ON THE DIRECTED TRIPLE SYSTEMS WITH A GIVEN AUTOMORPHISM

Biagio Micale *

Department of Mathematics – University of Catania – Italy

and

Mario Pennisi *

Department of S.A.V.A. – University of Molise – Italy

Abstract. A directed triple system of order v, denoted DTS(v), is said to be f–bicyclic if it admits an automorphism consisting of f fixed points and two disjoint cycles. In this paper, we give necessary and sufficient conditions for the existence of f–bicyclic DTS(v)s.

1. Introduction

A *directed triple* is a set of three ordered pairs of the form {(x,y), (y,z), (x,z)}, that we will denote by [x, y, z]. A *directed triple system* of order v, denoted DTS(v), is a pair (V, β), where V is a v–set and β is a set of directed triples of elements of V, called *blocks*, such that any ordered pair of distinct elements of V occurs in exactly one block of β. A DTS(v) exists if and only if v ≡ 0 or 1 (mod 3) [4].

An automorphism of a DTS(v) is a permutation π of V which fixes β. The *orbit* of a block under π is the image of the block under the powers of π. A set of blocks β' is said to be *a set of base blocks for a* DTS(v) *under the permutation* π if the orbits of the blocks of β' produce the DTS(v) and exactly one block of β' occurs in each orbit.

Several types of automorphisms have been studied for the question: "For what values

* Work supported by M.U.R.S.T. and G.N.S.A.G.A.

of v does there exist a DTS(v) admitting an automorphism of the given type?". In particular, a DTS(v) admitting an automorphism consisting of a single cycle is said to be *cyclic*; a cyclic DTS(v) exists if and only if $v \equiv 1$, 4 or 7 (mod 12) [2]. A DTS(v) admitting an automorphism consisting of a fixed point and a cycle of length $v-1$ is said to be *rotational*; a rotational DTS(v) exists if and only if $v \equiv 0 \pmod 3$ [1]. A DTS(v) admitting an automorphism consisting of f fixed points and a single cycle of length ≥ 2 will be said to be f–*cyclic*; an f–cyclic DTS(v), with $f \geq 2$, exists if and only if $v \geq 2f + 1$ and, further, $v \equiv 0 \pmod 3$ and $f \equiv 1 \pmod 3$ or $v \equiv 1 \pmod 3$ and $f \equiv 0 \pmod 3$ [5]. A DTS(v) admitting an automorphism consisting of two distinct cycles is said to be *bicyclic*; a bicyclic DTS(v) admitting an automorphism consisting of two cycles of the same length exists if and only if $v \equiv 4 \pmod 6$; a bicyclic DTS(v) admitting an automorphism consisting of a cycle of length M and a cycle of length N, where $M > N$, exists if and only if $N \equiv 1$, 4 or 7 (mod 12) and $M = kN$, with $k \equiv 2 \pmod 3$ [3].

A DTS(v) admitting an automorphism consisting of f fixed points and two disjoint cycles will be said to be f–*bicyclic*. The purpose of this paper is to present necessary and sufficient conditions for the existence of f–bicyclic DTS(v)s. We break this into two cases: in the first we assume that the two cycles have the same length, and in the second case we assume that the cycles have different lengths.

2. Automorphism consisting of f fixed points and two cycles of the same length

In this section, we will consider f–bicyclic DTS(v)s, in which the two cycles have the same length N, with vertex set $Z_N \times \{0,1\} \cup \{\alpha_1, \alpha_2, \ldots, \alpha_f\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_f$ are the fixed points of the automorphism π. We will represent $(x,0) \in Z_N \times \{0\}$ as x_0 and $(x,1) \in Z_N \times \{1\}$ as x_1, therefore we have:

$$\pi = (\alpha_1)\ (\alpha_2)\ \ldots\ (\alpha_f)\ (0_0, 1_0, \ldots, (N-1)_0)\ \ (0_1, 1_1, \ldots, (N-1)_1)$$

It is easy to prove the following preliminary lemma.

