we shall be using two pairs of complementary sequences having different weights in a similar fashion, to obtain $OD(4n; w, w, v, v)$.

More recently, we have introduced the notion of *orthogonal sets* (see [3], [4], [5], [8], [9], [11]). In this article we shall require only orthogonal sets $OS(n; nk; 1_k)$ (according to the notation of [3][1]), which are sets of k (± 1)-matrices $C_1, \ldots, C_k$ satisfying

$$C_1 {C_1}^t + \cdots + C_k {C_k}^t = nkI \quad (3)$$

$$C_i {C_j}^t = 0, \; i \neq j. \quad (4)$$

Therefore, we use a more compact notation for these, denoting $\{C_1, \ldots, C_k\}$ by (n, k)-OS.

An orthogonal set is *symmetric* if all its component matrices are symmetric. The most classical examples of orthogonal sets are (n, n)-OS, which are called *Kharaghani sets*. Though Kharaghani sets are not necessarily symmetric, they always coexist with symmetric ones. Also, a Kharaghani set of order n is equivalent to an Hadamard matrix of order n. The basic method of constructing weighing matrices (and in particular, Hadamard matrices) using these orthogonal sets may be stated as follows.

Lemma 4 *If $\{C_1, \ldots, C_k\} = (n, k)$-OS, and $W_1, \ldots, W_k$ are pairwise disjoint weighing matrices $W(m, w)$, then $W = C_1 \otimes W_1 + \cdots + C_k \otimes W_k = W(mn, nkw)$.*

It was Kharaghani who first showed how such sets may be used to construct special classes of Hadamard matrices and weighing matrices, using latin squares rather than disjoint weighing matrices. Here is a result which gives a plentiful supply of orthogonal sets.

Lemma 5 (See [3]; the last item reflects the main result of [16]) *The following orthogonal sets exist:*

1. *(mnk, k)-OS, given that there are Hadamard matrices of orders mk and nk;*
2. *$(2mnk, k)$-OS, given that there are Hadamard matrices of order $2mk$ and $2nk$;*
3. *(mn, kl)-OS, given that there are orthogonal sets (m, k)-OS and (n, l)-OS;*
4. *$(mn, 2kl)$-OS, given that there are orthogonal sets $(m, 2k)$-OS and $(n, 2l)$-OS.*

Moreover, if $m = n$ in parts 1 and 2, the matrices making up the resulting orthogonal set may be chosen to be symmetric. This is also true in parts 3 and 4 if the original orthogonal sets are symmetric, or if $m = n$, $k = l$. There also exist orthogonal sets:

[1] Our methods can be modified to exploit more general orthogonal sets; if not all coweights are 1, one must provide sequences with weights appropriate to match these. Another generalization would be to use orthogonal sets $OS(n, w, 1_k)$, $w < nk$, in which case, weighing matrices result.

5. $(2n,2)$-*OS for* $n = 3^{2r_0}p_1^{4r_1}\cdots p_k^{4r_k}$, *each* p_i *a prime congruent to* 3 *modulo* 4, *each* $r_i \geq 0$.

Orthogonal sets may also be used in a natural way to construct orthogonal designs, and vice versa. For more information on orthogonal sets and disjoint weighing matrices, and how they are obtained and used, see [3], [4], [11].

This next Lemma is derived from [4][Theorem 4.2], and it gives the underlying principle for our new constructions for Hadamard matrices. Notice that it generalizes Lemmas 1 and 4; it might be called the "Master Product Lemma for Orthogonal Matrices".

Lemma 6 *Let* $D_1,\ldots,D_u$ *be a set of* $n\times n$ *weighing matrices,* $D_i = W(m,w_i)$, *and* $S_1,\ldots,S_u$ *be* $(0,\pm1)$*-matrices such that* $w_1S_1{S_1}^t+\cdots+w_uS_u{S_u}^t = wI$. *If, for each* $i\neq j$ *and each property* $\Phi\in\{$ *disjointness, amicability, antiamicability* $\}$, *one of the pairs* (D_i,D_j) *and* (S_i,S_j) *satisfies* Φ, *then*

$$W = \sum_{i=1}^{u} D_i\otimes S_i = W(mn,w).$$

Proof: The terms in the sum, W, are disjoint, so W is a $(0,\pm1)$-matrix. They are also antiamicable, so cross terms in WW^t cancel, leaving

$$WW^t = \sum_{i=1}^{u} D_i{D_i}^t\otimes S_i{S_i}^t = I\otimes\sum_{i=1}^{u} w_iS_i{S_i}^t = wI.$$

□

2 Examining the method of Kharaghani

Kharaghani [13], [14], [10], [12] has developed methods using sequences of square matrices (rather than numbers or indeterminates), built from certain orthogonal sets, which give new classes of Hadamard matrices and orthogonal designs, as in the following result.

Theorem 7 (Kharaghani, [13]) *If there is an Hadamard matrix of order* 2^nm, *then there is an Hadamard matrix of order* $4^{n+1}m^2(r+4^n+1)$, *where* r *is a Golay number.*

Let us now present an alternate demonstration of this result, designed to illuminate the interplay between the use of sequences, orthogonal designs and orthogonal sets in Kharaghani's method. This approach also allows us to achieve a great deal more generality.

