The intersection problem for cubes

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Abstract

For all integers m, n and t, we determine necessary and sufficient conditions for the existence of (1) a pair of 3-cube decompositions of K_n having precisely t common 3-cubes; and (2) a pair of 3-cube decompositions of $K_{m,n}$ having precisely t common 3-cubes.

1 Introduction

A great deal of work has been done in recent years on the *intersection problem* for combinatorial designs. The question addressed in intersection problems is: given two designs based on the same underlying set of elements, how many blocks may they have in common? The intersection problem has been considered for many classes of designs, including Steiner triple systems (see [5]), m-cycle systems ([2]) and Steiner quadruple systems ([4]). For a fine survey on the intersection problem, the reader is directed to Billington [1], and the references therein.

In this paper we settle the intersection problem for 3-cube decompositions of the complete graph K_n and of the complete bipartite graph $K_{m,n}$. A 3-*cube*, henceforth simply a *cube*, is the graph C whose vertex set is $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and whose edge set consists of the edges of two 4-cycles (x_1, x_2, x_3, x_4) and (x_5, x_6, x_7, x_8) and the edges $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$ and $\{x_4, x_8\}$. We denote this graph by

*Research supported by Australian Research Council grant A49532750
†Research supported by Illinois State University Research Office

$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)_c$. Given a graph G and subgraph H, let $G \setminus H$ be the graph with vertex set $V(G \setminus H) = V(G)$ and edge set $E(G \setminus H) = E(G) \setminus E(H)$. The graph $K_v \setminus K_u$ is called the *complete graph on v vertices with a hole of size u*, with the vertices of K_u forming the *hole*.

Given graphs G_1 and G_2, with $V(G_1) \cap V(G_2) = \emptyset$, let $G_1 \vee G_2$ be the graph with vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{\{a, b\} : a \in V(G_1) \text{ and } b \in V(G_2)\}$. Given graphs G_1 and G_2, with $E(G_1) \cap E(G_2) = \emptyset$, let $G_1 + G_2$ be the graph with vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. We will use the notation $G_1 \vee G_2$ and $G_1 + G_2$ **only** when $V(G_1) \cap V(G_2) = \emptyset$ and $E(G_1) \cap E(G_2) = \emptyset$ respectively.

A C-decomposition of a graph G is a set D of cubes whose edge sets form a partition of the edge set of G. Hence a set D of edge-disjoint cubes forms a C-decomposition of G if and only if $\sum_{c \in D} C = G$. If there exists a C-decomposition of G, then we say C *divides* G and denote that by $C|G$.

For a graph G, let $I(G)$ denote the set of integers t for which there exist two C-decompositions of G with exactly t cubes in common. We define $J(G)$ to be the set of expected intersection numbers. That is, $J(G) = \{0, 1, 2, \ldots, b\} \setminus \{b - 1\}$, where $b = |E(G)|/12$ if there exists a C-decomposition of G, and $J(G) = \emptyset$ otherwise. Let $I(n), J(n), I(m, n)$ and $J(m, n)$ denote $I(K_n), J(K_n), I(K_{m,n})$ and $J(K_{m,n})$, respectively.

In [3], Bryant *et al* showed that there is a C-decomposition of K_n if and only if $n \equiv 1$ or $16 \pmod{24}$. Thus, $J(n) = \{0, 1, 2, \ldots, b\} \setminus \{b-1\}$, where $b = n(n-1)/24$ if $n \equiv 1$ or $16 \pmod{24}$, and $J(n) = \emptyset$ otherwise.

Also in [3], it was shown that for $m \leq n$, there is a C-decomposition of $K_{m,n}$ if and only if $m \equiv n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 4$ and $m \geq 4$. Thus, $J(m, n) = \{0, 1, 2, \ldots, b\} \setminus \{b - 1\}$, where $b = mn/12$ if $m \equiv n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 4$ and $m \geq 4$, and $J(m, n) = \emptyset$ otherwise.

Lemmas 1.1 and 1.2 follow immediately from the definitions of $J(n)$ and $J(m, n)$.

Lemma 1.1 For all m, n, $I(m, n) \subseteq J(m, n)$.

Lemma 1.2 For all n, $I(n) \subseteq J(n)$.

In Sections 2 and 3 we show that $J(m, n) \subseteq I(m, n)$ and $J(n) \subseteq I(n)$ respectively, thus obtaining the following two theorems:

Theorem 1.1 For all m and n, $I(m, n) = J(m, n)$.

