Remarks on regular factors in vertex-deleted subgraphs of regular graphs

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Abstract

Let G be a 2r-regular, 2r-edge-connected graph of odd order and let m be an integer such that $1 \leq m \leq r-1$. For any vertex u of G, the graph $G-\{u\}$ has an m-factor which contains none of $r-m$ given edges.

All graphs considered are finite. We shall allow graphs to contain multiple edges and we refer the reader to [2] for graph theoretic terms not defined in this paper.

Let G be a graph. We denote by $\kappa(G)$ the connectivity of G, which is defined to be the minimum number of vertices whose removal disconnects G or reduces it to K_1. If G is disconnected $\kappa(G)=0$ and if $G \cong K_n$, $\kappa(G) = n-1$, since we must remove $n-1$ vertices to reduce G to K_1. G is said to be k-connected if $\kappa(G) \geq k$.

We denote by $\lambda(G)$ the edge connectivity of G, which is defined to be the minimum number of edges whose removal disconnects G. Clearly if G is disconnected $\lambda(G)=0$ and if $G \cong K_n$, $\lambda(K_n) = n-1$. G is said to be k-edge-connected if $\lambda(G) \geq k$.

If $S, T \subseteq V(G)$ then $e_G(S,T)$ denotes the number and $E_G(S,T)$ the set of edges having one end-vertex in S and the other in T. In the case when S is the vertex-set of a subgraph H of G, sometimes we write $e_G(H,T)$ and $E_G(H,T)$ instead of $e_G(V(H),T)$ and $E_G(V(H),T)$. The number of components of G is denoted by $\omega(G)$.

The minimum degree of the vertices of G is denoted by $\delta(G)$. If $u \in V(G)$, $N_G(u)$ denotes the set of vertices adjacent to u.

A k-factor is a k-regular spanning subgraph of G. Thus a Hamilton cycle of a graph is a connected 2-factor and a perfect matching of a graph is a 1-factor.

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The next theorem, due to Petersen, is chronologically the first result on the existence of 1-factors.

Theorem 1: (Petersen [6]). Every 3-regular, 2-edge-connected graph has a 1-factor.

In 1934, Schönberger proved the following more general result.

Theorem 2: (Schönberger [9]). Let G be a 3-regular, 2-edge-connected graph. Then there exists a 1-factor of G that contains no two given edges of G.

We now mention two theorems which are generalizations of Schönberger's result.

Theorem 3: (Plesnik [7]). Every k-regular, $(k-1)$-edge-connected graph of even order has a 1-factor that contains no $k-1$ given edges.

Theorem 4: (Katerinis [4]). Let G be a k-regular, $(k-1)$-edge connected graph of even order and let m be an integer such that $1 \leq m \leq k-1$. Then there exists an m-factor of G that contains no $k-m$ given edges of G.

The next theorem examines the existence of a 1-factor in vertex-deleted subgraphs of a regular graph.

Theorem 5:((1): Grant et al. [3], (2): Plesnik [8])

Let G be a 2r-edge-connected, 2r-regular graph of odd order and let u be any vertex of G. Then

(1) $G-\{u\}$ has a 1-factor

(2) $G-\{u\}$ has a 1-factor which contains none of $r-1$ given edges.

A few years ago, we generalized Theorem 5(1) in the following way:

Theorem 6:(Katerinis [5]). Let G be a 2r-regular, 2r-edge-connected graph of odd order and m be an integer such that $1 \leq m \leq r$. Then for every $u \in V(G)$, the graph $G-\{u\}$ has an m-factor.

The purpose of this paper is to prove the following result which is a generalization of Theorem 5(2).

Theorem 7: Let G be a 2r-regular, 2r-edge-connected graph of odd order and let m be an integer such that $1 \leq m \leq r-1$. For any vertex u of G, the graph $G - \{u\}$ has an m-factor which contains none of $r - m$ given edges.

For the proof of Theorem 7, we will use the k-factor Theorem of Belck and Tutte and a combination of ideas used in [7] and [5].

The k-factor Theorem : (Belck [1], Tutte [10]). A graph G has a k-factor if and only if

$$q_G(D, S; k) + \sum_{x \in S}(k - d_{G-D}(x)) \leq k \mid D \mid$$

for all $D, S \subseteq V(G)$, $D \cap S = \emptyset$ where $q_G(D, S; k)$ denotes the number of components C of $((G - D) - S)$ such that $e_G(V(C), S) + k \mid V(C) \mid$ is odd. (Sometimes C is called an odd component).