Lemma 2.1. *The fixed points of an automorphism of a* DTS(v) *form a subsystem.*

We have the following necessary conditions:

Lemma 2.2. *If there exists an* f–*bicyclic* DTS(v) *admitting an automorphism* π *in which the two cycles have the same length, then* $v \geq 2f + 1$, $v \equiv 0$ *or* 1 (*mod* 3), $f \equiv 0$ *or* 1 (*mod* 3) *and* $v + f \equiv 4$ (*mod* 6).

Proof. A basic condition for the existence of an f–bicyclic DTS(v) is $v \equiv 0$ or 1 (mod 3), since this is the spectrum for DTS(v)s. Further, from Lemma 2.1 it follows that $f \equiv 0$ or 1 (mod 3).

Since the automorphism π has two cycles of length $\frac{v-f}{2}$, we have that $v - f$ is even. Further, if α is a fixed point, then there does exist two blocks starter containing α as only fixed point, $[0_0,\alpha,x_0]$ and $[0_1,\alpha,x_1]$, or $[0_0,\alpha,x_1]$ and $[0_1,\alpha,x_0]$. It follows that, using the standard idea of difference methods, we have that the number of fixed points can't be greater than the half of the number of differences, i. e. $f \leq v - f - 1$, therefore we have $v \geq 2f + 1$. Finally, the number of blocks of fixed points is $\frac{f(f-1)}{3}$ and the length of the orbit of each other block is $\frac{v-f}{2}$; since the number of blocks in a DTS(v) is $\frac{v(v-1)}{3}$, we have that $\frac{v-f}{2}$ divides $\frac{v(v-1)}{3} - \frac{f(f-1)}{3}$, and therefore $v + f \equiv 1$ (mod 3). Since $v - f$ is even, we have $v + f \equiv 4$ (mod 6). ❑

Theorem 2.1. *An* f–*bicyclic* DTS(v) *admitting an automorphism* π *in which the two cycles have the same length, exists if and only if* $v \geq 2f + 1$, $v \equiv 0$ or 1 (mod 3), $f \equiv 0$ *or* 1 (*mod* 3) *and* $v + f \equiv 4$ (*mod* 6).

Proof. If $v \geq 2f + 1$, $v \equiv 0$ or 1 (mod 3), $f \equiv 0$ or 1 (mod 3) and $v + f \equiv 4$ (mod 6), then there is a DTS(v) which admits an automorphism π consisting of f fixed points and a cycle of length v–f. By considering π^2 we see that this DTS(v) is also f–bicyclic. This shows that the necessary conditions of Lemma 2.2 are also sufficient. ❑

3. Automorphism consisting of f fixed points and two cycles of different lengths

In this section, we will consider f–bicyclic DTS(v)s, $f > 0$, in which the two cycles have lengths M and N, M>N, with vertex set $Z_N \times \{0\} \cup Z_M \times \{1\} \cup \{\alpha_1, \alpha_2, \ldots, \alpha_f\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_f$ are the fixed points of the automorphism π. We will represent $(x,0) \in Z_N \times \{0\}$ as x_0 and $(x,1) \in Z_M \times \{1\}$ as x_1, therefore we have:

$$\pi = (\alpha_1)\ (\alpha_2)\ \ldots\ (\alpha_f)\ (0_0, 1_0, \ldots, (N-1)_0)\ (0_1, 1_1, \ldots, (M-1)_1)$$

We have the following necessary conditions:

Lemma 3.1. *If there exists an* f–*bicyclic* DTS(v), $f > 0$, *admitting an automorphism* π *in which the two cycles have lengths* M *and* N *respectively,* $M > N$, *then* $N \geq f + 1$, $M = kN$, $k \equiv 2\ (mod\ 3)$ *and, further,* $N \equiv 1\ (mod\ 3)$ *and* $f \equiv 0\ (mod\ 3)$ *or* $N \equiv 2\ (mod\ 3)$ *and* $f \equiv 1\ (mod\ 3)$.

Proof. Suppose that there is an f–bicyclic DTS(v) with the vertex set and the automorphism as described above.

