The Maximum Degree of a Critical Graph of Diameter Two

Louis Caccetta and Jing Huang

School of Mathematics and Statistics
GPO Box U. 1987, Curtin University
Perth, 6001, Australia

Abstract

A graph of diameter k is *critical* if the deletion of any vertex results in a graph of diameter at least $k+1$. It was conjectured by Boals et. al. that the maximum degree of a critical graph of diameter 2 on n vertices is at most $\frac{1}{2}n$. Here we disprove the conjecture by showing that for $n \geq 7$ the maximum degree Δ_n possible in such a graph is $n-4$ when n is odd and $n-5$ when n is even. We specifically construct for each $n \geq 7$ a critical graph of maximum degree Δ_n. Our graphs provide a good lower bound for the maximum number of edges possible in such a graph. We also show, by construction, that for any integers n and p with $n \geq 7$ and $min\{\sqrt{4n+5}-2, \frac{1}{4}(n+3)\} \leq p \leq \Delta_n$, there exists a critical graph on n vertices of diameter 2 and maximum degree p. This implies that for any $p \geq 5$ there exists a critical graph of diameter 2 and maximum degree p on $\lfloor \frac{1}{4}(p^2+4p-1) \rfloor$ vertices.

1 Introduction

Modern communication networks often face message delay problems caused by the unavailability (through failure or occupancy) of its components or junctions. Graph theory concepts are useful in analyzing the efficiency and reliability of such networks. In many network applications, the problem that arises is to construct a network which satisfies certain requirements and which is optimal according to some criterion such as cost, output, or performance. Graph theory is particularly useful when the requirements of the network can be expressed in terms of graph parameters. For example, the diameter $d(G)$ of a graph G, which is defined as the maximum distance in G, provides a measure of the efficiency of the underlying network; $d(G)$ is just the maximum number of links needed to connect any two points (components) in the network. Thus studying graph parameters can provide useful information. In this paper, we focus on the parameter $d(G)$. In characterizing graphs with prescribed diameter, it is fruitful to consider a subclass of graphs, the so called critical graphs.

Australasian Journal of Combinatorics 16(1997), pp.245–258

Let G be a connected graph. The distance, $d_G(u,v)$, between the vertices u and v in G is the length of a shortest (u,v)-path in G. The *diameter* $d(G)$ is thus

$$d(G) = max\{d_G(u,v) : \ u,v \in V(G)\}$$

A vertex v of G is *critical* if $d(G-v) > d(G)$. If every vertex of G is critical, then G is called *critical*. Critical graphs have been studied in [1] - [6]. Gliviak [4] observed that a graph G with minimum degree at least two and girth (the length of a shortest cycle) at least $d(G)+3$ is a critical graph. A well-known result is that every critical graph is a block [1, 4, 6].

Critical graphs of small diameters have received particular attention. In this paper, we mainly consider those of diameter 2. Denote by $\mathcal{G}(n,p)$ the class of all critical graphs on n vertices of diameter 2 and maximum degree p. For simplicity, we shall refer to each graph in $\mathcal{G}(n,p)$ as an (n,p)-*critical* graph. One natural question to ask is:

For which values of n and p, is $\mathcal{G}(n,p) \neq \emptyset$?

Boals et. al. [1] conjectured that $\mathcal{G}(n,p) = \emptyset$ for all $p > \frac{1}{2}n$. An affirmative answer to the conjecture would imply that every critical graph contains at most $\frac{1}{4}n^2$ edges. Unfortunately the conjecture is false. For $n \geq 7$, we show that the maximum number Δ_n for which $\mathcal{G}(n,\Delta_n) \neq \emptyset$ is $n-4$ when n is odd and $n-5$ when n is even. In general, we show, by construction, that $\mathcal{G}(n,p) \neq \emptyset$ for all n and p with $n \geq 7$ and $min\{\sqrt{4n+5}-2, \frac{1}{4}(n+3)\} \leq p \leq \Delta_n$.

Let G be a graph. Two adjacent vertices x and y will be denoted by $x \sim y$. For each vertex v, we shall use $L_i(v)$ $(i \geq 1)$ to denote the set of all vertices which have distance i from v and shall call each $L_i(v)$ a *distance level* of v. Suppose that G is a graph of $\mathcal{G}(n,p)$. Let $L_1(v), L_2(v), \ldots, L_k(v)$ be the non-empty distance levels of a vertex v. Then we must have $k=2$. Thus for each vertex $u \neq v$, $u \in L_1(v) \cup L_2(v)$. Since u is critical, there are two vertices x and y such that $x \sim u \sim y$ is the *only* path of length ≤ 2 joining x and y. This implies that x and y are not adjacent and at least one of them must be in $L_2(v)$. This fact will be frequently used throughout our proofs.