They also noted that for any graph G and any positive integer k

$$q_G(D, S; k) + \sum_{x \in S}(k - d_{G-D}(x)) - k \mid D \mid \equiv k \mid V(G) \mid (mod 2). \qquad (1)$$

Proof of Theorem 7 :

Suppose that there exist a vertex u, an integer m, where $1 \leq m \leq r-1$, and a set X of edges of G with $r - m$ elements, such that the graph $G_1 = (G - \{u\}) - X$ does not have an m-factor. Then by the k-factor Theorem and (1), there exist $D, S \subseteq V(G_1)$, $D \cap S = \emptyset$ such that

$$q_{G_1}(D, S; m) + \sum_{x \in S}(m - d_{G_1-D}(x)) \geq m \mid D \mid +2 . \qquad (2)$$

Let $S' = S \cup \{u\}$ and $W = (G_1 - D) - S$.

We consider the following two cases:

Case 1: $\omega(W) \geq 1$

The graph G is 2r-regular, so

$$2r \mid D \mid \geq e_G(D, G - D) = e_G(D, W) + e_G(D, S'). \qquad (3)$$

We also have

$$e_G(D, W) = e_G(D \cup S', W) - e_G(S', W). \qquad (4)$$

Now let $W_1, W_2, \ldots, W_k$ be the components of W.

Define

$$\left.\begin{array}{l} \mid E_G(W_i, D \cup S') \mid = l_i' \\ \mid E_G(W_i, W - V(W_i)) \mid = l_i'' \end{array}\right\} \text{where } i = 1, 2, \ldots, k$$

and X_0 to be the set of edges having end-vertices in different components of W.Clearly $X_0 \subseteq X$ and $\sum_{i=1}^{k} l_i'' = 2 \mid X_0 \mid$ since every element of X_0 has been counted twice. Then since G is 2r-edge-connected we have

$$\sum_{i=1}^{k} l_i' + \sum_{i=1}^{k} l_i'' \geq 2r\omega(W), \text{ and so}$$

$$2r\omega(W) - \sum_{i=1}^{k} l_i'' \leq \sum_{i=1}^{k} l_i' = e_G(D \cup S', W). \text{ Thus}$$

$e_G(D \cup S', W) \geq 2r\omega(W) - 2 \mid X_0 \mid$ and hence (4) implies

$$e_G(D, W) \geq 2r\omega(W) - 2 \mid X_0 \mid - e_G(S', W). \tag{5}$$

But $e_G(D, S') = \sum_{x \in S'} d_G(x) - \sum_{x \in S'} d_{G-D}(x)$ and

since G is 2r-regular

$$e_G(D, S') = 2r \mid S' \mid - \sum_{x \in S'} d_{G-D}(x). \tag{6}$$

Substituting (6) and (5) in (3), we have

$$2r \mid D \mid \geq 2r\omega(W) - 2 \mid X_0 \mid - e_G(S', W) + 2r \mid S' \mid - \sum_{x \in S'} d_{G-D}(x). \tag{7}$$

Therefore since $\sum_{x \in S'} d_{G-D}(x) = 2e_G(S', S') + e_G(S', W)$, (7) implies

$$2r \mid D \mid \geq 2r\omega(W) - 2 \mid X_0 \mid + 2r \mid S' \mid - 2\sum_{x \in S'} d_{G-D}(x) + 2e_G(S', S'). \tag{8}$$

Now define $X_1 = E_G(S, S) \cap X$, $X_2 = E_G(S, W) \cap X$, $\mid X_1 \mid = x_1$ and $\mid X_2 \mid = x_2$. Then we have $\sum_{x \in S'} d_{G-D}(x) = \sum_{x \in S} d_{G-D}(x) + d_{G-D}(u) = \sum_{x \in S} d_{G_1-D}(x) + e_G(u, S) + 2x_1 + x_2 + d_{G-D}(u)$.

Hence (8) becomes $2r \mid D \mid \geq 2r\omega(W) - 2 \mid X_0 \mid + 2r \mid S' \mid - 2\sum_{x \in S} d_{G_1-D}(x) - 2e_G(u, S) - 4x_1 - 2x_2 - 2d_{G-D}(u) + 2e_G(S', S')$. But $d_{G-D}(u) \leq 2r, e_G(S', S') \geq e_G(u, S) + \mid E_G(S, S) \cap X \mid = e_G(u, S) + x_1$ and $x_1 + \mid X_0 \mid + x_2 \leq \mid X \mid = r - m$.