The set of fixed points of π^N is precisely the set $Z_N \times \{0\} \cup \{\alpha_1, \alpha_2, \ldots, \alpha_f\}$, therefore by using the Lemma 2.1 we obtain that this set forms an f–cyclic DTS(N + f). Therefore, $N \geq f + 1$ and, further, $N + f \equiv 0$ (mod 3) and $f \equiv 1$ (mod 3) or $N + f \equiv 1$ (mod 3) and $f \equiv 0$ (mod 3). So there must be some blocks of the f–bicyclic DTS(v) with one vertex from $Z_N \times \{0\}$ and two vertices from $Z_M \times \{1\}$. Since such blocks are fixed under π^M, we have that N divides M, say $M = kN$.

The number of blocks of the DTS(v) is $\frac{v(v-1)}{3}$ and the number of blocks of the f–cyclic DTS(N + f) with vertex set $Z_N \times \{0\} \cup \{\alpha_1, \alpha_2, \ldots, \alpha_f\}$ is $\frac{(N+f)(N+f-1)}{3}$. The length of the orbit of each other block of the DTS(v) is M, therefore M divides $\frac{v(v-1)}{3} - \frac{(N+f)(N+f-1)}{3}$. It follows that $M + 2(N + f) \equiv 1$ (mod 3). If $N + f \equiv 1$ (mod 3), then $kN \equiv 2$ (mod 3); therefore, $N \equiv 1$ (mod 3), $f \equiv 0$ (mod 3) and $k \equiv 2$ (mod 3). If $N + f \equiv 0$ (mod 3), then $kN \equiv 1$ (mod 3); therefore, $N \equiv 2$ (mod 3), $f \equiv 1$ (mod 3) and $k \equiv 2$ (mod 3). ❑

We now show that the necessary conditions of Lemma 3.1 are also sufficient.

We require the use of two structures. An (A, n)–*system* is a collection of ordered pairs

(a_r, b_r), r = 1, 2, ..., n, that partition the set {1,2,...,2n} with the property that $b_r = a_r + r$, for every r. It is proved in [7] that an (A, n)–system exists if and only if n ≡ 0 or 1 (mod 4). A (B, n)–*system* is a collection of ordered pairs (a_r, b_r), r = 1,2,...,n, that partition the set {1,2,...,2n − 1,2n + 1} with the property that $b_r = a_r + r$ for every r. It is proved in [6] that a (B, n)–system exists if and only if n ≡ 2 or 3 (mod 4).

Lemma 3.2. *An* f*–bicyclic* DTS(v), f > 0, *admitting an automorphism* π *in which the two cycles have lengths* M *and* N *respectively,* M > N, *exists if* N ≥ f + 1, M = kN, k ≡ 2 (*mod* 3), N ≡ 1 (*mod* 3) *and* f ≡ 0 (*mod* 3).

Proof. Let N ≡ 1 (mod 3), f ≡ 0 (mod 3), f > 0, N ≥ f + 1, M = kN, with k ≡ 2 (mod 3) and let $h = \dfrac{M-N-f-1}{3}$.

We define a set of blocks, β_1, and we consider two cases:

1) If h ≡ 0 or 1 (mod 4), let:

$$\beta_1 = \{[0_1, r_1, (b_r + h)_1]\colon r = 1,2,\ldots,h \text{ and } b_r \text{ from an } (A,h)-\text{system } (\text{omit these blocks if } h = 0)\} \cup \{[0_1, \alpha_i, (3h+i)_1]\colon i = 1,2,\ldots,f-2\}$$

2) If h ≡ 2 or 3 (mod 4), let:

$$\beta_1 = \{[0_1, r_1, (b_r + h)_1]\colon r = 1,2,\ldots,h \text{ and } b_r \text{ from a } (B,h)-\text{system}\} \cup \{[0_1, \alpha_1, (3h)_1]\} \cup \cup \{[0_1, \alpha_i, (3h+i)_1]\colon i = 2,3,\ldots,f-2 \text{ (omit these blocks if } f = 3)\}$$