2 The lower bound

In this section, we shall construct, for each $n \geq 7$, a graph in $\mathcal{G}(n, n-4)$ when n is odd and a graph in $\mathcal{G}(n,n-5)$ when n is even. Note that $\mathcal{G}(6,p) = \emptyset$ for any p and $\mathcal{G}(5,p) = \emptyset$ for any $p \geq 3$. The only graph contained in $\mathcal{G}(5,2)$ is the cycle of length 5.

The graph H displayed in Figure 1 shows that $\mathcal{G}(n, \frac{n-1}{2}) \neq \emptyset$ when n is odd and at least 5. In particular, when $n=7$, $\mathcal{G}(n,n-4) \neq \emptyset$. Using H as a building block, we construct an $(n,n-4)$-critical graph for each odd $n \geq 9$ (Figure 2) and an $(n,n-5)$-critical graph for each even $n \geq 10$ (Figure 3).

When $n=8$, Figure 4 shows an $(n,n-5)$-critical graph. Therefore we have the following:

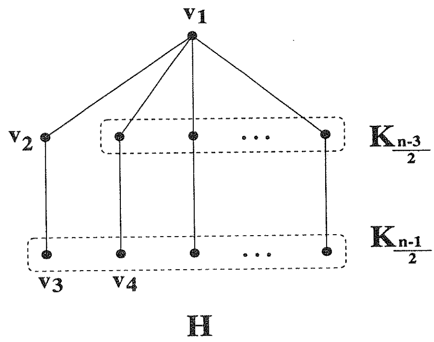

Figure 1: An $(n, \frac{n-1}{2})$-critical graph with $n \geq 5$ odd.

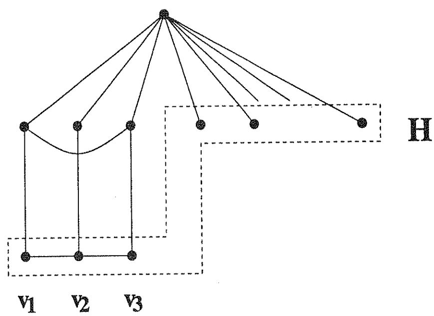

Figure 2: An $(n, n-4)$-critical graph with $n \geq 7$ odd.

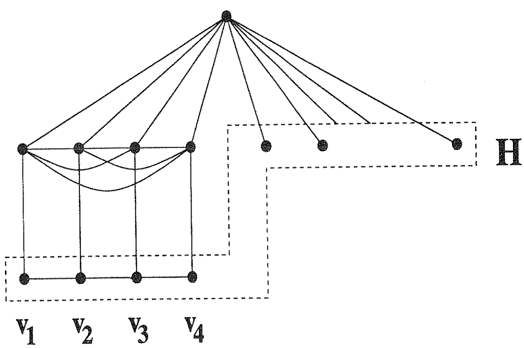

Figure 3: An $(n, n-5)$-critical graph with $n \geq 10$ even.

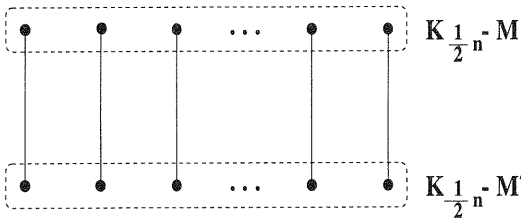

Figure 4: An $(n, \frac{n-2}{2})$-critical graph with $n \equiv 0(\text{mod}4)$, where M and M' are two edge-disjoint perfect matchings in $K_{\frac{1}{2}n}$.

Proposition 2.1 *For each $n \geq 7$,*

$$\Delta_n \geq \begin{cases} n-4 & \text{when } n \text{ is odd} \\ n-5 & \text{when } n \text{ is even.} \end{cases}$$

□

Since $n-4 > \frac{1}{2}n$ for odd $n \geq 9$ and $n-5 \geq \frac{1}{2}n$ for even $n \geq 12$, the conjecture of Boals et. al. is false for $n = 9$ and for all $n \geq 11$.

Note that the graph in Figure 1 has $\frac{1}{4}(n^2-2n+5)$ edges and the graph in Figure 3 has $\frac{1}{4}(n^2-8n+60)$ edges. For $n \equiv 0 \pmod 4$, the graph in Figure 4 has $\frac{1}{4}(n^2-2n)$ edges. All these numbers are asymptotically equal to $\frac{1}{4}n^2$, which is conjectured to be the upper bound for the maximum number of edges possible in a critical graph [6].

3 The upper bound

In this section, we shall establish the exact value of Δ_n. First we have the following.