Theorem 1.2 For all n, $I(n) = J(n)$.

We will make frequent use of the following straightforward lemmas. The proof of the first one is obvious.

Lemma 1.3 If $G = G_1 + G_2 + \ldots + G_n$ and there is a pair of C-decompositions of G_i with exactly t_i common cubes (for $i = 1, 2, \ldots, n$) then there is a pair of C-decompositions of G with $t_1 + t_2 + \ldots + t_n$ common cubes.

Lemma 1.4 Let G_1 and G_2 be edge-disjoint graphs such that $|E(G_1)| \geq 36$ and $|E(G_2)| \geq 24$. Suppose $C|G_1$ and $C|G_2$ with $I(G_1) = J(G_1)$ and $I(G_2) = J(G_2)$. Then $I(G_1 + G_2) = J(G_1 + G_2)$.

Proof: Let $n_1 = |E(G_1)|/12$ and let $n_2 = |E(G_2)|/12$. Without loss of generality, assume that $n_1 \geq n_2$. Let $t \in J(G_1 + G_2)$.

If $t \leq n_1 - 2$, then there exist two C-decompositions of G_1 with exactly t cubes in common and there exist two C-decompositions of G_2 with exactly zero cubes in common. Thus by Lemma 1.3, there exist two C-decompositions of $G_1 + G_2$ with exactly t cubes in common.

If $t = n_1 - 1$, then there exist two C-decompositions of G_1 with exactly $n_1 - 2$ ($n_1 - 3$ if $n_2 = 2$) cubes in common and there exist two C-decompositions of G_2 with exactly 1 (2 if $n_2 = 2$) cube in common. Thus, there exist two C-decompositions of $G_1 + G_2$ with exactly t cubes in common.

Else, if $t \geq n_1$, then there exist two C-decompositions of G_1 with exactly n_1 cubes in common and there exist two C-decompositions of G_2 with exactly $t - n_1$ cube in common. Thus, there exist two C-decompositions of $G_1 + G_2$ with exactly t cubes in common.

Therefore, $t \in I(G_1 + G_2)$ and thus, $I(G_1 + G_2) = J(G_1 + G_2)$. □

2 The complete bipartite graph

In this section, we prove that $I(m, n) = J(m, n)$ for all positive integers m and n. We first show that $I(6, 6) = J(6, 6)$ and that $I(9, 12) = J(9, 12)$.

We will present decompositions of $K_{m,n}$ where the vertex set of $K_{m,n}$ is $(\mathbf{Z}_m \times \{0\}) \bigcup (\mathbf{Z}_n \times \{1\})$ (with the obvious bipartition) and the ordered pair (x, y) of this vertex set is represented by x_y.

Lemma 2.1 $I(6, 6) = J(6, 6)$.

Proof: Let the vertex set of $K_{6,6}$ be $\{0_0, \ldots, 5_0\} \cup \{0_1, \ldots, 5_1\}$, with the obvious vertex partition. Let D, D_0 and D_1 denote the following designs, respectively.

$$\begin{aligned} D = \;& \{(0_0, 0_1, 2_0, 3_1, 2_1, 3_0, 1_1, 1_0)_c, \quad (2_0, 2_1, 4_0, 5_1, 4_1, 5_0, 3_1, 3_0)_c, \\ & (0_0, 1_1, 5_0, 5_1, 4_1, 4_0, 0_1, 1_0)_c\}, \end{aligned}$$

$$\begin{aligned} D_0 = \;& \{(0_0, 0_1, 2_0, 2_1, 1_1, 1_0, 3_1, 3_0)_c, \quad (0_0, 3_1, 5_0, 5_1, 4_1, 4_0, 2_1, 1_0)_c, \\ & (2_0, 1_1, 4_0, 5_1, 4_1, 5_0, 0_1, 3_0)_c\}, \end{aligned}$$

and

$$\begin{aligned} D_1 = \;& \{(0_0, 0_1, 2_0, 3_1, 2_1, 3_0, 1_1, 1_0)_c, \quad (0_0, 1_1, 4_0, 5_1, 4_1, 5_0, 2_1, 2_0)_c, \\ & (1_0, 0_1, 5_0, 5_1, 4_1, 4_0, 3_1, 3_0)_c\}. \end{aligned}$$

Clearly, each of D, D_0 and D_1 forms a C-decomposition of $K_{6,6}$. Note that $|D_0 \cap D| = 0$, $|D_1 \cap D| = 1$ and $|D \cap D| = 3$. □

Lemma 2.2 $I(9, 12) = J(9, 12)$.