Proof: Let U', V' be a pair of Golay sequences of length r. Writing $U = \frac{1}{2}(U' + V')$, $V = \frac{1}{2}(U' - V')$, we consider the circulant matrices

$$\begin{aligned} W &= circ(aU + bV, c, d_1, \ldots, d_k) \\ X &= circ(aU + bV, -c, -d_1, \ldots, -d_k) \\ Y &= circ(bU - aV, c, -d_1, \ldots, -d_k) \\ Z &= circ(bU - aV, -c, d_1, \ldots, d_k), \end{aligned} \tag{5}$$

of order $r+k+1$, where $a, b, c, d_1, \ldots, d_k$ are commuting indeterminates (k is a positive integer, as yet unspecified). Substituting W, X, Y, Z into the Goethals-Seidel array, we obtain an array of the form

$$aA + bB + cC + \sum_{i=1}^{k} d_i D_i, \tag{6}$$

where for each i,

$$aA + bB + cC + d_i D_i = OD(4(r + k + 1); 2r, 2r, 4, 4). \tag{7}$$

Now suppose that we have pairwise amicable (± 1)-matrices $E_1, E_2, E_3, F_1, \ldots, F_k$ of order N that satisfy

$$F_i F_j^t = 0, i \neq j, \tag{8}$$

and

$$\frac{r}{2} E_1 E_1^t + \frac{r}{2} E_2 E_2^t + E_3 E_3^t + \sum_{i=1}^{k} F_i F_i^t = N(r + k + 1) I_N. \tag{9}$$

Then the matrix

$$E_1 \otimes A + E_2 \otimes B + E_3 \otimes C + \sum_{j=1}^{k} F_j \otimes D_j \tag{10}$$

is an Hadamard matrix of order $4N(r + k + 1)$.

In particular, if we take $E_1 = E_2 = E_3 = H$, an Hadamard matrix of order N, and $\{F_1, \ldots, F_k\} = (N, k)$-OS, with H amicable to each F_i, then the construction applies. We shall refer to this as the "basic method". It is the starting point for our generalizations.

We construct two sets of matrices: $\{P_i\}_{i=1}^{2^n}$, consisting of all possible n-fold Kronecker products of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\{Q_i\}_{i=1}^{2^n}$, formed by all possible n-fold Kronecker products of the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}$. These matrices are all amicable (since the 2×2 matrices from which they are constructed are amicable). The Q_i's comprise a $(2^n, 2^n)$-OS, and the P_i's are disjoint permutation matrices.

By assumption, there is an Hadamard matrix of order $2^n m$. So by Lemma 5, there is an orthogonal set, $(2^n m^2, 2^n)$-OS, $\{H_i\}_{i=1}^{2^n}$. Let us define matrices

$$\begin{aligned} H &= P_1 \otimes H_1 + \cdots + P_{2^n} \otimes H_{2^n}, &(11)\\ F_{i(s,t)} &= Q_s \otimes H_t,\ i = 1, \ldots, 4^n. &(12)\end{aligned}$$

H is an Hadamard matrix of order $4^n m^2$, amicable to each of the matrices F_i, which comprise a $(4^n m^2, 4^n)$-OS. We now take A, B, C, D_i as in (6) and (7), with $k = 4^n$. It follows immediately from Theorem 6 that the matrix

$$H \otimes (A + B + C) + \sum_{i=1}^{k} F_i \otimes D_i \qquad (13)$$

is an Hadamard matrix of order $4^{n+1} m^2 (r + 4^n + 1)$, which proves the Theorem. □

Here is one way to modify this construction. Suppose we have an Hadamard matrix of order $2^n m$, $n > 0$. Lemma 5 then supplies us with $(2^n m^2, 2^{n-1})$-OS. Now the 2^{n-1} matrices Q_i, of order 2^{n-1} formed by all possible (n-1)-fold Kronecker products of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}$, comprise an $(2^{n-1}, 2^{n-1})$-OS. The 2^{n-1} matrices P_i, formed by all possible (n-1)-fold Kronecker products of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are pairwise disjoint permutation matrices of order 2^{n-1}, amicable to each of the Q_i's and to each other. The matrices

$$\begin{aligned} H &= P_1 \otimes H_1 + \cdots + P_{2^{n-1}} \otimes H_{2^{n-1}},\\ F_{i(s,t)} &= Q_s \otimes H_t,\ i = 1, \ldots, 4^{n-1}\end{aligned}$$

are amicable, with $H = H(N)$, $\{F_i\}_{i=1}^{k} = (N, k)$-OS, where $N = 2^{2n-1} m^2$ and $k = 4^{n-1}$, which provides us with the following result.

Theorem 8 *If there is an Hadamard matrix of order $2^n m$, then there is an Hadamard matrix of order $2 \cdot 4^n m^2 (r + 4^{n-1} + 1)$, where r is any Golay number.*

Writing the order of the hypothesized Hadamard matrix as $2^{n-1}(2m)$ and applying Theorem 7, we get an Hadamard matrix of order $4^n (2m)^2 (r + 4^{n-1} + 1)$, which is twice the order of the corresponding result given by Theorem 8. On the other hand, Theorem 8 can only produce a matrix with the factor $r + 4^n + 1$ by starting with an Hadamard matrix of order $2^{n+1} m$, and the resulting matrix will be twice the order of the one produced by Theorem 7 if an Hadamard matrix of order $2^n m$ exists. So it appears that neither result is implied by the other.

3 Using other sequences

Let us now turn to constructions for arrays similar to (6), consisting of weighing matrices, with patterns of embedded orthogonal designs arising as in (7).

The sequences used in the basic method are not the Golay sequences U', V' as such, but U, V. This leads us to apply these ideas to other pairs of complementary sequences.

In the following, let U, V be disjoint complementary sequences of length $w+k+1$ and weight w. Let $S, T_1, \ldots, T_k$ be the $k+1$ monomial sequences of length $w+k+1$ disjoint from U, V. From these we construct circulant matrices,

$$\begin{aligned} W &= circ(aU + bV + cS + d_1T_1 + \cdots + d_kT_k), \\ X &= circ(aU + bV - cS - d_1T_1 - \cdots - d_kT_k), \\ Y &= circ(bU - aV + cS - d_1T_1 - \cdots - d_kT_k), \\ Z &= circ(bU - aV - cS + d_1T_1 + \cdots + d_kT_k), \end{aligned} \tag{14}$$

suitable for substitution into the Goethals-Seidel array. This construction differs from (5) in that it provides more flexibility in distributing the symbols $c, d_1, \ldots, d_k$ among the positions disjoint from complementary sequences U, V—whose weight w can take on more values than in the basic method.