Therefore

$$2r \mid D \mid \geq 2r\omega(W) - 2(r-m) + 2r \mid S \mid + 2r - 2\sum_{x\in S} d_{G_1-D}(x) - (2)(2r)$$
$$\geq 2r\omega(W) - 4r + 2m + 2r \mid S \mid - 2\sum_{x\in S} d_{G_1-D}(x).$$

Thus

$$2r \mid D \mid \geq (2r-2)\omega(W) + 2\omega(W) - 2\sum_{x\in S} d_{G_1-D}(x) - 4r + 2m + 2r \mid S \mid. \quad (9)$$

Hence using (2) and the fact that $\omega(W) \geq q_{G_1}(D,S;m)$, (9) implies

$$2r \mid D \mid \geq 2m \mid D \mid + 4 - 2m \mid S \mid + 2m - 4r + (2r-2)\omega(W) + 2r \mid S \mid.$$

Therefore

$$\begin{aligned}(2r-2m) \mid D \mid &\geq (2r-2m) \mid S \mid + 4 + 2m - 4r + 2r - 2 \\ &\geq (2r-2m) \mid S \mid + 2 - 2r + 2m. \end{aligned} \quad (10)$$

Now suppose that $\mid S \mid \geq \mid D \mid + 1$. Then we have from (10)

$$(2r-2m) \mid D \mid \geq (2r-2m)(\mid D \mid + 1) + 2 - 2r + 2m.$$

Thus $0 \geq 2$, which is a contradiction. Therefore we may assume that

$$\mid D \mid \geq \mid S \mid. \quad (11)$$

At this point we consider the following two subcases.

Case 1a: m is even.
For every odd component C of W the integer $m \mid V(C) \mid + e_{G_1}(V(C), S)$ is odd and since m is even the integer $e_{G_1}(V(C), S)$ must be odd. Thus $e_{G_1}(V(C), S) \geq 1$ and so $\sum_{x\in S} d_{G_1-D}(x) \geq q_{G_1}(D,S;m)$. Hence (2) implies $\mid S \mid \geq \mid D \mid + 1$. But this contradicts (11). Thus Case 1a cannot occur.

Case 1b: m is odd.
By Theorem 5(2), G_1 has a 1-factor so using the k-factor Theorem

$$q_{G_1}(D,S;1) + \sum_{x\in S}(1 - d_{G_1-D}(x)) \leq \mid D \mid.$$

But since m is odd $q_{G_1}(D,S;1) = q_{G_1}(D,S;m)$, so

$$q_{G_1}(D,S;m) + \sum_{x\in S}(1 - d_{G_1-D}(x)) \leq \mid D \mid. \quad (12)$$

Substituting (12) in (2) we have $\mid S \mid \geq \mid D \mid +1$. Again this result contradicts (11). Thus Case 1b cannot occur, either.

Case 2: $\omega(W) = 0$.
Using the fact that $\omega(W) \geq q_{G_1}(D, S; m)$, (2) implies

$$\sum_{x \in S} (m - d_{G_1 - D}(x)) \geq m \mid D \mid +2. \tag{13}$$

Hence $m \mid S \mid \geq m \mid D \mid +2$ and thus $\mid S \mid > \mid D \mid$. But $V(G_1) = D \cup S$ and $\mid V(G_1) \mid$ is even, so that

$$\mid S \mid \geq \mid D \mid +2. \tag{14}$$

Now since G is $2r$-regular, by the k-factor theorem we have $\sum_{x \in S'} (2r - d_{G-D}(x)) \leq 2r \mid D \mid$, which implies
$2r(\mid S \mid +1) - \sum_{x \in S'} d_{G-D}(x) \leq 2r \mid D \mid$ and so

$$(2r - m) \mid S \mid +m \mid S \mid +2r - \sum_{x \in S'} d_{G-D}(x) \leq m \mid D \mid +(2r - m) \mid D \mid. \tag{15}$$