Proposition 3.1 *For each $n > 5$, $\mathcal{G}(n,p) \neq \emptyset$ implies that $p \leq n-4$.*

Proof: Let G be a graph in $\mathcal{G}(n,p)$ and let v be a vertex of G of degree p. Denote by $L_1(v), L_2(v)$ the distance levels of v. It is easy to see that $L_2(v)$ must contain at least two vertices, as otherwise G contains a vertex which is not critical, a contradiction. Suppose that $L_2(v)$ contains exactly two vertices, say u_1 and u_2. Since u_1 is critical, there exists a vertex $u_1' \in L_1(v)$ such that $u_1' \sim u_1 \sim u_2$ is the only path of length ≤ 2 joining u_1' and u_2. Similarly there exists a vertex $u_2' \in L_1(v)$ such that $u_2' \sim u_2 \sim u_1$ is the only path of length ≤ 2 joining u_2' and u_1. Note that $u_1' \neq u_2'$. Let x be a vertex in $L_1(v) - \{u_1', u_2'\}$. (Such a vertex exists because $n > 5$.) Since x is critical, there are two vertices y and z such that $y \sim x \sim z$ is the only path of length ≤ 2 joining y and z. One of y and z is in $L_2(v) = \{u_1, u_2\}$. Without loss of generality, assume that $y \in L_2(v)$. Since u_1 is adjacent to u_2, $z \notin L_2(v)$ and z is not adjacent to either of u_1 and u_2. Thus z can not be critical, a contradiction. Therefore $L_2(v)$ contains at least three vertices and $p \leq n-4$. □

In the case when n is even, the above upper bound can be sharpened as follows.

Proposition 3.2 *Suppose that $n > 5$ is even. Then $\mathcal{G}(n,p) \neq \emptyset$ implies that $p \leq n-5$.*

Proof: By Proposition 3.1, $p \leq n-4$. Suppose that $\mathcal{G}(n, n-4)$ contains a graph G. Let v be a vertex of G of degree $n-4$ and let $L_1(v), L_2(v)$ be the distance levels of v. Then $L_2(v)$ consists of three vertices which are denoted by u_1, u_2, u_3. The criticality of G implies that each vertex of $L_1(v)$ must have at least one neighbour in $L_2(v)$. It also implies that $L_2(v)$ must induce a connected subgraph of G. Without loss of generality, assume that u_1 is adjacent to both u_2 and u_3.

Suppose that u_2 is adjacent to u_3. We claim that each of u_1, u_2, and u_3 has its own unique neighbour in $L_1(v)$. Consider the vertex u_1. The criticality of u_1 implies the existence of two vertices u_1' and u_1'' such that $u_1' \sim u_1 \sim u_1''$ is the only path of length ≤ 2 joining u_1' and u_1''. One of u_1' and u_1'', say u_1', is in $L_2(v)$. Since any two vertices of $L_2(v)$ are adjacent, u_1'' is in $L_1(v)$. Clearly u_1'' is not adjacent to either of u_2 and u_3. Since u_1'' is critical, there is a vertex u_1''' such that $u_1''' \sim u_1'' \sim u_1$ is the only path of length ≤ 2 joining u_1''' and u_1. Thus u_1''' is not adjacent to any of u_1, u_2, u_3 and we must have $u_1''' = v$, as otherwise $u_1''' \in L_1(v)$ and u''' can not be critical, a contradiction. This implies that u_1'' is the only neighbour of u_1 in $L_1(v)$. A similar argument shows that each of u_2 and u_3 has its own unique neighbour in $L_1(v)$. Hence $L_1(v)$ contains precisely three vertices and we have $n = 7$, contradicting the assumption that n is even.

Suppose now that u_2 is not adjacent to u_3. For $i = 1, 2, 3$, let $S_i \subset L_1(v)$ consists of vertices whose only neighbour in $L_2(v)$ is u_i. We show that $S_1 = \emptyset$. So suppose that u_1' is a vertex of S_1. Then the criticality of u_1' implies that there exists a vertex x such that $x \sim u_1' \sim u_1$ is the only path of length ≤ 2 joining x and u_1. Thus x is not adjacent to u_1, u_2, or u_3 and hence we must have $x = v$. This also implies that u_1' is the only neighbour of u_1 in $L_1(v)$. Since u_2 is critical, there exists a vertex $u_2' \in S_2$ such that $u_2' \sim u_2 \sim u_1$ is the only path of length ≤ 2 joining u_2' and u_1. Similarly there is a vertex $u_3' \in S_3$ such that $u_3' \sim u_3 \sim u_1$ is the only path of length ≤ 2 joining u_3' and u_1. Hence $S_2 \neq \emptyset$ and $S_3 \neq \emptyset$. Take a vertex $y \in S_2$ (possibly $y = u_2'$). The criticality of y implies that there exists a vertex z such that $z \sim y \sim u_2$ is the only path of length ≤ 2 joining z and u_2. There are only two possibilities: Either $z = v$ or z is in S_3.

