BOUNDS OF EDGE-NEIGHBOR-INTEGRITY OF GRAPHS

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Abstract. Let G be a graph. An edge subversion strategy of G is a set of edges T in G whose incident vertices are deleted from G. The survival-subgraph is denoted by G/T. The edge-neighbor-integrity of G, ENI(G), is defined to be ENI(G) $= \min_{T \subseteq E(G)} \{|T| + \omega(G/T)\}$, where T is any edge subversion strategy of G, and $\omega(G/T)$ is the maximum order of the components of G/T. In this paper, we find the lower and upper bounds of ENI for all graphs related to some well-known graphic parameters, and we also discuss some properties of the graphs with ENI equal to the bounds.

I. Introduction

The integrity and the edge-integrity were introduced by Barefoot, Entringer, and Swart as a measure of the vulnerability of graphs to disruption caused by the removal of vertices or edges. [1,2] Goddard and Swart investigated further the bounds and properties of the integrity of the graphs. [8]

A spy network can be modeled by a graph whose vertices represent the stations and whose edges represent the lines of communication. If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to network as a whole. [9] Therefore instead of considering the integrity of a communication graph, in [6,7] we discussed the vertex-neighbor-integrity of graphs — a measure of the vulnerability of graphs to disruption caused by the removal of vertices and all of their adjacent vertices. Similarly, we can consider the edge analogue of (vertex)-neighbor-integrity — a measure of the vulnerability of graphs to disruption caused by the removal of edges, their incident vertices, and all of their incident edges. [4]

Let G = (V,E) be a graph. The *integrity* of G, I(G), is defined to be

$$I(G) = \min_{S \subseteq V(G)} \{|S| + m(G - S)\},$$

where $m(G - S)$ is the maximum order of the components of G−S. A subset S′ of

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V is called an *I-set* of G if $I(G) = |S'| + m(G - S')$. The *edge-integrity* of G, $I'(G)$, is defined to be

$$I'(G) = \min_{T \subseteq E(G)} \{|T| + m(G - T)\}.$$

A subset T' of E is called an *I'-set* of G if $I'(G) = |T'| + m(G - T')$.

Let u be a vertex in G. $N(u) = \{v \in V(G) | v \neq u,\ v \text{ and } u \text{ are adjacent}\}$ is the *open neighborhood* of u, and $N[u] = \{u\} \cup N(u)$ denotes the *closed neighborhood* of u. A vertex u in G is said to be *subverted* if the closed neighborhood $N[u]$ is deleted from G. A set of vertices $S = \{u_1, u_2, ..., u_m\}$ is called a *vertex subversion strategy* of G if each of the vertices in S has been subverted from G. Let G/S be the *survival-subgraph* when S has been a vertex subversion strategy of G. The *closed neighborhood of a vertex subset* S, N[S], is $\cup_{u \in S} N[u]$. Hence $G/S = G - N[S] = G - (\cup_{u \in S} N[u])$. The *vertex-neighbor-integrity* of a graph G, VNI(G), is defined to be

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\},$$

where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S. A subset S^* of V is called a *VNI-set* of G if $VNI(G) = |S^*| + \omega(G/S^*)$.

Let $e = [v, w]$ be an edge in G. The edge $e = [v, w]$ is said to be *subverted* if the edge e, all of its incident edges, and the two ends of e, v and w, are removed from G. (For simplicity, an edge $e = [v, w]$ is subverted if the two ends of the edge e, v and w, are deleted from G.) A set of edges $T = \{e_1, e_2, ..., e_r\}$ is called an *edge subversion strategy* of G if each of the edges in T has been subverted from G. Let G/T be the survival-subgraph when T has been an edge subversion strategy of G. The *edge-neighbor-integrity* of a graph G, ENI(G), is defined to be

$$ENI(G) = \min_{T \subseteq E(G)} \{|T| + \omega(G/T)\},$$

where T is any edge subversion strategy of G, and $\omega(G/T)$ is the maximum order of the components of G/T. A subset T^* of E is called an *ENI-set* of G if $ENI(G) = |T^*| + \omega(G/T^*)$.