Theorem 9 *Suppose there is a disjoint complementary pair U, V of length $w+k+1$, with weight w. Then there exists an array,*

$$aA + bB + cC + \sum_{j=1}^{k} d_jD_j,$$

where all matrices A, B, C, D_j are disjoint, and for each j,

$$aA + bB + cC + d_jD_j = OD(4(w+k+1); 2w, 2w, 4, 4).$$

We shall find uses for more general arrays, where not only the last part (ie, D_j) is subscripted, giving several embedded orthogonal designs, but the other parts (ie, A, B, C) are as well.

We can vary the construction for Theorem 9 by omitting S and c, using sequences of length $w+k$ (of most interest when $w+k$ is odd). More generally, we can use sequences of length $w+h+k$, replacing cS in (14) with $c_1S_1 + \cdots + c_hS_h$, where $S_1, \ldots, S_h$ are monomial sequences disjoint from each other and all the others.

Theorem 10 *Suppose there is a disjoint complementary pair U, V of length $w+h+k$, with weight w. Then there exists an array,*

$$aA + bB + \sum_{i=1}^{h} c_iC_i + \sum_{j=1}^{k} d_jD_j,$$

where all matrices A, B, C_i, D_j are disjoint, and for each i, j,

$$aA + bB + c_iC_i + d_jD_j = OD(4(w+h+k); 2w, 2w, 4, 4).$$

Let us go one step further by using t disjoint complementary pairs $U_1, V_1, \ldots, U_t, V_t$ of length $tw+h+k$, each pair U_s, V_s having weight w. Now let $T_1, \ldots, T_h, S_1, \ldots, S_k$ be the $h+k$ monomial sequences disjoint from all the U_i's and V_i's.

Circulant matrices formed using the sequences

$$\sum_{s=1}^{t}(a_sU_s + b_sV_s) + \sum_{i=1}^{h}(c_iS_i) + \sum_{j=1}^{k}(d_jT_j)$$

$$\sum_{s=1}^{t}(a_sU_s + b_sV_s) - \sum_{i=1}^{h}(c_iS_i) - \sum_{j=1}^{k}(d_jT_j)$$

$$\sum_{s=1}^{t}(a_sU_s - b_sV_s) + \sum_{i=1}^{h}(c_iS_i) - \sum_{j=1}^{k}(d_jT_j)$$

$$\sum_{s=1}^{t}(a_sU_s - b_sV_s) - \sum_{i=1}^{h}(c_iS_i) + \sum_{j=1}^{k}(d_jT_j)$$

may be used in the Goethals-Seidel array to obtain the following result.

Theorem 11 *Suppose there are t disjoint complementary pairs of length $tw+h+k$ and weight w. Then there exists an array,*

$$\sum_{s=1}^{t}(a_sA_s + b_sB_s) + \sum_{i=1}^{h} c_iC_i + \sum_{j=1}^{k} d_jD_j,$$

where all matrices A_s, B_s, C_i, D_j are disjoint, and for each s, i, j,

$$a_sA_s + b_sB_s + c_iC_i + d_jD_j = OD(4(tw+h+k); 2w, 2w, 4, 4).$$

So far our constructions are based on disjoint complementary pairs (or several disjoint pairs) and any number of monomial sequences. Since there are essentially two different series of these, we can use S_j, T_j in tandem, as with A_i, B_i, sacrificing the independence of the monomial sequences so that complementary pairs with larger weights may be used instead.

We begin as before with $h+k$ disjoint complementary pairs U_i, V_i, $i = 1, \ldots, h$, S_j, T_j, $j = 1, \ldots, k$, each pair U_i, V_i having weight w and each pair S_j, T_j having weight v. The sequences

$$\sum_{i=1}^{h}(a_iU_i + b_iV_i) + \sum_{j=1}^{k}(c_jS_j + d_jT_j)$$

$$\sum_{i=1}^{h}(a_iU_i + b_iV_i) - \sum_{j=1}^{k}(c_jS_j + d_jT_j)$$

$$\sum_{i=1}^{h}(a_iU_i - b_iV_i) + \sum_{j=1}^{k}(c_jS_j - d_jT_j)$$

$$\sum_{i=1}^{h}(a_iU_i - b_iV_i) - \sum_{j=1}^{k}(c_jS_j - d_jT_j)$$

give circulant matrices that may be used in the Goethals-Seidel array.

Theorem 12 *Suppose there are $h+k$ disjoint complementary pairs of length $hw+kv$, h pairs with weight w and k with weight v. Then there exists an array,*

$$\sum_{i=1}^{h}(a_iA_i + b_iB_i) + \sum_{j=1}^{k}(c_jC_j + d_jD_j),$$

where all matrices A_i, B_i, C_j, D_j are disjoint, and for each i, j,

$$a_iA_i + b_iB_i + c_jC_j + d_jD_j = OD(4(hw + kv); 2w, 2w, 2v, 2v).$$

4 Constructing and using seed matrices

Let us now consider how matrices suitable for combining with the arrays given by Theorems 9, 10, 11 and 12 can arise. In the basic method, we used matrices $H, F_1, \ldots, F_k$ such that:

- $H = H(N)$;
- $\{F_i\}_{i=1}^{k} = OS(N, k)$;
- H is amicable to each F_i.

