SMALLEST DEFINING SETS FOR 2-(10,5,4) DESIGNS

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Abstract

A set of blocks which is a subset of blocks of only one design is called a defining set of that design. In this paper we determine smallest defining sets of the 21 nonisomorphic 2-(10, 5, 4) designs.

1. Introduction

Let v, k, t, and λ be a set of natural numbers such that $\lambda > 0$ and v > k > t > 0. A t- (v, k, λ) design D is an ordered pair (X, \mathbb{B}) where X is a v-set and \mathbb{B} is a collection of k-subsets (called blocks) of X with the property that every t-subset of X appears in exactly λ blocks. Each element of X is contained in exactly r blocks.

Let S be a collection of k-subsets of X. Then we define

$$Ext(S) = \{D | D \text{ is a } t - (v, k, \lambda) \text{ design and } S \subset D\}.$$

If $Ext(S) = \{D\}$, then S is called a *defining set* for D, and is denoted by d(D). A defining set S for D with minimum cardinality among defining sets is called a *smallest defining set for D*. For further definitions and results the interested reader is referred to Gray [1,2]. Following some progress on the subject, Greenhill [3] devised an algorithm for determining a smallest defining set for a given design. This algorithm does not seem to be very efficient for large v's. The lemma which constitutes the main ingredient of this algorithm is given in Section 2. To obtain the desired results we have utilized the concept of basic trades, which is also described in Section 2. But before that we need a few more definitions and some simple lemmas.

If the full automorphism group of a t- (v, k, λ) design D does not contain any transposition, then D is called a Single-Transposition-Free (STF) design.

If $D = (X, \mathbb{B})$ is a t- (v, k, λ) design, then $\overline{D} = (X, \overline{\mathbb{B}})$, where $\overline{\mathbb{B}} = \{X - B | B \in \mathbb{B}\}$ is a t- $(v, v - k, v - 2r + \lambda)$ design. \overline{D} is called the *complement* of D.

Lemma 1 [1]. If S is a defining set of a $D = (X, \mathbb{B})$, then $\overline{S} = \{X - B | B \in S\}$ is a defining set of \overline{D} .

Lemma 2. If S is a defining set for D and $S^c = \{B | B \notin S\}$, then $\operatorname{Aut}(S) = \operatorname{Aut}(S^c)$.

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2. Trades and defining sets

A t-(v, k) trade $T = (T^+, T^-)$ consists of two collections of blocks, T^+ and T^- , such that $T^+ \cap T^- = \phi$, and for every t-subset $A \subset X$, the number of blocks containing A in T^+ and T^- is the same. The number of blocks of $T^+(T^-)$ is called the *volume* of T. It is clear that the sets of elements of X appearing in $T^+(T^-)$ must be the same, and is called the *foundation set* of T. We say that the trade $T = (T^+, T^-)$ is embedded in design $D = (X, \mathbb{B})$ if $T^+(T^-) \subseteq \mathbb{B}$. The following lemma demonstrates the connection between trades and defining sets.

Lemma 3 [1]. The intersection of a defining set of a design D with any trade in D is nonempty.

So, naturally, if one could determine the set of all trades of D, then smallest defining sets would be in reach. But finding all the trades in a design is a very hard task, since our knowledge about trades with different volumes and foundation sets, at the moment, is very limited. Nevertheless, the structure of the family of trades with volume equal to 2^t , called *basic* trades, has been completely determined [4,5,6]. With every such trade, we associate the following polynomial:

$$T = (T^+, T^-) = (S_1 - S_2)(S_3 - S_4) \cdots (S_{2t+1} - S_{2t+2})S_{2t+3},$$

where

$$S_{2i+3} \subset X, \emptyset \neq S_i \subseteq X, \quad 1 \le i < 2t + 3, S_i \cap S_j = \phi, \quad i \ne j, |S_{2i-1}| = |S_{2i}|, \quad 1 \le i \le t + 1, \sum_{i=1}^{i+2} |S_{2i-1}| = k.$$

By multiplying out and removing the parentheses, we take the positive terms as T^+ and negative terms as T^- . The factor S_{2t+3} in (2) is called the *tail* of the trade. The size of the tail varies from 0 to k - t - 1. Clearly changing the tail does not affect the volume of a (v, k, t) trade. This fact is the basis of the classification of basic trades.