We also have $\sum_{x \in S'} d_{G-D}(x) = \sum_{x \in S} d_{G-D}(x) + d_{G-D}(u)$ and $\sum_{x \in S} d_{G-D}(x) =$
$\sum_{x \in S} d_{G_1 - D}(x) + e_G(u, S) + 2 \mid E_G(S, S) \cap X \mid$

$$\leq \sum_{x \in S} d_{G_1 - D}(x) + e_G(u, S) + 2(r - m).$$

Thus (15) becomes

$$(2r - m) \mid S \mid +m \mid S \mid +2r - \sum_{x \in S} d_{G_1 - D}(x) - e_G(u, S) - 2(r - m) - d_{G-D}(u) \leq m \mid D \mid +(2r - m) \mid D \mid.$$

But $d_{G-D}(u) = e_G(u, S) \leq 2r$. Hence we have

$$(2r - m) \mid S \mid +m \mid S \mid - \sum_{x \in S} d_{G_1 - D}(x) - e_G(u, S) - 2(r - m) \leq m \mid D \mid +(2r - m) \mid D \mid.$$

Therefore using (13), we get
$(2r - m) \mid S \mid +2 - e_G(u, S) - 2(r - m) \leq (2r - m) \mid D \mid$. Therefore (14) implies $(2r - m)(\mid D \mid +2) + 2 - e_G(u, S) - 2(r - m) \leq (2r - m) \mid D \mid$ and thus since $e_G(u, S) \leq 2r$, $2 \leq 0$ which is of course a contradiction.

This completes the proof of Theorem 7.

The conditions of Theorem 7 imposed on G and m are necessary and best possible in the sense that if one of them is dropped then the remaining conditions are no longer sufficient for the existence of an m-factor containing none of $r-m$ given edges. In fact, as was shown in [5], if one of these conditions is dropped then the remaining conditions are not even sufficient for the conclusion of Theorem 6.

Finally we will show that we can not increase the cardinality of the set of edges that are not contained in the m-factor. For this purpose, we will describe a graph G of odd order which is 2r-regular, 2r-edge-connected having a vertex u and a set X of $r-m+1$ edges, where $m \leq r-1$, such that the deletion of u and X from G results in a graph which does not possess an m-factor.

For the construction of G we work as follows. We start from a complete bipartite graph H_0 with bipartition (U_0, V_0) where $U_0 = \{u_1, u_2, \ldots, u_{2r}\}$ and $V_0 = \{v_1, v_2, \ldots, v_{2r}\}$. We delete from H_0 the edges of the cycle $v_1 u_1 v_2 u_2 \ldots v_r u_r v_1$ and the independent edges $u_{r+1}v_{r+1}, u_{r+2}v_{r+2}, \ldots, u_{2r}v_{2r}$. We consider two new vertices v_0, u_0 which are joined to all the vertices of U_0 and V_0 respectively. Let us call H_1 the graph obtained after the completion of the above procedure. Clearly H_1 is a bipartite graph with bipartition (X_1, Y_1) where $X_1 = U_0 \cup \{u_0\}$ and $Y_1 = V_0 \cup \{v_0\}$. The vertices $u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_r$ have degree $2r-1$ in H_1 and the vertices $u_0, u_{r+1}, u_{r+2}, \ldots, u_{2r}, v_0, v_{r+1}, v_{r+2}, \ldots, v_{2r}$ have degree $2r$.

We also consider a simple graph T having the following properties: (i) $|V(T)|$ is odd, (ii) T has $2r$ vertices of degree $2r-1$ and all the other vertices are of degree $2r$, (iii) $\lambda(T) = 2r-1$. Let $\{w_1, w_2, \ldots, w_{2r}\}$ be the set of vertices of T having degree $2r-1$.

Now we take the graphs H_1 and T, and we add the independent edges $w_1u_1, w_2u_2, \ldots, w_ru_r, w_{r+1}v_1, w_{r+2}v_2, \ldots, w_{2r}v_r$. The resulting graph G is 2r-regular, of odd order and we will also show that G is $2r$-edge-connected.

In order to prove that $\lambda(G) = 2r$, we shall need the following Lemmas

Lemma 1 [11]: For any graph G

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Lemma 2: $\lambda(H_1) = 2r-1$.

Proof: Let S be an edge-cutset of H_1 such that $|S| = \lambda(H_1)$ and suppose that $\lambda(H_1) = |S| \leq 2r-2$. Let us also call A, B the two components of $H_1 - S$ and assume that A is the smaller one.

Clearly

$$|V(A)| \leq \frac{|V(H_1)|}{2} = 2r+1. \tag{16}$$

Define

$$A_x = V(A) \cap X_1$$
$$A_y = V(A) \cap Y_1.$$

We will first prove that $A_x \neq \emptyset$ and $A_y \neq \emptyset$. Suppose that $A_x = \emptyset$ and let $v \in A_y$. Then $N_{H_1}(v) \subseteq V(B)$ and hence $| N_{H_1}(v) | = | E_{H_1}(v, B) | \leq | S |$.

