Isomorphisms and Automorphism Groups of a Class of Cayley Digraphs on Abelian Groups*

Qiongxiang Huang

Department of Mathematics of Xinjiang University
Urumuqi, Xinjiang 830046, P.R.China†
and
Mathematics and Physics Institute of Xinjiang University
Urumuqi, Xinjiang 830046, P.R.China

Abstract

In this paper, we investigate problems about isomorphisms and automorphism groups of Cayley digraphs. A class of Cayley digraphs, corresponding to the so-called CDI-subsets, for which the isomorphisms are uniquely determined by the group automorphisms is characterized. Their automorphism groups are also characterized.

1. Introduction

The groups considered in this paper are finite abelian groups with the operation $+$ and identity denoted 0. Let G be a group and for each $S \subset G$ $(0 \notin S)$, the Cayley digraph $C(G,S)$ on G with the arc symbol set S is defined as follows: Its vertices are the elements of G, and (u,v) is an arc if and only if $v-u \in S$. Commonly, $C(G,S)$ is said to be a Cayley graph if $S=-S=\{-s \mid s \in S\}$. Since a Cayley graph is a special Cayley digraph, normally we don't distinguish them. When G is a cyclic group Z_n, we call $C(G,S)$ a circulant digraph. In this case, we use $C_n(S)$ instead of $C(Z_n,S)$.

Denote by $AutG$ the automorphism group of G. For $\tau \in AutG$ and $S \subset G$, set $\tau(S)=\{\tau(s) \mid s \in S\}$. We call two subsets S and T of G equivalent if there exists $\tau \in AutG$ such that $\tau(S)=T$. It is easy to see that $C(G,S) \cong C(G,T)$ if S and T are equivalent. But the converse is not true. We call S a CDI-subset of G if for any $C(G,T)$ isomorphic to $C(G,S)$, S and T are equivalent. CDI-subset of G is an abbreviation for "Cayley digraph isomorphism" which follows the terminology due to Babai [2]. Similarly, a CDI-subset S is said to be a CI-subset if $S=-S$.

Characterizing the CDI-subsets is a topic on circulant digraphs arising from Ádám's conjecture [1] that $C_n(S) \cong C_n(T)$ if and only if there exists an integer

*This work is supported by NSFC.

†Mailing address.

λ relatively prime to n such that $T = \lambda S = \{\lambda s \mid s \in S\}$. Although this conjecture was disproved by a counterexample due to Elspas and Turner [3], there is considerable work in this area [2-10]. This is because Ádám's conjecture suggests an interrelation between isomorphisms on groups and graphs.

In [3] Elspas and Turner posed the problem of characterizing those circulant digraphs for which isomorphism is equivalent to having equivalent arc symbol sets. It natually suggests a similar problem on Cayley digraphs, that is, to characterize the CDI-subsets. Sun Liang [11] proved Boesch's conjecture [12] that every subset S with $|S| = 4$ and $S = -S$ is a CI-subset of Z_n. Delorme et al. [13] obtained the same result as above for abelian groups.

It seems difficult to determine fully the CDI-subsets for a given group G. But we believe that most subsets of G are CDI-subsets.

It is well-known that G can be decomposed into a direct product of cyclic groups. Let

$$G = Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k} \tag{1}$$

be such a decomposition. For each element a of Z_{n_i}, we use the residue modulo n_i satisfying $0 \leq a < n_i$. Let $S_i \subset Z_{n_i}$ $(i = 1, 2, ..., k)$. Define

$$D(S_i) = max\{s_i \mid s_i \in S_i\} - min\{s_i \mid s_i \in S_i\}.$$

It is clear that $0 \leq D(S_i) < n_i$. We select a generating subset S_i of Z_{n_i} with $D(S_i) < \lceil \frac{n_i}{2} \rceil$, and then define a subset S_0 of G as follows:

$$S_0 = S_1 \times S_2 \times \cdots \times S_k \backslash \{(\underbrace{0, 0, ..., 0}_{k})\}.$$

Our object in this work is to prove the following.

Main Result: Let $S \subseteq S_0$. Then S is a CDI-subset of G if and only if S generates G.