Proof: Let the vertex set of $K_{9,12}$ be $\{0_0, \ldots, 8_0\} \cup \{0_1, \ldots, 11_1\}$, with the obvious vertex partition. Let D, D_0, D_1, D_2 and D_3 denote the following designs, respectively.

$$\begin{array}{rll} D = & \{(0_0, 0_1, 2_0, 2_1, 1_1, 1_0, 3_1, 3_0)_c, & (0_0, 4_1, 2_0, 6_1, 5_1, 1_0, 7_1, 3_0)_c, \\ & (0_0, 8_1, 2_0, 10_1, 9_1, 1_0, 11_1, 3_0)_c, & (0_0, 3_1, 5_0, 11_1, 7_1, 4_0, 0_1, 6_0)_c, \\ & (1_0, 2_1, 5_0, 10_1, 6_1, 4_0, 1_1, 6_0)_c, & (2_0, 1_1, 8_0, 9_1, 5_1, 7_0, 2_1, 6_0)_c, \\ & (3_0, 0_1, 8_0, 8_1, 4_1, 7_0, 3_1, 6_0)_c, & (4_0, 4_1, 8_0, 10_1, 8_1, 5_0, 6_1, 7_0)_c, \\ & (4_0, 5_1, 8_0, 11_1, 9_1, 5_0, 7_1, 7_0)_c\}, & \end{array}$$

$$\begin{array}{rll} D_0 = & \{(0_0, 1_1, 1_0, 4_1, 2_1, 3_0, 3_1, 2_0)_c, & (0_0, 5_1, 1_0, 8_1, 6_1, 3_0, 7_1, 2_0)_c, \\ & (0_0, 0_1, 2_0, 10_1, 9_1, 1_0, 11_1, 3_0)_c, & (0_0, 3_1, 5_0, 11_1, 7_1, 4_0, 1_1, 6_0)_c, \\ & (1_0, 2_1, 5_0, 10_1, 6_1, 4_0, 0_1, 6_0)_c, & (2_0, 1_1, 8_0, 9_1, 5_1, 7_0, 3_1, 6_0)_c, \\ & (3_0, 0_1, 8_0, 8_1, 4_1, 7_0, 2_1, 6_0)_c, & (4_0, 5_1, 8_0, 10_1, 8_1, 5_0, 6_1, 7_0)_c, \\ & (4_0, 4_1, 8_0, 11_1, 9_1, 5_0, 7_1, 7_0)_c\}, & \end{array}$$

$$\begin{array}{rll} D_1 = & \{(0_0, 1_1, 1_0, 4_1, 2_1, 3_0, 3_1, 2_0)_c, & (0_0, 0_1, 2_0, 6_1, 5_1, 1_0, 7_1, 3_0)_c, \\ & (0_0, 8_1, 2_0, 10_1, 9_1, 1_0, 11_1, 3_0)_c, & (0_0, 3_1, 5_0, 11_1, 7_1, 4_0, 1_1, 6_0)_c, \\ & (1_0, 2_1, 5_0, 10_1, 6_1, 4_0, 0_1, 6_0)_c, & (2_0, 1_1, 8_0, 9_1, 5_1, 7_0, 3_1, 6_0)_c, \\ & (3_0, 0_1, 8_0, 8_1, 4_1, 7_0, 2_1, 6_0)_c, & (4_0, 5_1, 8_0, 10_1, 8_1, 5_0, 6_1, 7_0)_c, \\ & (4_0, 4_1, 8_0, 11_1, 9_1, 5_0, 7_1, 7_0)_c\}, & \end{array}$$

$$\begin{array}{rll} D_2 = & \{(0_0, 1_1, 1_0, 4_1, 2_1, 3_0, 3_1, 2_0)_c, & (0_0, 5_1, 1_0, 8_1, 6_1, 3_0, 7_1, 2_0)_c, \\ & (0_0, 0_1, 2_0, 10_1, 9_1, 2_0, 11_1, 3_0)_c, & (0_0, 3_1, 5_0, 11_1, 7_1, 4_0, 0_1, 6_0)_c, \\ & (1_0, 2_1, 5_0, 10_1, 6_1, 3_0, 1_1, 6_0)_c, & (2_0, 1_1, 8_0, 9_1, 5_1, 7_0, 3_1, 6_0)_c, \\ & (3_0, 0_1, 8_0, 8_1, 4_1, 7_0, 2_1, 6_0)_c, & (4_0, 5_1, 8_0, 10_1, 8_1, 5_0, 6_1, 7_0)_c, \\ & (4_0, 4_1, 8_0, 11_1, 9_1, 5_0, 7_1, 7_0)_c\}, & \end{array}$$