But $| N_{H_1}(v) | \geq \delta(H_1) = 2r - 1$. So $| S | \geq 2r - 1$ contradicting the original hypothesis that $| S | \leq 2r - 2$. The proof that $A_y \neq \emptyset$ can be obtained using similar arguments.

Now let $u \in A_x$ and $v \in A_y$. Then

$$\begin{aligned} | S | &\geq e_{H_1}(u, B) + e_{H_1}(v, B) \\ &\geq (\delta(H_1) - | A_y |) + (\delta(H_1) - | A_x |) \\ &\geq 4r - 2 - | V(A) | \qquad \text{since } \delta(H_1) = 2r - 1 \,. \end{aligned}$$

But $| S | \leq 2r - 2$. Hence $| V(A) | \geq 2r$ and so by (16),

$$2r \leq | V(A) | \leq 2r + 1. \tag{17}$$

At this point we may assume without loss of generality that $| A_x | \leq | A_y |$. Then using (17),

$$| A_x | \leq r \ \text{ and } \ | A_y | \geq r \,. \tag{18}$$

We also have

$$\begin{aligned} e_{H_1}(A_y, B) &\geq \sum_{v \in A_y} (d_{H_1}(v) - | A_x |) \\ &\geq \delta(H_1) \, | A_y | \; - \; | A_x | \, | A_y | \\ &\geq [(2r - 1) - r] \; r \\ &\geq 2r - 1 \quad \text{for} \quad r \geq 3 \,. \end{aligned}$$

But $E_{H_1}(A_y, B) \subseteq S$ and hence $| S | \geq 2r - 1$, contradicting again the hypothesis that $| S | \leq 2r - 2$. So $\lambda(H_1) \geq 2r - 1$ and since $\delta(H_1) = 2r - 1$, Lemma 1 implies $\lambda(H_1) = 2r - 1$ for $r \geq 3$.

Finally in the case when $r = 2$, we can check easily that the lemma also holds.

Lemma 3 : G is $2r-1$-edge-connected.
Proof: Let S be a set of edges of G such that $|S| \leq 2r-2$.

Then since $\lambda(H_1) = \lambda(T) = 2r-1$, the graphs induced by $V(H_1)$ and $V(T)$ in $G-S$ will be connected graphs. We will also have that H_1 and T are joined in G by $2r$ edges, so since $|S| \leq 2r-2$, the graphs induced by $V(H_1)$ and $V(T)$ in $G-S$ will remain joined in $G-S$. Therefore S cannot be an edge-cutset of G and hence G is $(2r-1)$-edge-connected.

Lemma 4: Let G be a $2r$-regular graph. Then the edge connectivity of G is an even number.
Proof: Suppose that $\lambda(G)$ is odd and let S be an edge-cutset of G such that $\lambda(G) = |S|$. Let G_1 and G_2 be the two components of $G-S$. We have $\sum_{x \in V(G_1)} d_{G_1}(x) = \sum_{x \in V(G_1)} d_G(x) - \lambda(G)$.

But $\sum_{x \in V(G_1)} d_G(x)$ is an even number since G is 2r-regular, and $\lambda(G)$ is odd.

Hence $\sum_{x \in V(G_1)} d_{G_1}(x)$ is an odd integer, which is a contradiction. Therefore Lemma 4 holds.

Theorem 8: $\lambda(G) = 2r$.
Proof: It follows immediately using Lemma 3 and Lemma 4.

Now if we define M to be the set of edges $\{v_1 w_{r+1}, v_2 w_{r+2}, \ldots, v_{r-m+1} w_{2r-m+1}\}$,we will prove that the graph $G_1 = (G - \{u_0\}) - M$ does not have an m-factor.

Let $D = U_0$, $S = V_0 \cup \{v_0\}$. Then
$q_{G_1}(D, S; m) + \sum_{x \in S}(m - d_{G_1 - D}(x)) > m \mid D \mid$ because $|D| = 2r$,
$\sum_{x \in S}(m - d_{G_1 - D}(x)) = 2rm + 1$ and $q_{G_1}(D, S; m) = 1$. Therefore from the k-factor Theorem, G_1 does not have an m-factor.

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