If $z = v$, then y $(\in S_2)$ is the only neighbour of u_2 in $L_1(v)$. Now if u_3 has at least two neighbours in $L_1(v)$, then it is easy to see that none of them can be critical. Thus u_3 has exactly one neighbour in $L_1(v)$. Since each vertex of $L_1(v)$ must have at least one neighbour in $L_2(v)$, $L_1(v)$ contains precisely three vertices and $n = 7$ is odd, a contradiction. Therefore z must be in S_3. That is, each vertex y of S_2 corresponds to a vertex $z \in S_3$ in such a way that $z \sim y \sim u_2$ is the only path of length ≤ 2 joining z and u_2. Note that distinct vertices of S_2 must correspond to distinct vertices of S_3. A similar argument shows that each vertex a of S_3 corresponds to a vertex $b \in S_2$ in such a way that $b \sim a \sim u_3$ is the only path of length ≤ 2 joining a and u_3. Furthermore distinct vertices of S_3 must corresponds to distinct vertices in S_2. Hence $|S_2| = |S_3|$. Now if $L_1(v)$ contains a vertex adjacent to both u_2 and u_3, then clearly this vertex can not be critical. So $L_1(v) = S_2 \cup S_3 \cup \{u_1'\}$ and we have $n = 2|S_2| + 5$, contradicting the assumption. Therefore S_1 is empty.

The facts that S_1 is empty and u_1 is critical imply that u_1 is the only common neighbour of u_2 and u_3. Thus each vertex of $L_1(v)$ is adjacent to either u_2 or u_3 but not to both. Let $T_2 \subset L_1(v)$ consists of vertices which are adjacent to u_2 and $T_3 \subset L_1(v)$ consists of vertices which are adjacent to u_3. Then $T_2 \cup T_3 = L_1(v)$ and $T_2 \cap T_3 = \emptyset$. The vertex u_1 must have at least one neighbour in $L_1(v)$. Without loss of generality, assume that $c \in T_2$ is a neighbour of u_1 in $L_1(v)$. The criticality of c implies that c is in fact the only neighbour of u_1 in $L_1(v)$. Now a similar argument as above shows that each vertex y of $T_2 - \{c\}$ corresponds to a vertex z of T_3 in such a way that $z \sim y \sim u_2$ is the only path of length ≤ 2 joining z and u_2. If T_3 contains only one vertex, then $T_2 - \{c\}$ has only one vertex. Thus $L_1(v)$ contains precisely three vertices and $n = 7$ is odd, a contradiction. So suppose that T_3 contains at least two vertices. Then each vertex a of T_3 corresponds to a vertex b in $T_2 - \{c\}$ in such a way that $b \sim a \sim u_3$ is the only path of length ≤ 2 joining b and u_3. Hence we have $|T_2 - \{c\}| = |T_3|$ and n is odd, again a contradiction. $\square$

Combining propositions 3.1, 3.2, and 2.1 we have the following theorem.

Theorem 3.3 *For each $n \geq 7$,*

$$\Delta_n = \begin{cases} n-4 & \text{when } n \text{ is odd} \\ n-5 & \text{when } n \text{ is even.} \end{cases}$$

$\square$

As we explained at the begining of Section 2, the only critical graph on $n \leq 6$ vertices is the cycle of length 5.

4 Further analysis

Let us turn to the problem of determining values of n and p for which $\mathcal{G}(n,p) \neq \emptyset$, i.e., there exists an (n,p)-critical graph. By Theorem 3.3, $\mathcal{G}(n,p) \neq \emptyset$ implies that $p \leq \Delta_n$, and according to the Moore bound, $p \geq \sqrt{n-1}$. Our constructions below show that $\mathcal{G}(n,p) \neq \emptyset$ for all n and p with $n \geq 7$ and $min\{\sqrt{4n+5}-2, \frac{n+3}{4}\} \leq p \leq \Delta_n$. To describe our constructions, we define four basic graphs which are called *building blocks.*

In general, a building block is a graph obtained from some given graphs by adding appropriate adjacencies between pairs of them. These adjacencies include a perfect matching, the complement of a perfect matching, and the complete adjacency. In our diagrams these are denoted by a normal line, a broken bold line, and a bold line, respectively. To avoid ambiguity, we assume that the vertices of each given graph are ordered. If two graphs have the same number of vertices, then a perfect matching means the ith vertex of one graph is only adjacent to the ith vertex of the other. Thus the complement of a perfect matching means the ith vertex of one graph is adjacent to all vertices of the other except the ith one. Our first three building blocks are obtained from given graphs H_1, H_2, H_3, H_4 as follows:

1. Assume H_2 and H_3 have the same number of vertices. Then the building block $B_1(H_1, H_2, H_3, H_4)$ is the graph obtained from H_1, H_2, H_3, H_4 by adding a perfect matching between H_2 and H_3 and all edges between H_1 and H_2 and between H_3 and H_4 (see Figure 5).