and

$$\begin{array}{rll} D_3 = & \{(0_0, 0_1, 2_0, 2_1, 1_1, 1_0, 3_1, 3_0)_c, & (0_0, 4_1, 2_0, 6_1, 5_1, 1_0, 7_1, 3_0)_c, \\ & (0_0, 8_1, 2_0, 10_1, 9_1, 1_0, 11_1, 3_0)_c, & (0_0, 3_1, 5_0, 11_1, 7_1, 4_0, 1_1, 6_0)_c, \\ & (1_0, 2_1, 5_0, 10_1, 6_1, 4_0, 0_1, 6_0)_c, & (2_0, 1_1, 8_0, 9_1, 5_1, 7_0, 3_1, 6_0)_c, \\ & (3_0, 0_1, 8_0, 8_1, 4_1, 7_0, 2_1, 6_0)_c, & (4_0, 5_1, 8_0, 10_1, 8_1, 5_0, 6_1, 7_0)_c, \\ & (4_0, 4_1, 8_0, 11_1, 9_1, 5_0, 7_1, 7_0)_c\}, & \end{array}$$

Clearly, each of D, D_0, D_1, D_2 and D_1 forms a C-decomposition of $K_{9,12}$. Note that $|D_0 \cap D| = 0$, $|D_1 \cap D| = 1$, $|D_1 \cap D_0| = 7$, $|D_2 \cap D| = 2$, $|D_2 \cap D_1| = 5$, $|D_3 \cap D| = 3$, $|D_3 \cap D_0| = 6$, $|D_3 \cap D_2| = 4$ and $|D \cap D| = 9$. □

Theorem 2.3 $I(m,n) = J(m,n)$ for all positive integers m, n.

Proof: Let m and n be positive integers such that $m \leq n$. In [3], it was shown that there is a C-decomposition of $K_{m,n}$ if and only if $m \equiv n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 4$ and $m \geq 4$. Under these conditions, either

(C1) $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 6$,

(C2) $m \equiv 3 \pmod 6$ and $n \equiv 0 \pmod{12}$, or

(C3) $m \equiv 0 \pmod{12}$ and $n \equiv 3 \pmod 6$.

If (C1) is satisfied, then $K_{m,n}$ can clearly be decomposed into isomorphic copies of $K_{6,6}$. Since $K_{m,n}$ and $K_{n,m}$ are isomorphic, (C2) and (C3) are equivalent. In either case, $K_{m,n}$ can be decomposed into a collection of graphs, each of which is isomorphic to either $K_{6,6}$ or $K_{9,12}$. Thus by Lemma 1.4, it suffices to show that $I(6,6) = J(6,6)$ and $I(9,12) = J(9,12)$. □

3 The complete graph

3.1 Small cases

In this section we show that $I(n) = J(n)$ for $n = 16$ and 25.

Lemma 3.1 $\{0,1,2,3,4,5\} \subseteq I(16)$.

Proof: Let the vertex set of K_{16} be $\mathbb{Z}_{16}$. Then a C-decomposition of K_{16} is given by

$$\begin{aligned} D = \; & \{(1,2,5,6,8,3,4,7)_c, && (9,10,13,14,0,11,12,15)_c, \\ & (1,3,5,7,4,6,8,2)_c, && (9,11,13,15,12,14,0,10)_c, \\ & (1,9,2,10,5,13,6,14)_c, && (3,11,4,12,7,15,8,0)_c, \\ & (1,11,2,12,15,6,0,5)_c, && (1,13,2,14,0,3,15,4)_c, \\ & (5,9,6,10,11,8,12,7)_c, && (7,14,8,13,9,3,10,4)_c\}. \end{aligned}$$

Let D_0, D_1, D_2, D_3, D_4 and D_5 be the designs obtained by applying respectively each of the permutations (0 1 2 7), (0 1 2 3), (0 1), (0 13), (0 11) and (0 15) to D. Then it is straightforward to check that for $i \in \{0,1,2,3,4,5\}$, $|D \cap D_i| = i$. □

Lemma 3.2 $\{6,8\} \subseteq I(16)$.