We now define another set of blocks, β_2, and we consider four cases:

1) If N ≡ 1 (mod 12), say N = 12t + 1, let:

$$\beta_2 = \{[0_0, (3t+r)_1, (3t-r-1)_1], [0_0, (9t+r+1)_1, (9t-r-1)_1], [(9t-r-1)_1, (M-3t+r)_1, 0_0], [(3t-r-1)_1, (M-9t-1+r)_1, 0_0]\colon r = 0,1,\ldots,3t-1\} \cup \cup \{[(9t)_1, 0_0, (M-3t-1)_1]\} \cup \{[0_1, \alpha_{f-1}, (M-N-2)_1], [0_1, \alpha_f, (M-N-1)_1]\}$$

2) If N ≡ 4 (mod 12), say N = 12t + 4, let:

$$\beta_2 = \{[0_0, (3t+r+1)_1, (3t-r)_1], [(9t+2-r)_1, (M-3t+r)_1, 0_0]\colon r = 0,1,\ldots,3t\} \cup \cup \{[0_0, (9t+4+r)_1, (9t+2-r)_1], [(3t+1-r)_1, (M-9t-2+r)_1, 0_0]\colon r = 0,1,\ldots,3t-1\} \cup \cup \{[1_1, 0_0, (M-6t-2)_1], [(9t+3)_1, 0_0, (M-3t-1)_1]\} \cup \cup \{[0_1, \alpha_{f-1}, (M-N-2)_1], [0_1, \alpha_f, (M-N-1)_1]\}$$

3) If N ≡ 7 (mod 12), say N = 12t + 7, let:

$\beta_2=\{[0_0,(3t+2+r)_1,(3t+1-r)_1],[(3t+1-r)_1,(M-9t-5+r)_1,0_0]: r=0,1,\ldots,3t+1\}\cup$

$\cup\{[0_0,(9t+6+r)_1,(9t+4-r)_1],[(9t+4-r)_1,(M-3t-1+r)_1,0_0]: r=0,1,\ldots,3t\}\cup$

$\cup\{[(9t+5)_1,0_0,(M-3t-2)_1]\}\cup\{[0_1,\alpha_{f-1},(M-N-2)_1],[0_1,\alpha_f,(M-N-1)_1]\}$

4) If N ≡ 10 (mod 12), say N = 12t + 10, let:

$\beta_2=\{[0_0,(3t+2+r)_1,(3t+1-r)_1],[0_0,(9t+7+r)_1,(9t+5-r)_1]: r=0,1,\ldots,3t+1\}\cup$

$\cup\{[(9t+7-r)_1,(M-3t-1+r)_1,0_0]: r=0,1,\ldots,3t\}\cup$

$\cup\{[(3t+3-r)_1,(M-9t-6+r)_1,0_0]: r=0,1,\ldots,3t+2\}\cup$

$\cup\{[(9t+8)_1,0_0,(M-3t-4)_1],[(12t+10)_1,0_0,(M-1)_1]\}\cup$

$\cup\{[0_1,\alpha_{f-1},(M-N)_1],[0_1,\alpha_f,(M-6t-6)_1]\}$

In all cases, the union of the set $\beta_1\cup\beta_2$ with a set of base blocks for an f–cyclic DTS(N + f) on $Z_N\times\{0\}\cup\{\alpha_1,\alpha_2,\ldots,\alpha_f\}$ forms a set of base blocks for an f–bicyclic DTS(v). ❑

Lemma 3.3. *An* f*–bicyclic* DTS(v) *admitting an automorphism* π *in which the two cycles have lengths* M *and* N *respectively,* M > N, *exists if* N ≥ f + 1, M = kN, k ≡ 2 (*mod* 3), N ≡ 5, 8 *or* 11 (*mod* 12) *and* f ≡ 1 (*mod* 3).

Proof. Let N ≡ 5, 8 or 11 (mod 12), f ≡ 1 (mod 3), N ≥ f + 1, M = kN, with k ≡ 2 (mod 3) and let $h=\dfrac{M-N-f-1}{3}$.