$\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Example 1.1: $K_{1,n-1}$, where $n \geq 3$, is a star. By the definitions, it is clear that $I(K_{1,n-1}) = 2$, $I'(K_{1,n-1}) = n$, $VNI(K_{1,n-1}) = 1$, and $ENI(K_{1,n-1}) = 2$.

Example 1.2: P_n, where $n \geq 2$, is a path with n vertices. We have known that $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$ (ref. [1]), $I'(P_n) = \lceil 2\sqrt{n} \rceil - 1$ (ref. [1]), $VNI(P_n) = \lceil 2\sqrt{n+3} \rceil - 4$ (ref. [6]), and $ENI(P_n) = \lceil 2\sqrt{n+2} \rceil - 3$ (ref. [4]).

Example 1.3: K_n, where $n \geq 1$, is a complete graph. It is clear that $I(K_n) = I'(K_n) = n$, $VNI(K_n) = 1$, and $ENI(K_n) = \lceil n/2 \rceil$.

In Section III and Section IV, we find the lower and upper bounds of ENI for all graphs related to some well-known graphic parameters. (Hence for the completeness of the paper, we present the related graphic parameters and properties in Section II.) Furthermore, we discuss some properties of the graphs with ENI equal to the bounds.

II. Related Graphic Parameters and Basic Properties

In this section, we present the related graphic parameters and some basic properties. All other undefined terminology and notations are taken from [3].

Let $G = (V,E)$ be a graph and $T = \{e_1, e_2, ..., e_r\}$ be a subset of E. T is called an *edge cut strategy* of G if the survival-subgraph G/T is disconnected, or is a single vertex, or is $\emptyset$. The *edge-neighbor-connectivity* of G, $\Lambda(G)$, is defined to be the minimum size of all edge cut strategies T of G. [5]

A subset C of V is called a *covering* of G if every edge of G has at least one end in C. A covering C is a *minimum covering* if G has no covering C′ with $|C'| < |C|$. The *covering number* of G, $\alpha_0(G)$, is the number of vertices in a minimum covering of G.

A subset I of V is called an *independent set* of G if no two vertices of I are adjacent in G. An independent set I is *maximum* if G has no independent set I′ with $|I'| > |I|$. The *independence number* of G, $\beta_0(G)$, is the number of vertices in a maximum independent set of G.

A subset M of E is called a *matching* in G if no two edges of M are incident in G. A matching M is *maximum* if G has no matching M′ with $|M'| > |M|$. Let $\beta_1(G)$ be the number of edges in a maximum matching in G.

A subset L of E is called an *edge covering* of G if each vertex of G is an end of some edge in L. An edge covering L is a *minimum edge covering* if G has no edge covering L′ with $|L'| < |L|$. The *edge covering number* of G, $\alpha_1(G)$, is the number of edges in a minimum edge covering of G.

The following properties will be used later.

Lemma 2.1: For any graph G, $\alpha_0(G) + \beta_0(G) = |V(G)|$. [3]

Lemma 2.2: For any graph G, $\beta_1(G) \leq \alpha_0(G)$. [3]

Lemma 2.3: Let G = (V,E) be a graph, and T be an edge subset of E. Then $\Lambda(G) \leq \Lambda(G/T) + |T|$. [5]

Lemma 2.4: Let G = (V,E) be a graph. Then $\Lambda(G) \leq \lfloor |V|/2 \rfloor$. [5]

Lemma 2.5: Let G = (V,E) be a graph, and T^* be an edge subset of E. Then $ENI(G) \leq ENI(G/T^*) + |T^*|$.