Let $n_i = |S_{2i}|$ for $i = 1, 2, \dots, t+1$ and $l = |S_{2t+3}|$. Since $0 \le l \le k-t-1$, there are k-t nonisomorphic classes of basic trades. The number of nonisomorphic trades in each class is the number of integer solutions to the following system:

$$n_1 + n_2 + \dots + n_{t+1} = k - l$$

 $1 \le n_1 \le n_2 \le \dots \le n_{t+1}.$

This number is denoted by $p_{t+1}(k-l)$.

For given (k, t), the number of nonisomorphic basic trades is given by

$$\sum_{l=0}^{k-t-1} p_{t+1}(k-l).$$

3. 2-(10,5,4) designs and their defining sets

The family of 2-(10, 5, 4) designs has been completely classified [7,8]. There are exactly 21 nonisomorphic designs in this family, out of which there are eight pairs which are complements of each other. 13 designs are listed in Table 1, and the eight complementary ones are as follows:

 $D_2 = \overline{D}_1, \ D_4 = \overline{D}_3, \ D_7 = \overline{D}_6, \ D_9 = \overline{D}_8, \ D_{11} = \overline{D}_{10}, \ D_{13} = \overline{D}_{12}, \ D_{17} = \overline{D}_{16}, \ D_{21} = \overline{D}_{20}.$

All of these designs are STF (which is useful in efficiently employing Greenhill's Algorithm). For k = 5 and t = 2, there are 4 nonisomorphic basic trades as follows:

(i) $(x_1 - x_2)(x_3 - x_4)(x_5 - x_6)x_7x_8,$ (ii) $(x_1 - x_2)(x_3 - x_4)(x_5x_7 - x_6x_8)x_9,$ (iii) $(x_1 - x_2)(x_3x_4 - x_5x_6)(x_7x_8 - x_9x_{10}),$ (iv) $(x_1x_2x_3 - x_4x_5x_6)(x_7 - x_8)(x_9 - x_{10}),$

where $x_i \in X$, for $1 \leq i \leq 10$.

Via a computer program, all of the basic trades in the family of 2-(10, 5, 4) designs have been determined. Trades of the types (i) and (iv) do not exist in these designs. Utilizing the remaining types, (ii) and (iii), and Lemma 2, we have improved Greenhill's Algorithm, and consequently have determined smallest defining sets for the entire family.

In Table 1, blocks of smallest defining sets of each design are asterisked; and by Lemma 1, defining sets of the 8 other designs could easily be obtained. In Table 2, the cardinality of the automorphism group of each design, |G|, the number of basic trades in each design, T_n , and the blocks of the basic trades (i.e., blocks of T^+) of each design are given.

Example. D_{15} contains 18 basic trades, and a smallest set which intersects these trades has six blocks.

Employing Greenhill's Algorithm and Lemma 2 shows that sets of size 6 and 7 can not be defining sets for this design. But we find a set of blocks of size 8 which is a defining set for D_{15} and in fact a smallest one.

 $d_S(D_{15}) = \{12345, 12346, 12578, 12590, 13670, 13790, 23789, 24670\}.$

References

- K. Gray, On the minimum number of blocks defining a design, Bull. Austral. Math. Soc. 41(1990), 97-112.
- Defining sets of single-transposition-free designs, Utilitas Math. 38(1990), 97-103.
- C.S. Greenhill, An algorithm for finding smallest defining sets of t-designs, J. Combin. Math. Combin. Comput. 14(1993), 39-60.

- 4. H.L. Hwang, On the structure of (v,k,t) trades, J. Stat. Plann. Inference 13(1986), 179-191.
- 5. G.B. Khosrovshahi, D. Majumdar, and M. Widel, On the structure of basic trades, J. Combin. Information System Sci. 17(1992), 102-107.
- 6. ____, and N.M. Singhi, Further characterization of basic trades, J. Stat. Plann. Inference, to appear.
- N.M. Singhi, (19,9,4) Hadamard designs and their residual designs, J. Combin. Theory ser. A 16(1974), 241-252.
- 8. J.H. van Lint, and H.C.A. van Tilborg, and J.R.Wiekema, *Block designs* with v = 10, k = 5, $\lambda = 4$, J. Combin. Theory Ser. A **23**(1977), 105-115.