In addition, we also give a characterization of the automorphism group for such a $C(G, S)$.

2. Main result

First, we introduce some notation. Let $AutC(G, S)$ denote the automorphism group of $C(G, S)$ and $L(G) = \{\sigma_g \mid g \in G\}$, where $\sigma_g(a) = g + a$ for all $a \in G$. It is easy to check that $L(G)$ is a subgroup of $AutC(G, S)$ for each $S \subset G$ and acts transitively on the vertices of $C(G, S)$. Suppose $C(G, S) \cong C(G, T)$. Then there exists some isomorphism τ from $C(G, S)$ to $C(G, T)$ with $\tau(0) = 0$. Let $\Omega(S \to T)$ be the set consisting of all the isomorphisms from $C(G, S)$ to $C(G, T)$ with $\tau(0) = 0$ and $Aut_G(S \to T) = \{\tau \in AutG \mid \tau(S) = T\}$. Clearly, $Aut_G(S \to T) \subseteq \Omega(S \to T)$. Thus S is a CDI-subset of G if and only if $Aut_G(S \to T) \neq \emptyset$.

The following lemma is familiar to us and simple to prove.

Lemma 1 [14]. *$C(G, S)$ is strongly connected if and only if S generates G.*

The following lemma provides a necessary and sufficient condition for a Cayley digraph to satisfy $\Omega(S \to T) = Aut_G(S \to T)$. This is true of a large number of Cayley digraphs and plays, as will be seen later, an important role in the proof of our main result.

Lemma 2. *Let $C(G,S)$ be strongly connected and $C(G,T)$ be isomorphic to $C(G,S)$. Then $\Omega(S \to T) = Aut_G(S \to T)$ if and only if $\tau(a+b) = \tau(a) + \tau(b)$ for $a, b \in S$ and $\tau \in \Omega(S \to T)$.*

Proof. The necessity is obvious.

Let $\tau \in \Omega(S \to T)$ and $u \in G$. Since $\sigma_{-\tau(u)}\tau\sigma_u(0) = -\tau(u) + \tau(u) = 0$, $\sigma_{-\tau(u)}\tau\sigma_u \in \Omega(S \to T)$. By assumption,

$$\sigma_{-\tau(u)}\tau\sigma_u(a+b) = \sigma_{-\tau(u)}\tau\sigma_u(a) + \sigma_{-\tau(u)}\tau\sigma_u(b).$$

That is,

$$\tau(u+a+b) = \tau(u+a) - \tau(u) + \tau(u+b). \tag{2}$$

Set $u = \sum_{i=1} s_i$, where the s_i are elements of S (not necessarily distinct). In terms of (1), it is not difficult to show by induction that $\tau(\sum_{i=1} s_i) = \sum_{i=1} \tau(s_i)$. By Lemma 1, S generates G. Hence $\tau \in Aut_G(S \to T)$. This completes our proof.

Taking $u = b = a$ in (2), we have $\tau(3a) = 3\tau(a)$. Similarly, by setting $b = a$ and $u = a, 2a, 3a, \ldots$ respectively, we immediately get the following.

Corollary 1. *Let $a \in S$. If $\tau(2a) = 2\tau(a)$ for $\tau \in \Omega(S \to T)$, then $\tau(ia) = i\tau(a)$ for every integer i.*

In the following, we prove several lemmas which together achieve our object.

Let $g, u \in G$. Let $\langle g \rangle$ denote the subgroup generated by g so that $u + \langle g \rangle$ is a coset of $\langle g \rangle$. Let $\Re(S) = \{s + \langle g \rangle \mid s \in S \text{ and } g(\neq 0) \in G\}$ be the collection of cosets with respect to S. Saying that S contains no element of $\Re(S)$ means $s + \langle g \rangle \not\subseteq S$ for each $s + \langle g \rangle \in \Re(S)$. We have the following.