Proof: Let T' be an ENI-set of G/T^* and $T^{**} = T' \cup T^*$, then $|T^{**}| = |T'| + |T^*|$ and $G/T^{**} = G/(T' \cup T^*) = (G/T^*)/T'$.

$$\begin{aligned} ENI(G) &= \min_{T \subseteq E(G)} \{|T| + \omega(G/T)\} \\ &\leq |T^{**}| + \omega(G/T^{**}) \\ &= |T'| + |T^*| + \omega((G/T^*)/T') \\ &= ENI(G/T^*) + |T^*|. \end{aligned}$$

QED.

III. Lower Bounds of Edge-Neighbor-Integrity

For any graph G = (V,E), $\Lambda(G)$, VNI(G), and $\lceil I(G)/2 \rceil$ are all lower bounds of ENI(G).

Lemma 3.1: Let G = (V,E) be a graph and T^* be an ENI-set of G. Then T^* is an edge cut strategy of G.

Proof: If G is complete and T^* is an ENI-set of G, then G/T^* is a single vertex or $\emptyset$. Hence T^* is an edge cut strategy of G.

If G is incomplete and T^* is an ENI-set of G, we assume that T^* is not an edge cut strategy of G. So G/T^* is a connected graph with $|V(G/T^*)| \geq 2$. Then there is an edge e in G/T^* and $\omega(G/T^*) \geq \omega(G/(T^* \cup \{e\})) + 2$. Since $T^* \cup \{e\}$ is an edge subset of E(G), we have

$$\begin{aligned} ENI(G) &= \min_{T \subseteq E(G)} |T| + \omega(G/T) \\ &= |T^*| + \omega(G/T^*) \\ &\geq |T^*| + \omega(G/(T^* \cup \{e\})) + 2 \\ &= |T^* \cup \{e\}| + \omega(G/(T^* \cup \{e\})) + 1 \\ &\geq ENI(G) + 1 > ENI(G), \end{aligned}$$

a contradiction. Therefore T^* is an edge cut strategy of G. QED.

Theorem 3.2: For any graph G = (V,E), $\Lambda(G) \leq ENI(G)$.

Proof: Let T^* be an ENI-set of G. By Lemma 3.1, T^* is an edge cut strategy of G, so $\Lambda(G) \leq |T^*| \leq |T^*| + \omega(G/T^*) = ENI(G)$. QED.

Corollary 3.3: For any graph G = (V,E), if $ENI(G) = \Lambda(G)$, then every ENI-set T^* of G is a minimum edge cut strategy of G and $G/T^* = \emptyset$.

Proof: Let T^* be an ENI-set of G. By Lemma 3.1, T^* is an edge cut strategy of G, so $\Lambda(G) \leq |T^*|$.

Since $ENI(G) = \Lambda(G)$, we have $\Lambda(G) = |T^*| + \omega(G/T^*)$, and $|T^*| \leq \Lambda(G)$.

Therefore $|T^*| = \Lambda(G)$ and $\omega(G/T^*) = 0$. That is, T^* is a minimum edge cut strategy of G and $G/T^* = \emptyset$. QED.

Theorem 3.4: For any graph G = (V,E), $VNI(G) \leq ENI(G)$.

Proof: Let $T^* = \{[u_1, v_1], [u_2, v_2], ..., [u_r, v_r]\}$ be an ENI-set and S^* be a set of one end of each edge in T^*. Then $|S^*| \leq |T^*|$ and $\{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\} \subseteq N[S^*]$. Thus $G/S^* = G - N[S^*] \subseteq G - \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\} = G/T^*$, and $|S^*| + \omega(G/S^*) \leq |T^*| + \omega(G/T^*) = ENI(G)$. Therefore

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\} \leq |S^*| + \omega(G/S^*) \leq ENI(G). \qquad \text{QED.}$$

Corollary 3.5: If ENI(G) = VNI(G), then every ENI-set T^* of G must be a matching in G.