For brevity we shall call matrices satisfying properties 1,2,3 above (N, k)-*seed matrices.*

Theorem 13 *If there are (N, k)-seed matrices and*

1. *a disjoint complementary pair of length $w + k + 1$ and weight w, then there exists an Hadamard matrix of order $4N(w + k + 1)$;*
2. *k disjoint complementary pairs of length $kv + 1$ and weight v, then there exists an Hadamard matrix of order $4N(kv + 1)$.*

Proof: Let $H; F_1, \ldots, F_k$ be the seed matrices. If A, B, C, D_i are as obtained in Theorem 9, then it follows from Lemma 6 that (13) is an Hadamard matrix of order $4N(w+k+1)$. Similarly, we obtain from Theorem 11 an array $\sum_{i=1}^{k}(a_iA_i+b_iB_i)+cC$, and $\sum_{i=1}^{k}(A_i + B_i) \otimes F_i + C \otimes H$ is an Hadamard matrix of order $4N(kv + 1)$. □

We have already seen one way seed matrices arise; here are some others.

Theorem 14 *If there is a $(m, 2)$-OS and a $(n, 2)$-OS, then there are $(mn, 4)$-seed matrices.*

Proof: Let $\{U, V\}$ and $\{X, Y\}$ be the two orthogonal sets. The required seed matrices are $\frac{U+V}{2} \otimes X + \frac{U-V}{2} \otimes Y$; $U \otimes X$, $U \otimes Y$, $V \otimes X$, $V \otimes Y$. □

Theorem 15 *If there is a symmetric Hadamard matrix of order k and a (n, k)-OS, then there are (nk, k)-seed matrices.*

Proof: Let $K = H(k)$ be symmetric and $\{Q_1, \ldots, Q_k\} = (n, k)$-OS. Let $P_1, \ldots, P_k$ be disjoint permutation matrices of order k. Then we define $H = P_1 \otimes Q_1 + \cdots + P_k \otimes Q_k$, and $F_i = KP_i \otimes Q_i$, $i = 1, \ldots, k$. These are the required matrices. □

Applying Lemma 5, we obtain the following.

Corollary 16 *Suppose there is a symmetric Hadamard matrix of order k.*

1. *There are (k^2, k)-seed matrices.*
2. *If there are Hadamard matrices of orders mk and nk, then there are (mnk^2, k)-seed matrices.*
3. *If there are Hadamard matrices of orders $2mk$ and $2nk$, then there are $(2mnk^2, k)$-seed matrices.*

To illustrate this result, take $k = 2$. Let $K = H(2) = \begin{pmatrix} 1 & 1 \\ 1 & - \end{pmatrix}$. The construction for lemma 5, gives $\left\{ Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, Q_2 = \begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix} \right\} = (2,2)$-OS. Take $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $P_w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $H = P_1 \otimes Q_1 + P_2 \otimes Q_2 = \begin{pmatrix} Q_1 & Q_2 \\ Q_2 & Q_1 \end{pmatrix}$, $F_1 = KP_1 \otimes Q_1 = \begin{pmatrix} Q_1 & Q_1 \\ Q_1 & -Q_1 \end{pmatrix}$, $F_2 = KP_2 \otimes Q_2 = \begin{pmatrix} Q_2 & -Q_2 \\ Q_2 & Q_2 \end{pmatrix}$ form the $(4, 2)$-seed matrices given by part 1. These may be used in Theorem 13 to obtain Hadamard matrices of any order $16(r + 3)$, where r is a Golay number [12].

The idea behind Theorem 15 can be used more generally with symmetric orthogonal sets, which Lemma 5 provides in abundance.

Theorem 17 *If there is a symmetric orthogonal set $\{Q_1, \ldots, Q_h\} = (k, h)$-OS, and an orthogonal set $\{R_1, \ldots, R_k\} = (n, k)$-OS, then there are (nk, hk)-seed matrices.*

Proof: Let $P_1, \ldots, P_k$ be disjoint permutation matrices of order k. $H = \sum_{i=1}^{k} P_i \otimes R_i$ is an Hadamard matrix of order kn, amicable to each of $F_{i(s,t)} = Q_t P_s \otimes R_s$, which form a (kn, hk)-OS. □

Corollary 18 *If there is an Hadamard matrix of order n, there are (n^2, n^2)-seed matrices.*

To illustrate, let us take $k = 2$, Q_1, Q_2 (a symmetric $(2,2)$-OS obtained from $H(2)$) and P_1, P_2 as above, and $R_1 = Q_1$, $R_2 = Q_2$. Then $H = P_1 \otimes R_1 + P_2 \otimes R_2 = \begin{pmatrix} Q_1 & Q_2 \\ Q_2 & Q_1 \end{pmatrix}$, $F_1 = Q_1 P_1 \otimes R_1 = \begin{pmatrix} Q_1 & Q_1 \\ Q_1 & Q_1 \end{pmatrix}$, $F_2 = Q_1 P_2 \otimes R_2 = \begin{pmatrix} Q_2 & Q_2 \\ Q_2 & Q_2 \end{pmatrix}$, $F_3 = Q_2 P_1 \otimes R_1 = \begin{pmatrix} Q_1 & -Q_1 \\ -Q_1 & Q_1 \end{pmatrix}$, $F_4 = Q_2 P_2 \otimes R_2 = \begin{pmatrix} -Q_2 & Q_2 \\ Q_2 & -Q_2 \end{pmatrix}$ are a set of $(4,4)$-seed matrices.