Lemma 3. *Let $\langle S \rangle = G, a \in S$ and assume S contains no element of $\Re(S)$. If $\tau(2a) = 2\tau(a)$ for $\tau \in \Omega(S \to T)$, then $\tau(a+b) = \tau(a) + \tau(b)$ for $\tau \in \Omega(S \to T)$ and $b \in S$.*

Proof. For $\tau \in \Omega(S \to T)$, let $\tau(a) = t$ and $\tau(b) = t'$. Then $t, t' \in T$. Let τ^{-1} be the inverse of τ. By our assumption

$$\tau^{-1}(2t) = 2 \cdot \tau^{-1}(t).$$

Thus from Corollary 1, for each $\tau^{-1} \in \Omega(T \to S)$ and integer i we have

$$\tau^{-1}(it) = i \cdot \tau^{-1}(t).$$

Since $\sigma_{-\tau^{-1}(t')}\tau^{-1}\sigma_{t'} \in \Omega(T \to S)$, we have for every integer i

$$\sigma_{-\tau^{-1}(t')}\tau^{-1}\sigma_{t'}(it) = i \cdot \sigma_{-\tau^{-1}(t')}\tau^{-1}\sigma_{t'}(t).$$

That is,

$$\begin{aligned}\tau^{-1}(t'+it) &= \tau^{-1}(t')+i(-\tau^{-1}(t')+\tau^{-1}(t'+t))\\ &= \tau^{-1}(t'+t)+(i-1)(\tau^{-1}(t'+t)-\tau^{-1}(t')).\end{aligned}$$

Hence

$$\tau^{-1}(t'+it)-\tau^{-1}(it)=\tau^{-1}(t'+t)-\tau^{-1}(t)+(i-1)(\tau^{-1}(t'+t)-\tau^{-1}(t')-\tau^{-1}(t)).$$

Note that since $(it, t'+it)$ is an arc of $C(G,T)$, $(\tau^{-1}(it), \tau^{-1}(t'+it))$ is then an arc of $C(G,S)$. Thus

$$\tau^{-1}(t'+t)-\tau^{-1}(t)+(i-1)(\tau^{-1}(t+t')-\tau^{-1}(t')-\tau^{-1}(t))\in S.$$

Let $\tau^{-1}(t'+t)-\tau^{-1}(t)=s$ and $\tau^{-1}(t'+t)-\tau^{-1}(t')-\tau^{-1}(t)=g$. Then $s+(i-1)g\in S$ for every integer i and hence $s+\langle g\rangle\subseteq S$. Clearly $s\in S$, we deduce $g=0$ (since otherwise $s+\langle g\rangle$ is an element of $\Re(S)$, which contracts our assumption). That is, $\tau^{-1}(t'+t)=\tau^{-1}(t')+\tau^{-1}(t)$. By applying τ to both sides of this equation we obtain $\tau(a+b)=\tau(a)+\tau(b)$. This completes the proof.

According to (1), for each $g\in G$, g can be rewritten as $g=(g_1,g_2,...,g_k)$, where $g_i\in Z_{n_i}$ and $0\le g_i<n_i$ $(1\le i\le k)$. Set $|g|=\sum_{i=1}^{k} g_i$. Then $|g|$ is an integer. Let $u=(u_1,u_2,...,u_k), v=(v_1,v_2,...,v_k)\in G$. We say v is behind u if $|u|=|v|$ and there exists some t $(1\le t\le k)$ such that $u_t<v_t$ and $u_l=v_l$ if $l<t$. Now we define an ordering, also denoted by $<$, on the elements of G.

For each pair of elements u and v in G, $u<v$ if $|u|<|v|$ or v is behind u. Obviously, if $u<v$ and $v<w$, then $u<w$.