We define a set of blocks, β_1, and we consider two cases:

1) If h ≡ 0 or 1 (mod 4), let:

$\beta_1=\{[0_1,r_1,(b_r+h)_1]: r=1,2,\ldots,h \text{ and } b_r \text{ from an } (A,h)-\text{system}$

(omit these blocks if $h=0$)$\}\cup\{[0_1,\alpha_i,(3h+i)_1]: i=1,2,\ldots,f\}$

2) If h ≡ 2 or 3 (mod 4), let:

$\beta_1=\{[0_1,r_1,(b_r+h)_1]: r=1,2,\ldots,h \text{ and } b_r \text{ from a } (B,h)-\text{system}\}\cup\{[0_1,\alpha_1,(3h)_1]\}\cup$

$\cup\{[0_1,\alpha_i,(3h+i)_1]: i=2,3,\ldots,f$ (omit these blocks if $f=1$)$\}$

We now define another set, β_2, of blocks and we consider three cases:

1) If $N \equiv 5 \pmod{12}$, say $N = 12t + 5$, let:

$$\beta_2 = \{[0_0, (3t+1+r)_1, (3t-r)_1], [(9t+2-r)_1, (M-3t-1+r)_1, 0_0]: r = 0,1,\ldots,3t\} \cup$$
$$\cup \{[0_0, (9t+4+r)_1, (9t+2-r)_1], [(3t-r)_1, (M-9t-4+r)_1, 0_0]: r = 0,1,\ldots,3t\} \cup$$
$$\cup \{[(9t+3)_1, 0_0, (M-3t-2)_1]\}$$

2) If $N \equiv 8 \pmod{12}$, say $N = 12t + 8$, let:

$$\beta_2 = \{[0_0, (3t+2+r)_1, (3t+1-r)_1], [(9t+5-r)_1, (M-3t-1+r)_1, 0_0]: r = 0,1,\ldots,3t+1\} \cup$$
$$\cup \{[0_0, (9t+7+r)_1, (9t+5-r)_1], [(3t+2-r)_1, (M-9t-5+r)_1, 0_0]: r = 0,1,\ldots,3t\} \cup$$
$$\cup \{[(9t+6)_1, 0_0, (M-3t-2)_1], [1_1, 0_0, (M-6t-4)_1]\}$$

3) If $N \equiv 11 \pmod{12}$, say $N = 12t + 11$, let:

$$\beta_2 = \{[0_0, (3t+3+r)_1, (3t+2-r)_1]: r = 0,1,\ldots,3t+2\} \cup$$
$$\cup \{[0_0, (9t+9+r)_1, (9t+7-r)_1], [(9t+7-r)_1, (M-3t-2+r)_1, 0_0]: r = 0,1,\ldots,3t+1\} \cup$$
$$\cup \{[(3t+2-r)_1, (M-9t-8+r)_1, 0_0]: r = 0,1,\ldots,3t+2\} \cup$$
$$\cup \{[(9t+8)_1, 0_0, (M-3t-3)_1]\}$$

In all cases, the union of the set $\beta_1 \cup \beta_2$ with a set of base blocks for an f–cyclic DTS(N + f) on $Z_N \times \{0\} \cup \{\alpha_1, \alpha_2, \ldots, \alpha_f\}$ forms a set of base blocks for an f–bicyclic DTS(v). ❑

Lemma 3.4. *An* f–*bicyclic* DTS(v) *admitting an automorphism* π *in which the two cycles have lengths* M *and* N *respectively,* $M > N$, *exists if* $N \geq f + 1$, $M = kN$, $k \equiv 2 \pmod 3$, $N \equiv 2 \pmod{12}$ *and* $f \equiv 1 \pmod 3$.

Proof. Let $N \equiv 2 \pmod{12}$, say $N = 12t + 2$, $f \equiv 1 \pmod 3$, $N \geq f + 1$, $M = kN$, with $k \equiv 2 \pmod 3$ and let $h = \dfrac{M-N-f-1}{3}$.