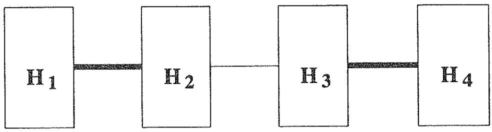

Figure 5: The building block $B_1(H_1, H_2, H_3, H_4)$ where bold lines mean complete adjacencies and normal lines mean perfect matchings.

2. Assume that H_1, H_2, H_3, H_4 have the same number of vertices. Then the building block $B_2(H_1, H_2, H_3, H_4)$ is the graph obtained from H_1, H_2, H_3, H_4 by adding all edges between H_1 and H_4 and then a perfect matching between H_i and H_{i+1} for each $i = 1, 2, 3$ (see Figure 6).

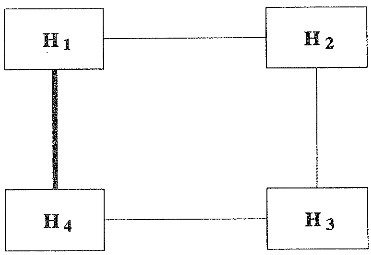

Figure 6: The building block $B_2(H_1, H_2, H_3, H_4)$.

3. Assume that H_1, H_2, H_3, H_4 have the same number of vertices. Then the building block $B_3(H_1, H_2, H_3, H_4)$ is the graph obtained from H_1, H_2, H_3, H_4 by adding a perfect matching between H_1 and H_4 and between H_i and H_{i+1} for each $i = 1, 2, 3$, and then the complement of a perfect matching between H_1 and H_3 and between H_2 and H_4 (see Figure 7).

Let $H_1, H_2, \ldots, H_k$ be graphs which have the same number of vertices.

4. The building block $B_4(H_1, H_2, \ldots, H_k)$ is obtained from $H_1, H_2, \ldots, H_k$ by adding a perfect matching between any pairs H_i and H_j with $i \neq j$ (see Figure 8).

We now use the above building blocks to construct the desired graphs in $\mathcal{G}(n, p)$. In all cases, we do not describe the routine checking that is needed to confirm the diameter property of the graph.

When $n \geq 7$ is odd and $\frac{1}{2}(n-1) \leq p \leq \Delta_n$, we construct a graph $G_1 \in \mathcal{G}(n, p)$ as follows: Take $H_1 = K_2$, $H_2 = \overline{K}_{\frac{1}{2}(n-7)}$, $H_3 = K_{\frac{1}{2}(n-7)}$, and $H_4 = \overline{K}_2$. Let G_1 be

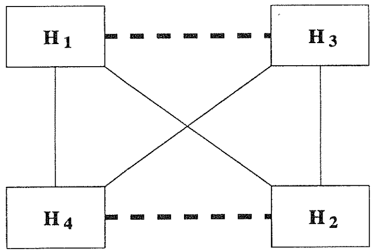

Figure 7: The building block $B_3(H_1, H_2, H_3, H_4)$ where broken bold lines mean the complements of perfect matchings.

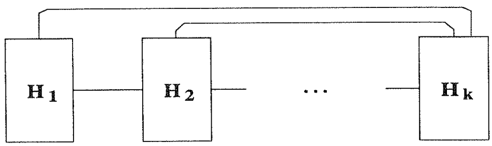

Figure 8: The building block $B_4(H_1, H_2, \ldots, H_k)$.

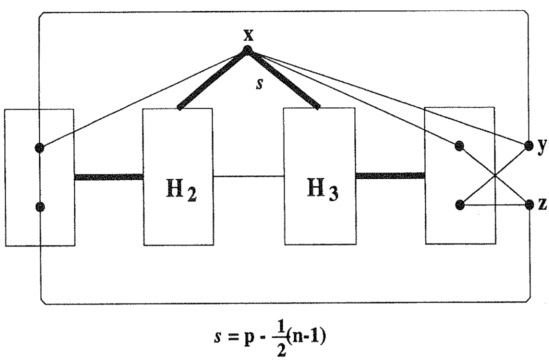

Figure 9: The graph G_1; s edges between x and H_3.

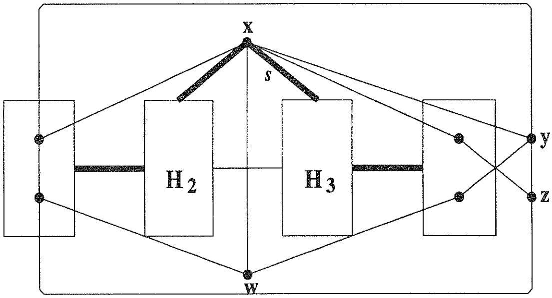

Figure 10: The graph G_2; s edges between x and H_3.

the graph obtained from $B_1(H_1, H_2, H_3, H_4)$ by adding three vertices x, y, z and the edges as shown in Figure 9; note that the bold edge labeled s means that there are s edges from x to H_3.