Seed matrices are suitable matrices for making block Golay sequences. For instance the above two examples provide Golay sequences of length 3 and 5 respectively both of block size 4. As such all the properties of Golay sequences apply (see [12] for details) and consequently there is a block Golay sequence of length 15 and block size 2^4. The above theorem alone provides many new classes of Hadamard matrices including all the results of [12], [13]. To present the strength of such a result, we shall show below the existence of an Hadamard matrix of order $2^6.2063$. The best known order so far is $2^7.2063$. To show the existence of such a matrix note that $2063 = 2048 + 15 = 2^{11} + 15$. The existence of a block Golay sequence of length 15 and block size 2^4 and a Golay sequence of length 2^{11} imply the existence of such an Hadamard matrix.

Here is another way to multiply seed matrices.

Theorem 19 *If there are (M,h)-seed matrices and (N,k)-seed matrices, then there are (MN,hk)-seed matrices. If there are $(M,2h)$-seed matrices and $(N,2k)$-seed matrices, then there are (MN,hk)-seed matrices.*

Proof: Let $H, \{Q_i\}_{i=1}^{h}$ be (M,h)-seed matrices and $K, \{R_j\}_{j=1}^{k}$ be (N,k)-seed matrices. If we set $L = H \otimes K$ and $F_{l(i,j)} = Q_i \otimes R_j$, then $L, \{F_l\}_{l=1}^{hk}$ are (MN,hk)-seed matrices. Similarly, if $H, \{Q_i\}_{i=1}^{2h}$ are $(M,2h)$-seed matrices and $K, \{R_j\}_{j=1}^{2k}$ are $(N,2k)$-seed matrices, we set $L = H \otimes K$ and $F_{l(i,j)} = \frac{1}{2}[Q_i + Q_{2h-i+1}] \otimes R_j + \frac{1}{2}[Q_i - Q_{2h-i+1}] \otimes R_{2k-j+1}$, $1 \le i \le h$, $1 \le j \le k$, to obtain (MN,hk)-seed matrices. □

Besides Theorem 13, there are other ways to use seed matrices. In the following, we give some simple applications that make use of information ignored in Theorem 19. This gives some idea of how Theorems 10, 11 and 12 can be exploited.

Theorem 20 *Suppose there are (M,h)-seed matrices and (N,k)-seed matrices.*

1. *If there is a disjoint complementary pair of length $w+h+k$ and weight w, then there is an Hadamard matrix of order $4MN(w+h+k)$.*
2. *If there is a disjoint complementary pair of length $w+h+hk$ and weight w, then there is an Hadamard matrix of order $4MN(w+h+hk)$.*
3. *If there are h disjoint complementary pairs of length $hw+k+1$ and weight w, then there is an Hadamard matrix of order $4MN(hw+k+1)$.*

4. *If there are h disjoint complementary pairs of length $hw + hk + k$ and weight w, then there is an Hadamard matrix of order $4MN(hw + hk + k)$.*

5. *If there are h disjoint complementary pairs of length $hw + h + k$ and weight w, then there is an Hadamard matrix of order $4MN(hw + h + k)$.*

6. *If there are h disjoint complementary pairs of length $hw + hk + 1$ and weight w, then there is an Hadamard matrix of order $4MN(hw + hk + 1)$.*

7. *If there are hk disjoint complementary pairs of length $hkw + h + 1$ and weight w, then there is an Hadamard matrix of order $4MN(hkw + h + 1)$.*

8. *If there are hk disjoint complementary pairs of length $hkw + h + k$ and weight w, then there is an Hadamard matrix of order $4MN(hkw + h + k)$.*

9. *If there are $h + k$ disjoint complementary sequences of length $hw + kv$, h with weight w and k with weight v, then there is an Hadamard matrix of order $4MN(hw + kv)$.*

Proof (sketch): Let $H, \{Q_i\}_{i=1}^h$ and $K, \{R_j\}_{j=1}^k$ be the (M,h)-seed matrices and (N,k)-seed matrices respectively. Note that $H \otimes K = H(MN)$, $\{Q_i \otimes K\}_{i=1}^h = (MN,h)$-OS, $\{H \otimes R_j\}_{j=1}^k = (MN,k)$-OS, and $\{Q_i \otimes R_j\}_{i=1..h,j=1..k} = (MN,hk)$-OS. Moreover, all these matrices are amicable (so among other things, we can isolate (MN,h), (MN,k) and (MN,hk)-seed matrices among these sets). Let us give the construction for part 8 to illustrate the general approach: let U_i, V_i, $i = 1,\ldots,hk$ be the hk complementary pairs. Theorem 11 (with $t = hk$) then provides us with an array involving A_s, B_s, C_i, D_j, $s = 1,\ldots,hk$, $i = 1,\ldots,h$, $j = 1,\ldots,k$; the required Hadamard matrix is

$$\sum_{s(i,j)=1}^{hk} (A_s + B_s) \otimes (Q_i \otimes R_j) + \sum_{i=1}^{h} C_i \otimes (Q_i \otimes K) + \sum_{j=1}^{k} D_j \otimes (H \otimes R_j).$$

The array given by Theorem 10 is similarly used for parts 1 and 2, the array given by Theorem 11 for parts 3 to 8, and the array given by Theorem 12 for part 9. □

We have not exhausted the possibilities here; for example, if instead of h and k we have $2h$ and $2k$, we may replace h, k and hk with $2h, 2k$ and hk in the Theorem, using the method of the second part of Theorem 19. In this way, we construct $OS(MN,t)$ for $t = 1, 2h, 2k$ and hk; we can also get $t = 4hk$ with the original approach, with all matrices amicable. This gives even more possible orders, such as $4MN(hkw + 4hk + 1)$, a variation of part 7 of Theorem 20.

