# SMALLEST DEFINING SETS FOR 2-( $10,5,4$ ) DESIGNS 

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#### Abstract

A set of blocks which is a subset of blocks of only one design is called a defining set of that design. In this paper we determine smallest defining sets of the 21 nonisomorphic $2-(10,5,4)$ designs.


## 1. Introduction

Let $v, k, t$, and $\lambda$ be a set of natural numbers such that $\lambda>0$ and $v>k>t>0$. A $t-(v, k, \lambda)$ design $D$ is an ordered pair $(X, \mathbb{B})$ where $X$ is a $v$-set and $\mathbb{B}$ is a collection of $k$-subsets (called blocks) of $X$ with the property that every $t$-subset of $X$ appears in exactly $\lambda$ blocks. Each element of $X$ is contained in exactly $r$ blocks.

Let $S$ be a collection of $k$-subsets of $X$. Then we define

$$
\operatorname{Ext}(S)=\{D \mid D \text { is a } t-(v, k, \lambda) \text { design and } S \subset D\}
$$

If $\operatorname{Ext}(S)=\{D\}$, then $S$ is called a defining set for $D$, and is denoted by $d(D)$. A defining set $S$ for $D$ with minimum cardinality among defining sets is called a smallest defining set for $D$. For further definitions and results the interested reader is referred to Gray [1,2]. Following some progress on the subject, Greenhill [3] devised an algorithm for determining a smallest defining set for a given design. This algorithm does not seem to be very efficient for large $v$ 's. The lemma which constitutes the main ingredient of this algorithm is given in Section 2. To obtain the desired results we have utilized the concept of basic trades, which is also described in Section 2. But before that we need a few more definitions and some simple lemmas.

If the full automorphism group of a $t-(v, k, \lambda)$ design $D$ does not contain any transposition, then $D$ is called a Single-Transposition-Free (STF) design.

If $D=(X, \mathbb{B})$ is a $t-(v, k, \lambda)$ design, then $\bar{D}=(X, \overline{\mathbb{B}})$, where $\overline{\mathbb{B}}=\{X-B \mid B \in \mathbb{B}\}$ is a $t-(v, v-k, v-2 r+\lambda)$ design. $\bar{D}$ is called the complement of $D$.
Lemma 1 [1]. If $S$ is a defining set of a $D=(X, \mathbb{B})$, then $\bar{S}=\{X-B \mid B \in S\}$ is a defining set of $\bar{D}$.
Lemma 2. If $S$ is a defining set for $D$ and $S^{c}=\{B \mid B \notin S\}$, then $\operatorname{Aut}(S)=\operatorname{Aut}\left(S^{c}\right)$.

## 2. Trades and defining sets

A $t-(v, k)$ trade $T=\left(T^{+}, T^{-}\right)$consists of two collections of blocks, $T^{+}$and $T^{-}$, such that $T^{+} \cap T^{-}=\phi$, and for every $t$-subset $A \subset X$, the number of blocks containing $A$ in $T^{+}$and $T^{-}$is the same. The number of blocks of $T^{+}\left(T^{-}\right)$is called the volume of $T$. It is clear that the sets of elements of $X$ appearing in $T^{+}\left(T^{-}\right)$must be the same, and is called the foundation set of $T$. We say that the trade $T=\left(T^{+}, T^{-}\right)$is embedded in design $D=(X, \mathbb{B})$ if $T^{+}\left(T^{-}\right) \subseteq \mathbb{B}$. The following lemma demonstrates the connection between trades and defining sets.

Lemma 3 [1]. The intersection of a defining set of a design $D$ with any trade in $D$ is nonempty.

So, naturally, if one could determine the set of all trades of $D$, then smallest defining sets would be in reach. But finding all the trades in a design is a very hard task, since our knowledge about trades with different volumes and foundation sets, at the moment, is very limited. Nevertheless, the structure of the family of trades with volume equal to $2^{t}$, called basic trades, has been completely determined [ $\left.4,5,6\right]$. With every such trade, we associate the following polynomial:

$$
T=\left(T^{+}, T^{-}\right)=\left(S_{1}-S_{2}\right)\left(S_{3}-S_{4}\right) \cdots\left(S_{2 t+1}-S_{2 t+2}\right) S_{2 t+3},
$$

where

$$
\begin{aligned}
& S_{2 t+3} \subset X, \\
& \emptyset \neq S_{i} \subseteq X, \quad 1 \leq i<2 t+3 \\
& S_{i} \cap S_{j}=\phi, \quad i \neq j \\
& \left|S_{2 i-1}\right|=\left|S_{2 i}\right|, \quad 1 \leq i \leq t+1 \\
& \sum_{i=1}^{t+2}\left|S_{2 i-1}\right|=k
\end{aligned}
$$

By multiplying out and removing the parentheses, we take the positive terms as $T^{+}$and negative terms as $T^{-}$. The factor $S_{2 t+3}$ in (2) is called the tail of the trade. The size of the tail varies from 0 to $k-t-1$. Clearly changing the tail does not affect the volume of a $(v, k, t)$ trade. This fact is the basis of the classification of basic trades.