When $n \geq 10$ is even and $\frac{1}{2}(n-2) \leq p \leq \Delta_n$, we construct a graph $G_2 \in \mathcal{G}(n,p)$ as follows: Take $H_1 = K_2$. $H_2 = \overline{K}_{\frac{1}{2}(n-8)}$, $H_3 = K_{\frac{1}{2}(n-8)}$, and $H_4 = \overline{K}_2$. Let G_2 be the graph obtained from $B_1(H_1, H_2, H_3, H_4)$ by adding four vertices x, y, z, w and the edges as shown in Figure 10. These two constructions together with the $(8,3)$-critical graph described in Section 2 give the following:

Proposition 4.1 *For any integers n and p with $n \geq 7$ and $\frac{1}{2}(n-1) \leq p \leq \Delta_n$, $\mathcal{G}(n,p) \neq \emptyset$.* □

Proposition 4.2 *For any integers n and p with $n \geq 7$ and $\frac{1}{4}(n+3) \leq p \leq \frac{1}{2}(n-1)$ except $n = 9$ and $p = 3$, $\mathcal{G}(n,p) \neq \emptyset$.*

Proof: Suppose that $n \equiv 0, 2 (\text{mod} 4)$ and $n \geq 8$. Let $H_1 = \overline{K}_{\lceil \frac{1}{4}(n-2) \rceil}$, $H_2 = H_3 = K_{\lceil \frac{1}{4}(n-2) \rceil}$, and let H_4 be any graph on $\lceil \frac{1}{4}(n-2) \rceil$ vertices of maximum degree $p - \lceil \frac{1}{4}(n+3) \rceil$. It is easy to see that $B_2(H_1, H_2, H_3, H_4) \in \mathcal{G}(n,p)$ when $n \equiv 0 (\text{mod} 4)$. When $n \equiv 2 (\text{mod} 4)$, a graph of $\mathcal{G}(n,p)$ can be obtained from $B_2(H_1, H_2, H_3, H_4)$ by adding two vertices x, y and the edges as shown in Figure 11.

Suppose now that $n \equiv 1, 3 (\text{mod} 4)$ and $n \geq 13$. Assume that $p < \frac{1}{2}(n-1)$. Let $H_2 = H_3 = H_4 = \overline{K}_{\lceil \frac{1}{4}(n-7) \rceil}$ and let H_1 be any graph on $\lceil \frac{1}{4}(n-7) \rceil$ vertices of maximum degree $p - \lceil \frac{1}{4}(n+3) \rceil$. When $n \equiv 1 (\text{mod} 4)$, a graph of $\mathcal{G}(n,p)$ can be obtained from $B_3(H_1, H_2, H_3, H_4)$ by adding five vertices u, x, y, z, w and the edges as shown in Figure 12. When $n \equiv 3 (\text{mod} 4)$, a graph of $\mathcal{G}(n,p)$ can be obtained from $B_3(H_1, H_2, H_3, H_4)$ by adding seven vertices and the edges as shown in Figure 13. In case when $p = \frac{1}{2}(n-1)$, we obtain a graph of $\mathcal{G}(n,p)$ from the constructed $(n, \frac{1}{2}(n-3))$-critical graph by adding the missing edges between H_1 and H_3.

A $(7,3)$-, a $(9,4)$-, and a $(11,5)$-critical graph are described in Section 2 and a $(11,4)$-critical graph is shown in Figure 14. □

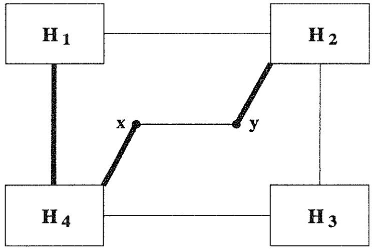

Figure 11: An (n,p)-critical graph with $n \equiv 2 (\mathrm{mod} 4)$ $(n \geq 10)$ and $\frac{1}{4}(n+3) \leq p \leq \frac{1}{2}(n-1)$.

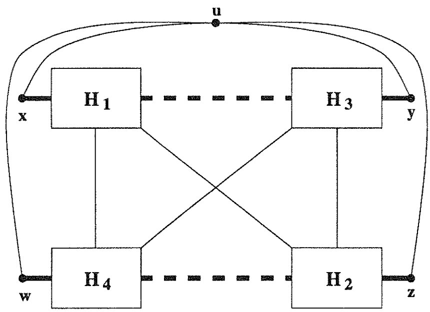

Figure 12: An (n,p)-critical graph with $n \equiv 1 (\mathrm{mod} 4)$ $(n \geq 13)$ and $\frac{1}{4}(n+3) \leq p \leq \frac{1}{2}(n-1)$.