Let S_i and S_0 be as specified in section 1. Let $a=(a_1,a_2,...,a_k)$, $b=(b_1,b_2,...,b_k)$, and $c=(c_1,c_2,...,c_k)$ be three elements in S_0. We have

Lemma 4. *If $2a=b+c$, then $b<a$ or $c<a$.*

Proof. For a contradiction, suppose $b>a$ and $c>a$. Select $s_i=min\{s\in S_i\}$, $i=1,2,...,k$. Take $s=(s_1,s_2,...,s_k)\in S_0$. Then

$$0\le a_i-s_i<\lceil\frac{n_i}{2}\rceil, 0\le b_i-s_i<\lceil\frac{n_i}{2}\rceil \text{ and } 0\le c_i-s_i<\lceil\frac{n_i}{2}\rceil, i=1,2,...,k.$$

By assumption

$$2(a-s)=(b-s)+(c-s). \tag{3}$$

In addition, it is easy to see from the definition that

$$b-s>a-s \text{ and } c-s>a-s. \tag{4}$$

If one of $|b-s|$ or $|c-s|$ is greater than $|a-s|$, then

$$|2(a-s)|<|b-s|+|c-s|=|(b-s)+(c-s)|.$$

Since $0\le 2(a_i-s_i)<n_i$ and $0\le(b_i-s_i)+(c_i-s_i)<n_i$ $(1\le i\le k)$, we deduce that $2(a-s)<(b-s)+(c-s)$. This is impossible due to (3). Thus we may further assume that

$$|b-s|=|c-s|=|a-s|.$$

According to (4), there exist t_1 and t_2 $(1 \leq t_1, t_2 \leq k)$ such that

$$\begin{array}{lll} b_{t_1} - s_{t_1} > a_{t_1} - s_{t_1} & \text{and} & b_l - s_l = a_l - s_l \text{ if } l < t_1 \\ c_{t_2} - s_{t_2} > a_{t_2} - s_{t_2} & \text{and} & c_l - s_l = a_l - s_l \text{ if } l < t_2\text{n}. \end{array}$$

Set $t = min\{t_1, t_2\}$. We have

$$(b_t - s_t) + (c_t - s_t) > 2(a_t - s_t) \text{ and } (b_l - s_l) + (c_l - s_l) = 2(a_l - s_l) \text{ if } l < t.$$

On the other hand,

$$|(b - s) + (c - s)| = |b - s| + |c - s| = |2(a - c)|.$$

Then by definition, $(b - s) + (c - s)$ is behind $2(a - s)$. Hence

$$(b - s) + (c - s) > 2(a - s).$$

This again leads a contradiction with (3). It completes our proof.

Let $S = \{d_i = (d_{i1}, d_{i2}, ..., d_{ik}) \mid d_{ij} \in Z_{n_j} \ \ (1 \leq j \leq k) \text{ and } i = 1, 2, ..., n\}$ be a subset of S_0. In the ordering of G defined above, we can assume that

$$d_1 < d_2 < \cdots < d_n. \tag{5}$$

Lemma 5. *Let $C(G, T)$ be any Cayley digraph isomorphic to $C(G, S)$. Then $\tau(2d_1) = 2\tau(d_1)$ for every $\tau \in \Omega(S \to T)$.*

Proof. Since $(d_1, 2d_1)$ is an arc of $C(G, S)$, $(\tau(d_1), \tau(2d_1))$ is an arc of $C(G, T)$. Thus there is $d_1' \in S$ such that $\tau(2d_1) = \tau(d_1) + \tau(d_1')$.

If $d_1' = d_1$, our proof has finished. Otherwise, $d_1' \neq d_1$. Then $\tau(2d_1)$ has two common in-adjacency vertices $\tau(d_1)$ and $\tau(d_1')$ in T, and therefore $2d_1$ has two common in-adjacency vertices in S of which at least one is different from d_1. Thus there are two elements d_i and d_j in S such that $2d_1 = d_i + d_j$. But from Lemma 4, we have $d_i < d_1$ or $d_j < d_1$. This contradicts (5).

Lemma 6. *Let $S \subseteq S_0$. Then S contains no element of $\Re(S)$.*

Proof. For a contradiction, suppose there is $a = (a_1, a_2, ...a_k) \in S$ and $g = (g_1, g_2, ..., g_k) \in G$ such that $a + \langle g \rangle \subseteq S$. Let $o(g)$ denote the order of g in G. Then, for $0 \leq n < o(g)$, $a + ng \in S \Longrightarrow a_i + ng_i \in S_i \subset Z_{n_i}$ $(1 \leq i \leq k)$. If $gcd(g_i, n_i) = 1$, then $S_i = Z_{n_i}$. This is impossible since $D(S_i) < \lceil \frac{n_i}{2} \rceil$. Suppose $gcd(g_i, n_i) = \alpha_i \neq 1$. Then $\langle \alpha_i \rangle = \langle g_i \rangle$. Therefore $D(S_i) \geq (\alpha_i(\frac{n_i}{\alpha_i} - 1) + a_i) - a_i \geq n_i - \alpha_i \geq \frac{n_i}{2}$. This leads a contradiction with the choice of S_i.