5 Calculations, consequences and remarks

Let us now consider our choices for complementary pairs. Unfortunately, there is not a great variety of weights available. We can always make complementary sequences disjoint by adding enough zeros and shifting, so the condition of disjointness is not a major consideration. A result of Eliahou, Kervaire and Saffari [6] eliminates all weights having a factor $\equiv 3 \bmod 4$. Of the weights not eliminated by this result, only a few are known to exist. Aside from those obtained from Golay sequences, there are disjoint complementary pairs of length 6 and weight 5:

$$(11-000), \quad (000101),$$

and length 14 and weight 13:

$$(1110-110-1-000), \quad (0001000-000101).$$

Such sequences may also be multiplied, as follows.

Theorem 21 *Let U, V be a disjoint complementary pair of length k and weight v. Let S, T be a disjoint complementary pair of length l and weight w. Then $U \times S + V \times T^*$, $U \times T - V \times S^*$ is a disjoint complementary pair of length kl and weight vw.*

Henceforth, we shall take $w = 2^a 5^b 10^c 13^d 26^e$, $v = 2^{a'} 5^{b'} 10^{c'} 13^{d'} 26^{e'}$, $l_w = 2^a 6^b 10^c 14^d 26^e$, and $l_v = 2^{a'} 6^{b'} 10^{c'} 14^{d'} 26^{e'}$, where $a, b, c, d, e, a', b', c', d', e'$ are nonnegative integers. The following differs from Corollary 5.8 of [15] only in that disjoint sequences are given.

Corollary 22 *There is a disjoint complementary pair of length l_w and weight w.*

No other weights are known (so the first unresolved weight is 17). As for getting several disjoint complementary pairs, U_i, V_i, we infer the following result from Corollary 22.

Corollary 23 *For any nonnegative integers h and k, there are $h + k$ disjoint complementary pairs of length $hl_w + kl_v$, h of weight w and k of weight v.*

Corollary 23 is far from best possible, for the larger b, d, b' and d' are, the more zeros there are in the resulting sequences. Convenient location of the zeros in these sequences may allow disjoint sequences of smaller length. For example, Corollary 23 gives four sequences of length 20, consisting of a complementary pair of weight 5 and a complementary pair of weight 13. But these weights are also attained by the following sequences of length 18:

1	1	1	0	−	1	1	0	−	1	−	0	0	0	0	0	0	0
0	0	0	1	0	0	0	−	0	0	0	1	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	−

Similarly, we can have two pairs of length 10, both weight 5, or length 26 and both weight 13. Combining these, we have h pairs of weight 5 and k pairs of weight 13, all disjoint and having length $5h + 13k$ when $h + k$ is even, or length $5h + 13k + 1$ otherwise. Applying Theorem 21 repeatedly, we can also infer part 3 of the following result, which summarizes these observations.

Theorem 24 *Let h, k be nonnegative integers.*

1. *if $h+k$ is even, then there are h complementary pairs of weight 5 and k complementary pairs of weight 13, with all sequences disjoint with length $5h+13k$.*

2. *if $h+k$ is odd, then there are h complementary pairs of weight 5 and k complementary pairs of weight 13, with all sequences disjoint with length $5h+13k+1$.*

3. *If 2^{t+1} divides h, then there are h complementary pairs of weight 5^t with length $5^t h$, or weight 13^t with length $13^t h$, with all sequences disjoint.*

Now let us consider some consequences to all that we have done. The following results do not exhaust the possibilities, by any means.

Theorem 25 *Suppose there are Hadamard matrices of orders m and n, $m, n > 1$.*

1. *If $m^2+n^2 \geq l_w - w$, then there is an Hadamard matrix of order $4m^2n^2(w+m^2+n^2)$.*

2. *If $m^2+m^2n^2 \geq l_w - w$, then there is an Hadamard matrix of order $4m^2n^2(w+m^2+m^2n^2)$.*

3. *If $n^2 \geq m^2(l_w-w)-1$, then there is an Hadamard matrix of order $4m^2n^2(m^2w+n^2+1)$.*

4. *If $n^2 \geq \frac{m^2}{1+m^2}(l_w-w)$, then there is an Hadamard matrix of order $4m^2n^2(m^2w+m^2n^2+n^2)$.*

5. *If $n^2 \geq m^2(l_w-w-1)$, then there is an Hadamard matrix of order $4m^2n^2(m^2w+m^2+n^2)$.*

6. *If $m^2+\frac{m^2n^2}{4} \geq l_w - w$, then there is an Hadamard matrix of order $4m^2n^2(w+m^2+\frac{m^2n^2}{4})$.*

7. *If $n^2 \geq \frac{4m^2}{4+m^2}(l_w-w)$, then there is an Hadamard matrix of order $4m^2n^2(m^2w+\frac{m^2n^2}{4}+n^2)$.*

8. *If $n^2 \geq 4(l_w-w)-\frac{4}{m^2}$, then there is an Hadamard matrix of order $4m^2n^2(m^2w+\frac{m^2n^2}{4}+1)$.*

9. *If 2^{t+1} divides $\frac{m^2n^2}{4}$ and $p=5^t$ or 13^t, then there is an Hadamard matrix of order $4m^2n^2(\frac{m^2n^2}{4}p+m^2n^2+1)$.*

Proof (sketch): By Corollary 18, there are (m^2, m^2)-seed matrices and (n^2, n^2)-seed matrices. We use parts 1–5 of Theorem 20 for the first five parts (we could have used other parts for even greater variety). For the next three parts, we modify parts 2, 4, and 6 of Theorem 20 according to the suggestions following that result (with

$h = \frac{m^2}{2}$, $k = \frac{n^2}{2}$). The last part uses the modified version of part 7 of theorem 20 together with the sequences in part 3 of Theorem 24. □

Theorem 26 *Suppose there are symmetric Hadamard matrices of orders $2m$ and $2n$.*

1. *If $mn \geq l_w - w - 1$, then there is an Hadamard matrix of order $64m^2n^2(w + mn + 1)$.*
2. *If $n \geq 2(l_w - w)$, then there is an Hadamard matrix of order $64m^2n^2(2mw + mn + 1)$.*

Proof (sketch): By Theorem 16, we have $(4m^2, 2m)$-seed matrices and $(4n^2, 2n)$-seed matrices. Theorem 19 then gives $(16m^2n^2, mn)$-seed matrices. The first Hadamard matrix is obtained by Theorem 13. The other is obtained by modifying part 6 of Theorem 20 according to the suggestion following that result. □

Compare the following result to the main result of [14] (note that orthogonal designs are constructed there; such constructions are also possible using most of the methods of this paper, but for simplicity we restrict our attention to Hadamard matrices).