We define a set of blocks, β_1, and we consider two cases:

1) If $h \equiv 0$ or $1 \pmod 4$, let:

$$\beta_1 = \{[0_1, r_1, (b_r + h)_1]: r = 1,2,\ldots,h \text{ and } b_r \text{ from an } (A,h)-\text{system}$$
$$\text{(omit these blocks if } h = 0)\} \cup \{[N_1, 0_0, (M-1)_1], [(6t+1)_1, 0_0, (M-2)_1]\} \cup$$
$$\cup \{[0_1, \alpha_i, (3h+i)_1]: i = 1,2,\ldots,f-1 \text{ (omit these blocks if } f = 1)\} \cup$$
$$\cup \{[0_1, \alpha_f, (M-6t-2)_1]\}$$

2) If $h \equiv 2$ or $3 \pmod 4$, let:

$\beta_1 = \{[0_1, r_1, (b_r + h)_1]: r = 1,2,\ldots, h \text{ and } b_r \text{ from a } (B,h)-\text{system}\} \cup$
$\cup \{[N_1, 0_0, (M-2)_1], [(6t+1)_1, 0_0, (M-1)_1]\} \cup \{[0_1, \alpha_f, (M-6t-3)_1]\} \cup$
$\cup \{[0_1, \alpha_1, (3h)_1], [0_1, \alpha_{f-1}, (M-N-1)_1] \text{ (omit these blocks if } f=1)\} \cup$
$\cup \{[0_1, \alpha_i, (3h+i)_1]: i = 2,3,\ldots, f-2 \text{ (omit these blocks if } f=1)\}$

Further, let:

$\beta_2 = \{[0_1, (1+r)_0, (M-6t-1+2r)_1]: r = 0,1,\ldots,3t\} \cup$
$\cup \{[(9t-r)_1, 0_0, (M-3t-1+r)_1]: r = 0,1,\ldots,3t-2\} \cup$
$\cup \{[(6t-r)_1, 0_0, (M+r)_1], [(3t-r)_1, 0_0, (M-9t-2+r)_1]: r = 0,1,\ldots,3t-1\}$

In all cases, the union of the set $\beta_1 \cup \beta_2$ with a set of base blocks for an f–cyclic DTS(N + f) on $Z_N \times \{0\} \cup \{\alpha_1, \alpha_2, \ldots, \alpha_f\}$ forms a set of base blocks for an f–bicyclic DTS(v). ❑

The results of this section combine to give us:

Theorem 3.1. *An* f–*bicyclic* DTS(v), *with* $f > 0$, *admitting an automorphism* π *in which the two cycles have lengths* M *and* N *respectively,* $M > N$, *exists if and only if* $N \geq f + 1$, $M = kN$, $k \equiv 2$ (*mod* 3) *and, further,* $N \equiv 1$ (*mod* 3) *and* $f \equiv 0$ (*mod* 3) *or* $N \equiv 2$ (*mod* 3) *and* $f \equiv 1$ (*mod* 3).

References

[1] C. J. Cho, Y. Chae and S. G. Hwang, *Rotational directed triple systems*, J. Korean Math. Soc. 24 (1987) 133–142.

[2] C. J. Colbourn and M. J. Colbourn, *The analysis of directed triple systems by refinement*, Annals of Discrete Math., 15 (1982) 93–103.

[3] R. Gardner, *Bicyclic directed triple systems*, Ars Combinatoria (to appear).

[4] S. H. Y. Hung and N. S. Mendelsohn, *Directed triple systems*, J. Combin. Theory Ser. A 14 (1973) 310–318.

[5] B. Micale and M. Pennisi, *The spectrum of d–cyclic oriented triple systems*, Ars Combinatoria (to appear).

[6] E. S. O'Keefe, *Verification of a conjecture of Th. Skolem*, Math. Scand. 9 (1961) 80–82.

[7] T. Skolem, *On certain distributions of integers in pairs with given differences*, Math. Scand. 5 (1957) 57–68.

(Received 19/7/96)