Lemma 7. *Let $S = \{d_i = (d_{i1}, d_{i2}, ..., d_{ik}) \mid d_1 < d_2 < \cdots < d_n\} \subseteq S_0$ and let $C(G, T)$ be any Cayley digraph isomorphic to $C(G, S)$. Then $\tau(2d_i) = 2\tau(d_i)$ for every $\tau \in \Omega(S \to T)$ and $d_i \in S$.*

Proof. We prove our result by induction on the index of $d_i \in S$. According to Lemma 5, $\tau(2d_1) = 2\tau(d_1)$. Suppose we have established that

$$\tau(2d_l) = 2\tau(d_l) \text{ for } d_l < d_i, \text{ where } i \geq 2.$$

Since $2d_j \neq 2d_{j'}$ for $j \neq j'$, it is easy to see that there is an odd number of vertices, say $2m+1$ vertices, of S which are out-adjacent to $2d_i$, and d_i is clearly such a vertex. Let $d_{i_l}, d_{j_l} (l = 1, 2, ..., m)$ be all these vertices other than d_i such that

$$2d_i = d_{i_1} + d_{j_1} = d_{i_2} + d_{j_2} = \cdots = d_{i_m} + d_{j_m}.$$

Because of Lemma 4, one can further assume that

$$d_{i_l} < d_i < d_{j_l}, \quad l = 1, 2, ..., m.$$

Thus by the induction hypothesis, we have $\tau(2d_{i_l}) = 2\tau(d_{i_l})$ for $\tau \in \Omega(S \to T)$. Then by combining Lemma 6 and Lemma 3, for every $\tau \in \Omega(S \to T)$, we have

$$\tau(d_{i_l} + d_{j_l}) = \tau(d_{i_l}) + \tau(d_{j_l}) \quad l = 1, 2, ..., m.$$

Now we consider $\tau(2d_i)$. It is clear that there exists $d_i' \in S$ such that $\tau(2d_i) = \tau(d_i) + \tau(d_i')$. In view of

$$\tau(d_i) + \tau(d_i') = \tau(2d_i) = \tau(d_{i_l}) + \tau(d_{j_l}), \quad l = 1, 2, ..., m.$$

we deduce that $d_i' \notin \{d_{i_l}, d_{j_l} \mid l = 1, 2, ..., m\}$. This means, by our assumption, that $d_i' = d_i$, i.e., $\tau(2d_i) = 2\tau(d_i)$. Thus, by induction, we complete our proof.

Theorem 1. *Let S be a subset of S_0 such that $\langle S \rangle = G$. Then S is a CDI-subset of G.*

Proof. Let $C(G,T)$ be any Cayley digraph isomorphic to $C(G,S)$. Set $S = \{d_i (i = 1, 2, ..., n) \mid d_1 < d_2 < \cdots d_n\}$. By Lemma 7 and Lemma 3, we have

$$\tau(d_i + d_j) = \tau(d_i) + \tau(d_j) \quad \text{for } \tau \in \Omega(S \to T) \text{ and } d_i, d_j \in S.$$

By Lemma 2, for each $\tau \in \Omega(S \to T)$, we have $\tau \in Aut_G(S \to T)$. The result follows readily.

Let S be a subset of S_0 which generates G and let $C(G,T)$ be any Cayley digraph isomorphic to $C(G,S)$. Then $\Omega(S \to T) \neq \emptyset$. It is worth mentioning that each isomorphism in $\Omega(S \to T)$ is a group isomorphism on G.