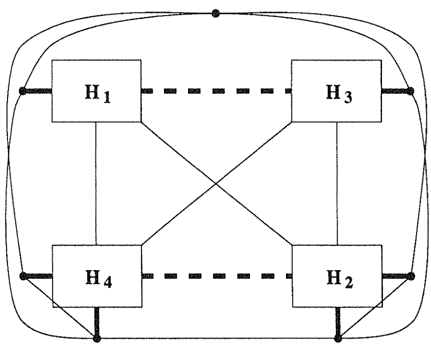

Figure 13: An (n,p)-critical graph with $n \equiv 3 (\mathrm{mod} 4)$ $(n \geq 15)$ and $\frac{1}{4}(n+3) \leq p \leq \frac{1}{2}(n-1)$.

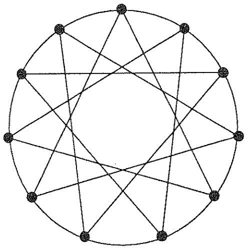

Figure 14: A $(11,4)$-critical graph.

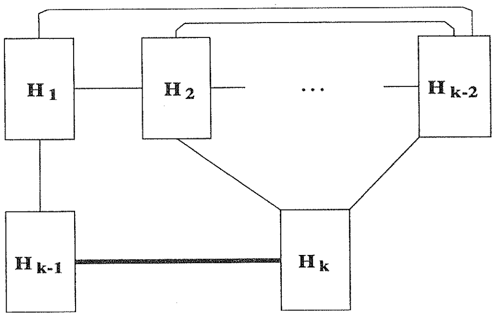

Figure 15: An (n,p)-critical graph with $n = kt$ ($k \geq 4$ and $t \geq 2$) and $p = t + k - 3$.

Lemma 4.3 *Let $n = kt$ with $k \geq 4$ and $t \geq 2$. Then $\mathcal{G}(n,p) \neq \emptyset$ for any p with $t + k - 3 \leq p \leq (t-1)(k-2) + 1$.*

Proof: We first show how to construct a graph $G \in \mathcal{G}(n, t+k-3)$. Let $H_i = K_t$ ($i = 1, 2, \ldots, k-2$) and let $H_{k-1} = H_k = \overline{K}_t$. Then a graph of $\mathcal{G}(n, t+k-3)$ can be obtained from $B_4(H_1, H_2, \ldots, H_{k-2})$, H_{k-1}, and H_k by adding the edges as shown in Figure 15. Now to construct a graph in $\mathcal{G}(n,p)$ for each p with $t+k-3 \leq p \leq (t-1)(k-2)+1$ we choose a vertex $x \in V(H_1)$ and add $p - (t+k-3)$ new edges between x and vertices in $V(H_2) \cup V(H_3) \cup \ldots \cup V(H_{k-2})$, and make sure that x is not adjacent to at least one vertex in each H_i ($2 \leq i \leq k-2$). □

Lemma 4.4 *Let $n = kt + 1$ with $k \geq 4$ and $t \geq 2$. Then $\mathcal{G}(n,p) \neq \emptyset$ for any p with $t + k - 2 \leq p \leq (t-1)(k-1) + 1$.*

Proof: Take $H_i = K_{t-1}$ ($i = 1, 2, \ldots, k$). To construct a graph $G \in \mathcal{G}(n, t+k-2)$, we add $k+1$ vertices and edges to $B_4(H_1, H_2, \ldots, H_k)$ as shown in Figure 16. Now a graph of $\mathcal{G}(n,p)$ for each p with $t+k-2 \leq p \leq (t-1)(k-1)+1$ can be obtained from G by choosing a vertex x from H_1 and adding $p - t + k - 2$ edges between x and vertices in $V(H_2) \cup \ldots \cup V(H_k)$ and make sure that x is not adjacent to at least one vertex in each H_i ($i \geq 2$). □

Lemma 4.5 *Let $n = kt + r$ with $k \geq 4$, $t \geq 2$, and $2 \leq r \leq k-2$. Then $\mathcal{G}(n,p) \neq \emptyset$ for any p with $t + k - 2 \leq p \leq (t-1)(k-2) + 1$.*

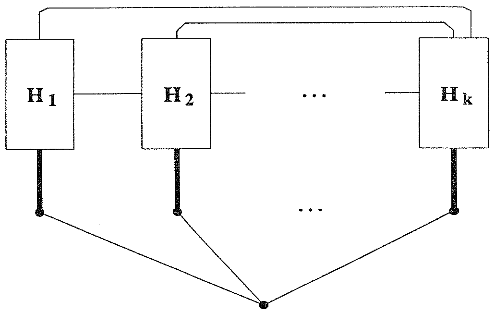

Figure 16: An (n,p)-critical graph with $n = kt+1$ ($k \geq 4$ and $t \geq 2$) and $p = t+k-3$.