Proof: Let

$$S_1 = \{(1,2,5,6,8,3,4,7)_c, (1,3,5,7,4,6,8,2)_c\},$$

and

$$S_2 = \{(1,4,5,8,2,3,6,7)_c, (1,3,5,7,6,8,2,4)_c\},$$

so that S_1 and S_2 are a pair of C-decompositions of $K_8 \setminus F$ (where F is a 1-factor) with zero common cubes. Also, let

$$\begin{aligned} R = \; & \{(0,1,8,9,10,11,2,3)_c, && (0,8,3,11,12,4,14,6)_c, \\ & (0,13,1,14,15,3,12,2)_c, && (1,9,2,10,15,5,13,7)_c, \\ & (4,5,12,13,15,14,7,6)_c, && (4,9,6,10,11,7,8,5)_c\}, \end{aligned}$$

so that R is a C-decomposition of $F \vee F$, where F is a 1-factor on eight vertices. Hence, by Lemma 1.3 (since $K_{16} = (K_8 \setminus F) + (F \vee F) + (K_8 \setminus F)$) there is a pair of C-decompositions of K_{16} with six common cubes and a pair of C-decompositions of K_{16} with eight common cubes. □

Lemma 3.3 $\{7\} \subseteq I(16)$.

Proof: Let the vertex set of K_{16} be $\mathbb{Z}_{16}$. Then two C-decompositions of K_{16} are given by

$$\begin{array}{rll} D_1 = & \{(2,4,8,10,1,5,7,11)_c, & (3,5,9,11,2,6,8,0)_c, \\ & (4,6,10,0,3,7,9,1)_c, & (0,3,6,9,12,8,1,4)_c, \\ & (0,5,2,7,13,10,12,1)_c, & (0,6,11,14,15,12,5,13)_c, \\ & (1,10,4,14,15,7,13,3)_c, & (2,8,11,13,14,5,15,6)_c, \\ & (2,9,12,11,15,14,7,4)_c, & (3,9,15,10,12,13,8,14)_c\}, \end{array}$$

and

$$\begin{array}{rll} D_2 = & \{(0,4,6,8,1,5,7,9)_c, & (2,4,8,10,1,3,7,11)_c, \\ & (11,3,5,9,0,2,6,10)_c, & (0,3,6,9,12,8,1,4)_c, \\ & (0,5,2,7,13,10,12,1)_c, & (0,6,11,14,15,12,5,13)_c, \\ & (1,10,4,14,15,7,13,3)_c, & (2,8,11,13,14,5,15,6)_c, \\ & (2,9,12,11,15,14,7,4)_c, & (3,9,15,10,12,13,8,14)_c\}. \end{array}$$

Clearly, $|D_1 \cap D_2| = 7$. □

The following lemma follows immediately from Lemmas 3.1, 3.2 and 3.3:

Lemma 3.4 $I(16) = J(16)$.

Before proving that $I(25) = J(25)$, we need one more result.

Lemma 3.5 There is a pair of C-decompositions of $K_{13} \setminus K_4$ having precisely zero common cubes, and a pair with precisely six common cubes.

Proof: Let the vertex set of $K_{13} \setminus K_4$ be $\{0,1,2,3\} \cup \{4,5,\ldots,12\}$, with the vertices $0,1,2,3$ in the hole. Then two C-decompositions of $K_{13} \setminus K_4$ are given by

$$\begin{array}{rll} D_1 = & \{(0,4,1,5,6,2,7,3)_c, & (0,7,4,8,9,5,6,10)_c, \\ & (0,10,2,11,12,5,8,6)_c, & (1,6,7,11,12,9,8,3)_c, \\ & (1,8,11,9,10,12,4,3)_c, & (2,5,4,9,12,11,10,7)_c\}, \end{array}$$

and

$$\begin{array}{rll} D_2 = & \{(1,4,2,5,6,0,7,3)_c, & (1,7,4,8,9,5,6,10)_c, \\ & (1,10,0,11,12,5,8,6)_c, & (2,6,7,11,12,9,8,3)_c, \\ & (2,8,11,9,10,12,4,3)_c, & (0,5,4,9,12,11,10,7)_c\}, \end{array}$$