Corollary 2. *Let $S \subseteq S_0$, $\langle S \rangle = G$ and $C(G,S) \cong C(G,T)$. Then $\Omega(S \to T) = Aut_G(S \to T)$.*

In particular, when $k = 1$, $G = Z_{n_1}$ is the cyclic group of integers modulo n_1. Theorem 1 includes the following result about isomorphisms of circulant digraphs.

Corollary 3. *Let S be a subset of Z_n with $D(S) < \lceil \frac{n}{2} \rceil$. Then $C_n(S)$ satisfies Ádám's conjecture.*

Proof. It is not difficult to see that $C_n(S)$ satisfies Ádám's conjecture if and only if its components satisfy Ádám's conjecture and the components of $C_n(S)$ are some copies of another strongly connected circulant digraph. Thus our result follows immediately by Theorem 1.

In the following, we give an example to illustrate that the condition $D(S) < \lceil \frac{n}{2} \rceil$ is necessary and in some sense is best possible.

Example 1. Let m be a positive integer divisible by 4 and put $n = 2m$. Set $S = \{1, m+1, 2\} \subset Z_n$. Then $C_n(S)$ does not satisfies Ádám's conjecture.

In fact, set $T = \{1, m+1, m+2\}$. For $u \in Z_n$, define

$$\tau(u) = u + im, \qquad \text{where } u \in \{2i, 2i+1\} \quad (0 \le i < m).$$

It is not difficult to verify that τ is an isomorphism from $C_n(S)$ to $C_n(T)$. But there is no integer λ relatively prime to n such that $T = \lambda S$.

3. Automorphism Group of $C(G, S)$

Let S be a subset of G. Then S generates a subgroup of G. It is not difficult to show that $C(\langle S\rangle, S)$ is a component of $C(G, S)$. In other words, $C(G, S)$ consists of r copies of $C(\langle S\rangle, S)$, where $r = [G : \langle S\rangle]$. In this case, the automorphism group of $C(G, S)$ is the wreath product of these r's $AutC(\langle S\rangle, S)$. Thus, without loss of generality, we assume in this section that $\langle S\rangle = G$.

Regarding $C(G, T)$ as $C(G, S)$, the isomorphism $\tau \in \Omega(S \to T)$ is the automorphism of $C(G, S)$ with $\tau(0) = 0$. In this case, $\Omega(S \to T)$ is refered to as $\Omega(S) = \{\tau \in AutC(G, S) \mid \tau(0) = 0\}$ and $Aut_G(S \to T)$ as $Aut_G(S) = \{\tau \in AutG \mid \tau(S) = S\}$. Based on the results in the last section, we give a characterization of $AutC(G, S)$ for $S \subsetneq S_0$.

First we cite a well-known result.

Lemma 8 [15]. *$AutC(G, S) = L(G)\Omega(S)$.*

From Corollary 3 and Lemma 8, we have

Theorem 2. *Let $S \subseteq S_0$ and $\langle S\rangle = G$. Then $AutC(G, S) = L(G)Aut_G(S)$.*

Generally speaking, $Aut_G(S)$ is a subgroup of $AutG$. To the best of our knowledge, $AutG$ is not known yet. So we can not give an explicit expression for $Aut_G(S)$. But given an $S \subset S_0$, it is not too difficult to determine $AutC(G, S)$ in terms of Theorem 2.

Let S_i and S_0 be as specified in section 1. If G is a direct product

$$G_0 = Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k},$$

where $n_1, n_2, \ldots, n_k$ are integers relatively prime to each other, we can describe $C(G_0, S_0)$ in detail.

The following lemma is familiar to us from group theory.

Lemma 9. *$AutG_0 = AutZ_{n_1} \times AutZ_{n_2} \times \cdots \times AutZ_{n_k}$, where n_i $(i = 1, 2, \ldots, k)$ are relatively prime to each other.*

Then by Lemma 9, we have

$$Aut_{G_0}(S_0) = Aut_{Z_{n_1}}(S_1) \times Aut_{Z_{n_2}}(S_2) \times \cdots \times Aut_{Z_{n_k}}(S_k).$$

Combining with Lemma 8, we derive the following.