Proof: Let $T^* = \{[u_1, v_1], [u_2, v_2], ..., [u_r, v_r]\}$ be an ENI-set of G. Assume that T^* is not a matching, so w.l.o.g., let $u_j = u_k$, for some $j \neq k$, and $S^* = \{u_i | [u_i, v_i] \in T^*$, where $i = 1, 2, ..., r\}$, so $|S^*| < |T^*| = r$, and $\{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\} \subseteq N[S^*]$.

$$\begin{aligned} VNI(G) &= \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\} \\ &\leq |S^*| + \omega(G/S^*) \\ &= |S^*| + m(G - N[S^*]) \\ &\leq |S^*| + m(G - \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}) \\ &< |T^*| + \omega(G/T^*) = ENI(G), \end{aligned}$$

a contradiction. Therefore T^* must be a matching in G. QED.

The converse of the above corollary is not true, see the following example:

Example 3.1: Let C_6 = (V,E), where $V = \{v_i | 1 \leq i \leq 6\}$, and $E = \{e_i | e_i = [v_i, v_{i+1}], 1 \leq i \leq 6, \text{the addition is taken modulo } 6\}$.

$T_1 = \{e_1, e_4\}$, $T_2 = \{e_2, e_5\}$, $T_3 = \{e_3, e_6\}$, $T_4 = \{e_1, e_3, e_5\}$, and $T_5 = \{e_2, e_4, e_6\}$ are all ENI-sets of G, and T_1, T_2, T_3, T_4, and T_5 are matchings in G. But VNI(G) = 2 $\neq$ ENI(G) = 3.

Theorem 3.5: For any graph G = (V,E), $\lceil I(G)/2 \rceil \leq ENI(G)$.

Proof: Let $T^*=\{[u_1, v_1], [u_2, v_2], ..., [u_r, v_r]\}$ be an ENI-set of G, so ENI(G) = $|T^*| + \omega(G/T^*) = r + \omega(G/T^*)$. Let $S^* = \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}$. Since T^* may not be a matching in G, $|S^*| \leq 2r$.

$$\begin{aligned} I(G) &= \min_{S \subseteq V(G)} \{|S| + m(G - S)\} \\ &\leq |S^*| + m(G - S^*) \\ &\leq 2r + \omega(G/T^*) \\ &\leq 2(r + \omega(G/T^*)) = 2 \cdot ENI(G). \end{aligned}$$

Therefore $\lceil I(G)/2 \rceil \leq ENI(G)$. QED.

IV. Upper Bounds of Edge-Neighbor-Integrity

The integrity and the edge-integrity are upper bounds of the edge-neighbor-integrity as described below:

Theorem 4.1: For any graph G = (V,E), $ENI(G) \leq I(G) \leq I'(G)$.

Proof: It is easy to obtain $I(G) \leq I'(G)$. [2]

If G is complete, then $ENI(G) = \lceil |V|/2 \rceil \leq |V| = I(G)$.

Now we assume that G is incomplete and let $S^* = \{u_1, u_2, ..., u_r\}$ be an I-set of G. Then S^* is a vertex cut-set of G (ref. [8]), and u_i, where $1 \leq i \leq r$, is not an isolated vertex of G. Let $T^* = \{[u_i, v_i] \in E(G) |$ for some vertex $v_i \in V, u_i \in S^*$, where $i = 1, 2, ..., r\}$, then $|T^*| = |S^*| = r$.

$$\begin{aligned} G/T^* &= G - \{u_1, u_2, ..., u_r, v_1, v_2, ... v_r\} \\ &= G - (S^* \cup \{v_i \in V(G) | [u_i, v_i] \in T^*, u_i \in S^*\}) \subseteq G - S^*, \end{aligned}$$

and hence $\omega(G/T^*) \leq m(G - S^*)$.

$$\begin{aligned} ENI(G) &= \min_{T \subseteq E(G)} \{|T| + \omega(G/T)\} \\ &\leq |T^*| + \omega(G/T^*) \\ &\leq |S^*| + m(G - S^*) = I(G). \end{aligned}$$

QED.