G	$T_n \mid$	The blocks numbers of basic trades in D_i^* .
1	11	16AG,17DG,25AF,27CF,38BF,39BH,46EH,56CD,57AC,67AD,89FH
1	10	19AI,25AF,27CF,38BF,39BH,46EH,56CD,57AC,67AD,89FH
2	8	2357,23AC,25AF,27CF,35CF,37AF,58BC,78AB
2	10	1358,15BF,34FG,38BF,48BG,57AC,57HI,59AE,79CE,ACHI
2	12	13BC,18CF,24BD,28DG,34FG,38BF,48BG,57AC,57HI,59AE,79CE,ACHI
6	12	23HI,28EH,29BI,29EF,38BF,38EI,39BH,56CD,57AC,67AD,89FH,BEFI
16	12	34CD,34FG,35CF,35DG,36CG,36DF,45CG,45DF,46CF,46DG,56CD,56FG
16	12	3456,34CD,35CF,35DG,36CG,36DF,45CG,45DF,46CF,46DG,56FG,CDFG
72	18	1368,14BD,16BG,18DF,25EI,27EH,29AI,29CH,34FG,36DF,38BG,16DG
		48BF,57AC,57HI,59AE,79CE,ACHI
8	14	25EI,27EH,29AI,29CH,34FG,36DF,38BG,46DG,48BF,57AC,57HI,59AE,
		79CE,ACHI
4	8	1357,13AC,15AG,17CG,35CG,37AG,45CF,47AF
8	10	34FG,36DF, 38BG,46DG,48BF,57AC,57HI,59AE,79CE,ACHI
9	9	15DI,16AH,2358,24BE,37CI,46CG,79AB,89FH,DEFG
	$ \begin{array}{r} 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 6 \\ 16 \\ 16 \\ 72 \\ 8 \\ 4 \\ 8 \end{array} $	$\begin{array}{cccccccccccccccccccccccccccccccccccc$

Table 2.

* 16AG, for example, means that blocks number 1,6, A, and G form a T^+ of a basic trade in D_1 .

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ſ		D_1	D_3	D_5	D_6	D_8	D ₁₀	D ₁₂	D ₁₄	D ₁₅	D ₁₆	D ₁₈	D ₁₉	D ₂₀
Γ	1	12345	12345*	12345^{*}	12345*	12345^{*}	12345^{*}	12345*	,12345*	12345*	12345*	12345*	12345*	12345*
	2	12346	12346*	12346*	12346*	12346^{*}	12346^{*}	12346*	12346*	12346*	12346	12346*	12346	12360*
	3	12678^{*}	12678*	12678*	12678*	12678*	12689*	12579^{*}	12579^{*}	12578*	12579^{*}	12570	12579*	12489*
	4	12890*	12890*	12890 *	12890	12890	12890	12580^{*}	12580*	12590*	12580*	12589	12580*	12780*
	5	13578^{*}	13580*	13580	13589*	13589*	13578^{*}	13679*	13670	13670*	13670*	13679*	13689*	13568
	6	13790*	13790	13790*	13790	13790	13790	13680	13689	13790*	13790	13890*	13790	13790*
	7	14590*	14579*	14570	14570*	14570*	14580*	14789*	14789*	14689	14689	14690*	14670	14567
	8	14689*	14689*	14689*	14679*	14690	14670*	14780*	14780*	14890	14890	14780	14890*	14590
	9	15670*	15670	15679	15680	15678	15679	15690	15690	15678	15678*	15678	15678*	16789*
	A	23580	23578	23589	23570	23570	23590	23790	23790	23689	23689	23678	23670*	23478
	B	23690	23690	23670	23690	23679^{*}	23670	23890	23890	23789*	23789	23790*	23789	23579
	C	24579	24590	24579	24589	24589	24579	24670	24670	24670*	24670	24680	24689	24690
	D	24780	24780	24780	24780	24780	24780	24689	24689	24780	24780	24789	24780	25670
	Е	25679	25679	25690	25679	25690	25678	25678	25678	25690	25690	25690	25690	25689
	F	34670	34670	34690	34680	34680	34689	34570	34579*	34579	34578	34570	34578	34679*
	G	34789	34789	34789	34789	34789	34789	34589	34580	34580	34590*	34589	34590	34680
	H	35689	35689	35678	35678	35680*	35680	35678	35678	35680	35680*	35680	35680	35890
	I	45680	45680	45680	45690*	45679	45690*	45690	45690	45679	45679	45679	45679	45780

Table 1. A list of 13 nonisomorphic 2-(10, 5, 4) designs.

Note: "" indicates that the block belongs to a smallest defining set of the design.

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