Theorem 27 *Let $N = 3^{2r_0}p_1^{4r_1}p_2^{4r_2}\cdots p_k^{4r_k}$, where each p_i is a prime of the form $4q+1$. If $4^t \geq l_w - w - 1$, then there is an Hadamard matrix of order $4^{t+1}N^t(w+4^t+1)$.*

Proof: By lemma 5, there is a $(2N, 2)$-OS and a $(2, 2)$-OS. By Theorem 14, there are $(4N, 4)$-seed matrices. By Theorem 19, there are $((4N)^t, 4^t)$-seed matrices. The result now follows by Theorem 13. □

The next result is combines Theorems 19 and 13, and corollaries 16 and 18.

Corollary 28 *If there is an Hadamard matrix of order $2m$, a symmetric Hadamard matrix of order $2n$, and a disjoint complementary pair of weight w and length $w + 2m^2n + 1$, then there is an Hadamard matrix of order $64m^2n^2(w + 2m^2n + 1)$.*

Theorem 29 *Suppose there is an Hadamard matrix of order m and a symmetric Hadamard matrix of order n.*

1. *If $m^2 > l_w - w$, then there exists an Hadamard matrix of order $4m^2(w+m^2+1)$.*
2. *If $n > l_v - v$, then there is an Hadamard matrix of order $4n^2(v + n + 1)$.*
3. *If n is large enough, then there are Hadamard matrices of orders $4m^2n^2q$, where q is any of $w+m^2n+n$, $w+m^2+n$, $w+m^2+m^2n$, m^2w+n+1, $m^2(w+1)+n$, $m^2w + m^2n + n$, $m^2w + m^2n + 1$.*

4. *If m is large enough, there are Hadamard matrices of orders $4m^2n^2q$, where q is any of $w+m^2+1$, $w+m^2+n$, $w+n+m^2n$, $w+m^2+m^2n$, $nw+m^2+1$, $n(w+1)+m^2$, $nw+m^2+m^2n$, $nw+m^2n+1$.*

Proof (sketch): Corollary 18 provides (m^2, m^2)-seed matrices, so which the first part follows from Theorem 13. Similarly, Theorem 16 gives (n^2, n)-seed matrices, which gives the second part. The remaining parts follow by using these two sets of seed matrices in Theorem 20, considering under what conditions the appropriate sequences are guaranteed to exist. □

Although we did not say exactly how large m and n must be in parts 3 and 4 of Theorem 29, these numbers are easy to produce from Theorem 20. For example, $q = m^2w + m^2n + 1$ may be obtained whenever $m^2w + m^2n + 1 \geq m^2 l_w$, that is when $n \geq l_w - w - \frac{1}{m^2}$. If we take $w = 169$, then $l_w = 196$, so as long as $n \geq 28$ (or 24 as long as $m \neq 1$), there exists an Hadamard matrix of order $4m^2n^2(169m^2 + m^2n + 1)$.

Theorem 30 *Suppose there are Hadamard matrices of orders $2mk$ and $2nk$ and a symmetric Hadamard matrix of order $k \geq l_w - w - 1$.*

1. *If $l_w \leq w + k + 1$ and there is a symmetric Hadamard matrix of order $2k$, then there is an Hadamard matrix of order $8mnk^2(w + 2k + 1)$.*

2. *If there is a symmetric Hadamard matrix of order k and $l_w \leq w + 2k$, then there is an Hadamard matrix of order $16mnk^2(w + 2k)$.*

3. *If there is a symmetric Hadamard matrix of order k and $l_w \leq w + k + 1$ then there is an Hadamard matrix of order $8mnk^2(w + k + 1)$.*

4. *If there is a symmetric Hadamard matrix of order k and $l_w \leq w + k$, then there is an Hadamard matrix of order $8mnk^2(w + k)$.*

Further, if there are Hadamard matrices of orders mk and nk, then there is an Hadamard matrix of order $4mnk^2(w + k + 1)$.

Proof: To construct the last matrix: Theorem 16 gives (mnk^2, k)-seed matrices $H, \{F_j\}_{j=1}^{k}$. Taking A, B, C, D_j as in Theorem 9, (13) is the required Hadamard matrix. The remaining matrices are obtained as follows. Part 1: By Theorem 16, $(2mnk^2, k)$-seed matrices H, F_i exist. Using A, B, C, D_j as given by Theorem 9, the desired Hadamard matrix is (13). Part 2: Similarly use the array of Theorem 10 with $h = 0$; Part 3: use $(2mnk^2, k)$-seed matrices and the array of Theorem 9; Part 4: use $(2mnk^2, k)$-seed matrices and the array of Theorem 10. □

Corollary 31 *Suppose that $p^2h^2 \geq l_w - w - 1$ and there exists an Hadamard matrix of order $2ph$, then:*

1. *If there are Hadamard matrices of order mp^2h and np^2h, then there are Hadamard matrices of orders $8mnp^4h^2(w + p^2h^2 + 1)$ and $8mnp^4h^2(w + p^2h^2)$.*