Lemma 4.2: Let G = (V,E) be a graph. Then $I(G) \le I(G - S_1) + |S_1|$, for any vertex subset $S_1 \subseteq V$.

Proof: Let S^* be an I-set of $G - S_1$. Then

$$\begin{aligned} I(G) &= \min_{S \subseteq V(G)} \{|S| + m(G - S)\} \\ &\le |S_1 \cup S^*| + m(G - (S_1 \cup S^*)) \\ &= |S_1| + |S^*| + m((G - S_1) - S^*) \\ &= |S_1| + I(G - S_1). \qquad \text{QED.} \end{aligned}$$

We can improve the upper bound of ENI(G), I(G), as described below. Let S be an I-set of G, and $M = \{e_1, e_2, ..., e_r\}$ be a maximum matching in $< S >$, the induced subgraph of G by S. Then we have the following theorem.

Theorem 4.3: For any graph G = (V,E), $ENI(G) \le I(G) - r$.

Proof: Let $e_i = [u_i, v_i]$, where u_i, v_i are in S, $i = 1, 2, , 3, ..., r$. Let $S' = S - \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}$.

$$\begin{aligned} G - S &= G - (S' \cup \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}) \\ &= (G - \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}) - S' = (G/M) - S'. \end{aligned}$$

Since S is an I-set of G, $I(G) = |S| + m(G - S) = |S'| + 2r + m((G/M) - S')$.

$$\begin{aligned} I(G/M) &= I(G - \{u_1, u_2, ..., u_r, v_1, v_2, ..., v_r\}) \\ &\ge I(G) - 2r \qquad \text{(by Lemma 4.2)} \\ &= |S'| + m((G/M) - S') \ge I(G/M). \end{aligned}$$

Hence $I(G/M) = |S'| + m((G/M) - S')$, and S' is an I-set of G/M.

$$\begin{aligned} ENI(G) &\le ENI(G/M) + r \qquad \text{(by Lemma 2.5)} \\ &\le I(G/M) + r \qquad \text{(by Theorem 4.1)} \\ &= |S'| + m((G/M) - S') + r \\ &= I(G) - r. \qquad \text{QED.} \end{aligned}$$

Corollary 4.4: Let G = (V,E) be a graph. If ENI(G) = I(G), then the induced subgraph of G, $< S >$, must be a null graph, where S is an I-set of G.

Proof: Let $S = \{v_1, v_2, ..., v_r\}$ be an I-set of G. If there is an edge in $< S >$, then by Theorem 4.3, $ENI(G) \le I(G) - 1$, a contradiction. Therefore $< S >$ must be a null graph. QED.

The converse of the above corollary is not true, as shown in the following example:

Example 4.1: The graph G is shown in Figure 4.1. S = $\{a, b, c\}$ is an I-set of G. I(G) = $|S| + m(G - S) = 3 + 2 = 5$. T = $\{e_1, e_2\}$ is an ENI-set of G. ENI(G)=$|T| + \omega(G/T) = 2 + 1 = 3$. $< S >$ is a null graph with 3 vertices, but ENI(G) $\neq$ I(G).

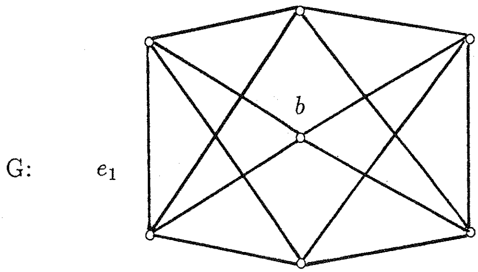

Figure 4.1

Next we describe the relationships between the edge-neighbor-integrity and the graphic parameters, α_0, α_1, β_0, and β_1.

Theorem 4.5: For any graph G = (V,E), ENI(G) $\leq \alpha_1$(G).