Let $n_{i}=\left|S_{2 i}\right|$ for $i=1,2, \cdots, t+1$ and $l=\left|S_{2 t+3}\right|$. Since $0 \leq l \leq k-t-1$, there are $k-t$ nonisomorphic classes of basic trades. The number of nonisomorphic trades in each class is the number of integer solutions to the following system:

$$
\begin{aligned}
& n_{1}+n_{2}+\cdots n_{t+1}=k-l \\
& 1 \leq n_{1} \leq n_{2} \leq \cdots \leq n_{t+1} .
\end{aligned}
$$

This number is denoted by $p_{t+1}(k-l)$.
For given $(k, t)$, the number of nonisomorphic basic trades is given by

$$
\sum_{l=0}^{k-t-1} p_{t+1}(k-l) .
$$

## 3. $2-(10,5,4)$ designs and their defining sets

The family of $2-(10,5,4)$ designs has been completely classified $[7,8]$. There are exactly 21 nonisomorphic designs in this family, out of which there are eight pairs which are complements of each other. 13 designs are listed in Table 1, and the eight complementary ones are as follows:

$$
D_{2}=\bar{D}_{1}, D_{4}=\bar{D}_{3}, D_{7}=\bar{D}_{6}, D_{9}=\bar{D}_{8}, D_{11}=\bar{D}_{10}, D_{13}=\bar{D}_{12}, D_{17}=\bar{D}_{16}, D_{21}=\bar{D}_{20}
$$

All of these designs are STF (which is useful in efficiently employing Greenhill's Algorithm). For $k=5$ and $t=2$, there are 4 nonisomorphic basic trades as follows:

$$
\begin{aligned}
\text { (i) } & \left(x_{1}-x_{2}\right)\left(x_{3}-x_{4}\right)\left(x_{5}-x_{6}\right) x_{7} x_{8}, \\
\text { (ii) } & \left(x_{1}-x_{2}\right)\left(x_{3}-x_{4}\right)\left(x_{5} x_{7}-x_{6} x_{8}\right) x_{9}, \\
\text { (iii) } & \left(x_{1}-x_{2}\right)\left(x_{3} x_{4}-x_{5} x_{6}\right)\left(x_{7} x_{8}-x_{9} x_{10}\right), \\
\text { (iv) } & \left(x_{1} x_{2} x_{3}-x_{4} x_{5} x_{6}\right)\left(x_{7}-x_{8}\right)\left(x_{9}-x_{10}\right),
\end{aligned}
$$

where $x_{i} \in X$, for $1 \leq i \leq 10$.
Via a computer program, all of the basic trades in the family of $2-(10,5,4)$ designs have been determined. Trades of the types $(i)$ and (iv) do not exist in these designs. Utilizing the remaining types, (ii) and (iii), and Lemma 2, we have improved Greenhill's Algorithm, and consequently have determined smallest defining sets for the entire family.

In Table 1, blocks of smallest defining sets of each design are asterisked; and by Lemma 1, defining sets of the 8 other designs could easily be obtained. In Table 2, the cardinality of the automorphism group of each design, $|G|$, the number of basic trades in each design, $T_{n}$, and the blocks of the basic trades (i.e., blocks of $T^{+}$) of each design are given.
Example. $D_{15}$ contains 18 basic trades, and a smallest set which intersects these trades has six blocks.

Employing Greenhill's Algorithm and Lemma 2 shows that sets of size 6 and 7 can not be defining sets for this design. But we find a set of blocks of size 8 which is a defining set for $D_{15}$ and in fact a smallest one.

$$
d_{S}\left(D_{15}\right)=\{12345,12346,12578,12590,13670,13790,23789,24670\}
$$

## References

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Table 2.