Clearly, $|D_1 \cap D_2| = 0$, and $|D_1 \cap D_1| = 6$. □

Lemma 3.6 $I(25) = J(25)$

Proof: Let V_1, V_2 and V_3 be three mutually disjoint vertex sets of sizes 12, 9 and 4 respectively. Let $G_1 \cong K_{16}$ have vertex set $V_1 \cup V_3$, let $G_2 \cong K_{9,12}$ have vertex set $V_1 \cup V_2$ (and the obvious bipartition), and let $G_3 \cong K_{13} \setminus K_4$ have vertex set $V_2 \cup V_3$ (with the vertices of V_3 in the hole). Then $K_{25} = G_1 + G_2 + G_3$.

Now using Lemma 1.3 with $t_1 \in I(16)$ (see Lemma 3.4), $t_2 \in \{0,9\}$ (see Lemma 2.2) and $t_3 \in \{0,6\}$ (see Lemma 3.5) it is straightforward to check that we have $I(25) = J(25)$. □

3.2 Constructions

In this section we give two similar constructions for the cases $n \equiv 1$ or $16 \pmod{24}$.

Lemma 3.7 If $n \equiv 1 \pmod{24}$ then $I(n) = J(n)$.

Proof: Let $n = 24r + 1$ and let $V_1, V_2, \ldots, V_r$ be r mutually disjoint vertex sets of size 24 and let $\infty \notin \bigcup_{i=1}^{r} V_i$. For each i, j with $1 \le i < j \le r$, let $G_{i,j} \cong K_{24,24}$ have vertex set $V_i \cup V_j$ (and the obvious vertex partition), and for each $i = 1, 2, \ldots, r$ let $G_i \cong K_{25}$ have vertex set $V_i \cup \{\infty\}$.

Then

$$K_n = \sum_{1 \le i < j \le r} G_{i,j} + \sum_{1 \le i \le r} G_i.$$

Since $I(24, 24) = J(24, 24)$ and $I(25) = J(25)$, we conclude by Lemma 1.4 that $I(n) = J(n)$. □

In the following lemma we use a similar construction to that described in [3].

Lemma 3.8 If $n \equiv 16 \pmod{24}$ then $I(n) = J(n)$.

Proof: Let $n = 24r + 16$, $A = \{x_1, x_2, \ldots, x_{16}\}$ and $V_1, V_2, \ldots, V_r$ be r mutually disjoint vertex sets of size 24 such that $V_i \cap A = \emptyset$ for all i. For each i, j with $1 \le i < j \le r$, let $G_{i,j} \cong K_{24,24}$ have vertex set $V_i \cup V_j$ (and the obvious vertex partition), for each $i = 1, 2, \ldots, r$ let $G_i \cong K_{25}$ have vertex set $V_i \cup \{x_1\}$, and let $G_i' \cong K_{15,24}$ have vertex set $V_i \cup \{x_2, x_3, \ldots, x_{16}\}$ (and the obvious vertex partition). Finally, let $G \cong K_{16}$ have vertex set A. Then

$$K_n = G + \sum_{1 \le i < j \le r} G_{i,j} + \sum_{1 \le i \le r} (G_i + G_i').$$

Since $I(16) = J(16)$, $I(24, 24) = J(24, 24)$, $I(25) = J(25)$ and $I(15, 24) = J(15, 24)$, we conclude by Lemma 1.4 that $I(n) = J(n)$. □

References

[1] E.J. Billington, *The intersection problem for combinatorial designs*, Congressus Numerantium, **92** (1993), 33–54.

[2] E.J. Billington, *The intersection problem for m-cycle systems*, Journal of Combinatorial Designs, **1**(6) (1993), 435–452.

[3] D.E. Bryant, S.I. El-Zanati and R.B. Gardner, *Decompositions of $K_{m,n}$ and K_n into cubes*, Australasian Journal of Combinatorics, **9** (1994), 285–290.

[4] A. Hartman and Z. Yehudai, *Intersections of Steiner quadruple systems*, Discrete Mathematics, **104** (1992), 227–244.

[5] C.C. Lindner and A. Rosa, *Steiner triple systems having a prescribed number of triples in common*, Canadian Journal of Mathematics, **27** (1975), 1166–1175. Corrigendum: Canadian Journal of Mathematics, **30** (1978), 896.

(Received 26/4/96)