Proof: Let L be a minimum edge covering of G. Since each vertex of G is an end of some edge in L, G/L=$\emptyset$ and $\omega(G/L) = 0$. Hence

$$\begin{aligned} \text{ENI}(G) &= \min_{T \subseteq E(G)} \{|T| + \omega(G/T)\} \\ &\leq |L| + \omega(G/L) = \alpha_1(G). \end{aligned}$$ QED.

Theorem 4.6: For any graph G = (V,E), ENI(G) $\leq \beta_1(G) + 1$.

Proof: Let M be a maximum matching in G. G/M=$\emptyset$ or a set of isolated vertices, since otherwise we may get a matching with the size larger than |M|. Thus,

$$\begin{aligned} \text{ENI}(G) &= \min_{T \subseteq E(G)} \{|T| + \omega(G/T)\} \\ &\leq |M| + \omega(G/M) \leq \beta_1(G) + 1. \end{aligned}$$ QED.

By the above theorem, it is easy to get an upper bound, $\lceil |V(G)|/2 \rceil$, of the edge-neighbor-integrity.

Corollary 4.7: For any graph G = (V,E), ENI(G) $\leq \lceil |V|/2 \rceil$.

Proof: Let M be a maximum matching, so $|M| = \beta_1(G) \leq \lfloor |V|/2 \rfloor$.

(i) If $\beta_1(G) = \lfloor |V|/2 \rfloor$, then G/M = $\emptyset$ (if |V| is even), or a single vertex (if |V| is odd), and

$$ENI(G) \leq |M| + \omega(G/M)$$

$$= \begin{cases} \lfloor \frac{|V|}{2} \rfloor = \lceil \frac{|V|}{2} \rceil, & \text{if } |V| \text{ is even;} \\ \lfloor \frac{|V|}{2} \rfloor + 1 = \lceil \frac{|V|}{2} \rceil, & \text{if } |V| \text{ is odd.} \end{cases}$$

(ii) If $\beta_1(G) < \lfloor |V|/2 \rfloor$, then by Theorem 4.6,

$$ENI(G) \leq \beta_1(G) + 1 < \lfloor \frac{|V|}{2} \rfloor + 1.$$

Therefore

$$ENI(G) \leq \lfloor \frac{|V|}{2} \rfloor \leq \lceil \frac{|V|}{2} \rceil. \qquad \text{QED.}$$

Theorem 4.8: For any graph G = (V,E), ENI(G) $\leq \alpha_0(G) + 1$.

Proof: By Lemma 2.2 and Theorem 4.6, we obtain that ENI(G)$\leq \alpha_0(G)+1$. QED.

As described above, α_1, $\alpha_0 + 1$, and $\beta_1 + 1$ are upper bounds of ENI. However, the independence number, β_0, has no such a relationship with ENI. See the following examples:

Example 4.2: K_n is a complete graph with n vertices. $\beta_0(K_n) = 1$ and ENI(K_n) = $\lceil n/2 \rceil$. ENI(K_n) $> \beta_0(K_n) + 1 > \beta_0(K_n)$, if $n \geq 5$.

Example 4.3: $K_{1,n-1}$, where $n \geq 3$, is a star. $\beta_0(K_{1,n-1}) = n-1$ and ENI($K_{1,n-1}$) = 2. ENI($K_{1,n-1}$) $< \beta_0(K_{1,n-1}) < \beta_0(K_{1,n-1}) + 1$, if $n \geq 4$.

Example 4.4: $K_{n,m}$ is a complete bipartite graph with a bipartition (X,Y), where $|X| = n$ and $|Y| = m$. $\beta_0(K_{n,m}) = \max(n, m)$ and

$$ENI(K_{n,m}) = \begin{cases} n = m = \beta_0(K_{n,m}), & \text{if } n = m; \\ \min(n, m) + 1 < \beta_0(K_{n,m}) + 1, & \text{if } n \neq m. \end{cases}$$

Example 4.5: ENI(C_7) $= 4 = \beta_0(C_7) + 1$.

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