Theorem 3. $AutC(G_0, S_0) = L(G_0) \times Aut_{Z_{n_1}}(S_1) \times \cdots \times Aut_{Z_{n_k}}(S_k)$.

Notice that $AutZ_{n_i} \cong \{\lambda \in Z_{n_i} \mid gcd(\lambda, n_i) = 1\} = Z_n^*$, so we have

$$Aut_{Z_{n_i}}(S_i) \cong \{\lambda \in Z_{n_i}^* \mid \lambda S_i = S_i\} \quad i = 1, 2, ..., k.$$

Thus we can easily obtain $AutC(G_0, S_0)$ from Theorem 3.

Let $m \geq 2$ and $\alpha \geq 2$ be two positive integers and $n = m^\alpha - 1$. Set $S_{m,\alpha} = \{1, m, m^2, ..., m^{\alpha-1}\} \subset Z_n$. Then $D(S_{m,\alpha}) < \lceil \frac{n}{2} \rceil$ and

$$\begin{aligned} Aut_{Z_n}(S_{m,\alpha}) \cong & \ \{\lambda \in Z_n^* \mid \lambda S_{m,\alpha} = S_{m,\alpha}\} \\ = & \ \{1, m, m^2, ..., m^\alpha\}. \end{aligned}$$

Thus $Aut_{Z_n}(S_{m,\alpha})$ is isomorphic to the group $\{1, m, m^2, ..., m^\alpha\}$ under multiplication, that is, the cyclic group Z_k. So we have

Example 2. $AutC_n(S_{m,\alpha}) \cong Z_n \times Z_\alpha$.

Since Z_α acting on $S_{m,\alpha}$ is vertex transitive, it is not difficult to see that $C_n(S_{m,\alpha})$ is a type of arc-transitive circulant digraph.

References

[1] A. Ádám , Research Problem 2-10, J.Combin. Theory 2 (1967) 393.

[2] L. Babai, Isomorphism Problem for a Class of Point-Symmetric Structures, Acta. Math. Acad. Sci. Hung., 29(1977) 329-336.

[3] B. Elspas and J. Turner, Graphs with Circulant Adjacency Matrices, J.Combin.Theory 9, (1970) 297-307.

[4] D. Z. Djokovic, Isomorphism Problem for a Special Class of Graphs, Acta. Math. Acad. Sci. Hung. 21 (1970) 267-270.

[5] J. Turner, Point Symmetric Groups with a Prime Number of Points, J. Combin. Theory, 3(1967),136-145.

[6] L. Babai and P. Frankl, Isomorphism of Cayley Graphs I, Collòq. Math. Soc. J. Bolyai, 18. Combinatorics, Keszthely, 1976; North-Holland, Amsterdam 35-52.

[7] L. Babai and P. Frankl, Isomorphism of Cayley Graphs II, Acta. Math. Acad. Sci. Hung. 34 (1979) 177-183.

[8] C.D. Godsil, On Cayley Graph Isomorphisms, Ars Combinatorica, 15(1983) 231-246.

[9] B. Alspach and T.D. parsons, Isomorphism of Circulant Graphs and Digraphs, Discrete Math. 25(1979) 97-108.

[10] S. Toida, A note on Ádám's Conjecture, J. Combin. Theory, Ser.B 23(1977) 239-246.

[11] Sun Liang, Isomorphisms of Circulant Graphs, Chinese. Ann. Math., 9A(5)(1988) 567-574 (in chinese).

[12] F. Boesch and R. Tindell, Circulants and Their Connectivities, Journal of Graph Theory, 8(1984) 487-499.

[13] C. Delorme, O. Fvavaron and M. Maheo, Isomorphisms of Cayley Multigraphs of Degree 4 on Finite Abelian Groups, Europ.J.Combinatorics, 13(1992) 5-8.

[14] D. Witte and J.A. Gallian, A survey: Hamiltonian cycles in Cayley graphs, Discrete Math. 51(1984) 283-304.

[15] N.L. Biggs, Algebraic Graph Theory, Cambridge Univ., Press (1974).

(Received 15/6/94; revised 2/12/96)