2. *If there are Hadamard matrices of order $2mp^2h$ and $2np^2h$, then there are Hadamard matrices of orders $16mnp^4h^2(w+p^2h^2+1)$ and $16mnp^4h^2(w+p^2h)$.*

Further, if there is an Hadamard matrix of order $2ph$, then:

3. *If there are Hadamard matrices of order mp^2h and np^2h, then there are Hadamard matrices of orders $4mnp^4h^2(w+p^2h^2+1)$ and $4mnp^4h^2(w+p^2h^2)$.*

4. *If there are Hadamard matrices of order $2mp^2h$ and $2np^2h$, then there are Hadamard matrices of orders $8mnp^4h^2(w+p^2h^2+1)$ and $8mnp^2h^2(w+p^2h^2)$.*

Proof (sketch): Part 3: Lemma 5 provides us with a symmetric (p^2h, h)-OS and a (mnp^2h, p^2h)-OS. Theorem 17 gives (mnp^4h^2, p^2h^2)-seed matrices, and so we have an Hadamard matrix of order $4mnp^4h^2(w+p^2h^2+1)$, using the array of Theorem 9 as before. Order $4mnp^4h^2(w+p^2h^2)$ is attained by omitting x and S as we suggested following Theorem 9. Parts 1,2 and 4 are done similarly, using different parts of Lemma 5. □

Notice that if we take $p=1$, $m=n$, w a Golay number and h a suitable power of 2 in part 1 of this result, we obtain Theorem 7 as a special case.

Theorem 32 *If there is an Hadamard matrix of order 2^sm, then there are Hadamard matrices of all orders $4^{s+1}m^2(5^t\cdot 4^sm^2+1)$ and $4^{s+1}m^2(13^t\cdot 4^sm^2+1)$, where $t\le 2s-1$. Further, if that matrix is symmetric, there are Hadamard matrices of all orders $4^{s+1}n^2(5^t\cdot 2^sn+1)$ and $4^{s+1}n^2(13^t\cdot 2^sn+1)$, where $t\le s-1$.*

Proof: From Corollary 18, we obtain $(4^sm^2, 4^sm^2)$-seed matrices (for the first part) and from Theorem 16, $(4^sn^2, 2^sn)$-seed matrices (for the second part). The result follows using part 2 of Theorem 13, using sequences given by Theorem 24. □

Theorem 20 has some rather exotic consequences. The following result makes use of Goldbach's conjecture [2].

Theorem 33 *If Goldbach's conjecture is valid, then for any odd number p there exists an Hadamard matrix of order $16ph^2k^2$, where h and k are even numbers such that $h+k=4p$ (and $\frac{h}{2}-1$, $\frac{k}{2}-1$ are prime).*

Proof: Assuming Goldbach's conjecture, any even number can be represented as a sum of two odd primes (counting 1 as prime). Let $2(p-1)=r+s$ be such a representation. Paley's construction (see [7]) provides symmetric Hadamard matrices of orders $h=2(r+1)$ and $k=2(s+1)$. Notice that $h+k=4p$, as required. Theorem 16 gives (h^2, h)-seed matrices and (k^2, k)-seed matrices, and part 1 of Theorem 20 (with $w=0$) gives the required Hadamard matrix. □

In all of the foregoing results, we can replace any occurrence of l_w and w with the length and weight of a known complementary pair.

References

[1] J. Cooper and J. S. Wallis. A construction for Hadamard arrays. *Bull. Austral. Math. Soc.*, 7:269–278, 1972.

[2] R. Courant and H. Robbins. *What is Mathematics?* Oxford University Press, New York, 1941.

[3] R. Craigen. Orthogonal sets and orthogonal designs. Technical Report CORR #90–19, University of Waterloo, 1990.

[4] R. Craigen. *Constructions for Orthogonal Matrices.* PhD thesis, University of Waterloo, March 1991.

[5] R. Craigen. Constructing Hadamard matrices with orthogonal pairs. *Ars Combin.*, 33:57–64, 1992.

[6] Shalom Eliahou, Michel Kervaire, and Benjamin Saffari. A new restriction on the lengths of Golay complementary sequences. *J. Combin. Theory Ser. A*, 55:49–59, 1990.

[7] A. Geramita and J. Seberry. *Quadratic Forms, Orthogonal Designs, and Hadamard Matrices*, volume 45 of *Lecture Notes in Pure and Applied Mathematics.* Marcel Dekker Inc., New York and Basel, 1979.

[8] H. Kharaghani. New class of weighing matrices. *Ars Combin.*, 19:69–72, 1985.

[9] H. Kharaghani. A new class of symmetric weighing matrices. *J. Austral. Math. Soc. Ser. A*, 41:44–46, 1986.

[10] H. Kharaghani. Construction of orthogonal designs. *Ars Combin.*, 24:149–151, 1987.

[11] H. Kharaghani. New classes of orthogonal designs. *Ars Combin.*, 29:187–192, 1990.

[12] H. Kharaghani. Block Golay sequences with applications. *Australas. J. Combin.*, 6:293–303, 1992.

[13] H. Kharaghani. A construction for Hadamard matrices. *Discrete Math.*, 120:115–120, 1993.

[14] H. Kharaghani. A new class of orthogonal designs. *Ars Combin.*, 35:129–134, 1993.

[15] J. Seberry and M. Yamada. Hadamard matrices, sequences, and block designs. In J. H. Dinitz and D. R. Stinson, editors, *Contemporary Design Theory: A Collection of Surveys*, pages 431–560. John Wiley & Sons, Inc., 1992.

[16] Ming-Yuan Xia. Some infinite classes of special Williamson matrices and difference sets. *J. Combin. Theory Ser. A*, 61(2):230–242, 1992.

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