| $D_{i}$ | $\|G\|$ | $T_{n}$ | The blocks numbers of basic trades in $D_{i}^{*}$ |
| :---: | ---: | ---: | :--- |
| 1 | 1 | 11 | $16 \mathrm{AG}, 17 \mathrm{DG}, 25 \mathrm{AF}, 27 \mathrm{CF}, 38 \mathrm{BF}, 39 \mathrm{BH}, 46 \mathrm{EH}, 56 \mathrm{CD}, 57 \mathrm{AC}, 67 \mathrm{AD}, 89 \mathrm{FH}$ |
| 3 | 1 | 10 | $19 \mathrm{AI}, 25 \mathrm{AF}, 27 \mathrm{CF}, 38 \mathrm{BF}, 39 \mathrm{BH}, 46 \mathrm{EH}, 56 \mathrm{CD}, 57 \mathrm{AC}, 67 \mathrm{AD}, 89 \mathrm{FH}$ |
| 5 | 2 | 8 | $2357,23 \mathrm{AC}, 25 \mathrm{AF}, 27 \mathrm{CF}, 35 \mathrm{CF}, 37 \mathrm{AF}, 58 \mathrm{BC}, 78 \mathrm{AB}$ |
| 6 | 2 | 10 | $1358,15 \mathrm{BF}, 34 \mathrm{FG}, 38 \mathrm{BF}, 48 \mathrm{BG}, 57 \mathrm{AC}, 57 \mathrm{HI}, 59 \mathrm{AE}, 79 \mathrm{CE}, \mathrm{ACHI}$ |
| 8 | 2 | 12 | $13 \mathrm{BC}, 18 \mathrm{CF}, 24 \mathrm{BD}, 28 \mathrm{DG}, 34 \mathrm{FG}, 38 \mathrm{BF}, 48 \mathrm{BG}, 57 \mathrm{AC}, 57 \mathrm{HI}, 59 \mathrm{AE}, 79 \mathrm{CE}, \mathrm{ACHI}$ |
| 10 | 6 | 12 | $23 \mathrm{HI}, 28 \mathrm{EH}, 29 \mathrm{BI}, 29 \mathrm{EFF}, 38 \mathrm{BF}, 38 \mathrm{EI}, 39 \mathrm{BH}, 56 \mathrm{CD}, 57 \mathrm{AC}, 67 \mathrm{AD}, 89 \mathrm{FH}, \mathrm{BEFI}$ |
| 12 | 16 | 12 | $31 \mathrm{CD}, 34 \mathrm{FG}, 35 \mathrm{CF}, 35 \mathrm{DG}, 36 \mathrm{CG}, 36 \mathrm{DF}, 45 \mathrm{CG}, 45 \mathrm{DF}, 46 \mathrm{CF}, 46 \mathrm{DG}, 56 \mathrm{CD}, 56 \mathrm{FG}$ |
| 14 | 16 | 12 | $3456,34 \mathrm{CD}, 35 \mathrm{CF}, 35 \mathrm{DG}, 36 \mathrm{CG}, 36 \mathrm{DF}, 45 \mathrm{CG}, 45 \mathrm{DF}, 46 \mathrm{CF}, 46 \mathrm{DG}, 56 \mathrm{FG}, \mathrm{CDFG}$ |
| 15 | 72 | 18 | $1368,14 \mathrm{BD}, 16 \mathrm{BG}, 18 \mathrm{DF}, 25 \mathrm{EI}, 27 \mathrm{EH}, 29 \mathrm{AI}, 29 \mathrm{CH}, 34 \mathrm{FG}, 36 \mathrm{DF}, 38 \mathrm{BG}, 16 \mathrm{DG}$ <br> $48 \mathrm{BF}, 57 \mathrm{AC}, 57 \mathrm{HI}, 59 \mathrm{AE}, 79 \mathrm{CE}, \mathrm{ACHI}$ |
| 16 | 8 | 14 | $25 \mathrm{EI}, 27 \mathrm{EH}, 29 \mathrm{AI}, 29 \mathrm{CH}, 34 \mathrm{FG}, 36 \mathrm{DF}, 38 \mathrm{BG}, 46 \mathrm{DG}, 48 \mathrm{BF}, 57 \mathrm{AC}, 57 \mathrm{HI}, 59 \mathrm{AE}$, <br> $79 \mathrm{CE}, \mathrm{ACHI}$ |
| 18 | 4 | 8 | $1357,13 \mathrm{AC}, 15 \mathrm{AG}, 17 \mathrm{CG}, 35 \mathrm{CG}, 37 \mathrm{AG}, 45 \mathrm{CF}, 47 \mathrm{AF}$ |
| 19 | 8 | 10 | $34 \mathrm{FG}, 36 \mathrm{DF}, 38 \mathrm{BG}, 46 \mathrm{DG}, 48 \mathrm{BF}, 57 \mathrm{AC}, 57 \mathrm{HI}, 59 \mathrm{AE}, 79 \mathrm{CE}, \mathrm{ACHI}$ |
| 20 | 9 | 9 | $15 \mathrm{DI}, 16 \mathrm{AH}, 2358,24 \mathrm{BE}, 37 \mathrm{CI}, 46 \mathrm{CG}, 79 \mathrm{AB}, 89 \mathrm{FH}, \mathrm{DEFG}$ |

* 16AG, for example, means that blocks number $1,6, \mathrm{~A}$, and G form a $T^{+}$of a basic trade in $D_{1}$.

Table 1. A list of 13 nonisomorphic $2-(10,5,4)$ designs.


Note: '*' indicates that the block belongs to a smallest defining set of the design.