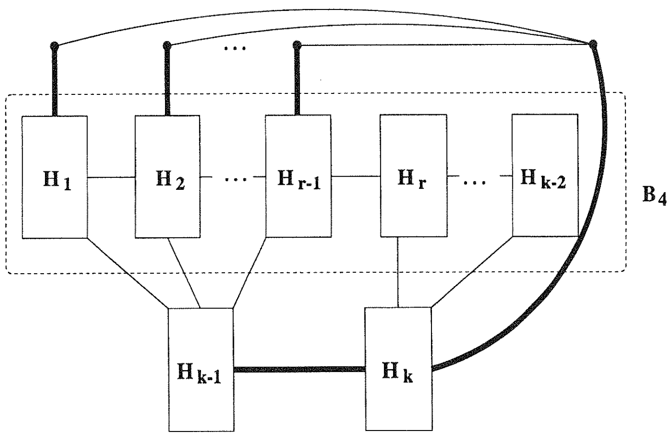

Figure 17: An (n,p)-critical graph with $n = kt + r$ ($k \geq 4$, $t \geq 2$, and $2 \leq r \leq k-2$) and $p = t + k - 2$.

Proof: Let $H_i = K_t$ for each $i = 1, 2, \ldots, k-2$ and let $H_{k-1} = H_k = \overline{K}_t$. Figure 17 shows how to construct an $(n, t+k-2)$-critical graph from $B_4(H_1, H_2, \ldots, H_{k-2})$. Now to construct a graph in $\mathcal{G}(n,p)$ for each p with $t+k-2 \leq p \leq (t-1)(k-2)+1$ we choose a vertex $x \in V(H_1)$ and add $p-(t+k-2)$ new edges between x and vertices in $V(H_2) \cup V(H_3) \cup \ldots V(H_{k-2})$, and make sure that x is not adjacent to at least one vertex in each H_i $(2 \leq i \leq k-2)$. □

Proposition 4.6 *For any integers n and p with $\sqrt{4n+5}-2 \leq p \leq \frac{1}{4}(n+3)$, $\mathcal{G}(n,p) \neq \emptyset$.*

Proof: Suppose that $n = t^2$ with $t \geq 4$. By Lemma 4.3, $\mathcal{G}(n,p) \neq \emptyset$ for any p with $2t-3 = 2\sqrt{n}-3 \leq p \leq (t-1)(t-2)+1$. Note that $2t-3 = 2\sqrt{n}-3 < \sqrt{4n+5}-2$ and $(t-1)(t-2)+1 = \frac{1}{4}(t^2+3(t-2)^2) \geq \frac{1}{4}(n+3)$. Suppose that $t^2 < n < (t+1)^2$. We consider the following cases:

Case 1. $n = t^2+r$ with $t \geq 4$ and $0 < r < t$. Assume that $r \neq t-1$. Then, by lemmas 4.4 and 4.5, $\mathcal{G}(n,p) \neq \emptyset$ for any p with $2t-2 \leq p \leq (t-1)(t-2)+1$. Note that in this case we have $2t-2 = 2\sqrt{n-r}-2 < \sqrt{4n+5}-2$ and $(t-1)(t-2)+1 = \frac{1}{4}(t^2+3(t-2)^2) = \frac{1}{4}(t^2+r+3(t-2)^2-r) \geq \frac{1}{4}(n+3)$. When $r = t-1$, we have $n = t^2+t-1 = (t-1)(t+2)+1$. Hence, by Lemma 4.4, $\mathcal{G}(n,p) \neq \emptyset$ for any p with $2t-1 = \sqrt{4n+5}-2 \leq p \leq (t-2)(t+1)+1$. In this case, we have $(t-2)(t+1)+1 = \frac{1}{4}(t^2+t-1+(2t+1)(t-3)+t^2) \geq \frac{1}{4}(n+3)$.

Case 2. $n = t(t+1)$ with $t \geq 3$. By Lemma 4.3, $\mathcal{G}(n,p) \neq \emptyset$ for any p with $2t-2 = \sqrt{4n+1}-2 \leq p \leq (t-1)(t-2)+1$. In this case, we have $2t-2 = \sqrt{4n+1}-2 < \sqrt{4n+5}-2$ and $(t-1)^2+1 = \frac{1}{4}(t(t+1)+3(t-1)(t-2)+2) \geq \frac{1}{4}(n+3)$.

Case 3. $n = t(t+1)+r$ with $t \geq 3$ and $1 \leq r < t$. By Lemma 4.5, $\mathcal{G}(n,p) \neq \emptyset$ for any p with $2t-1 \leq p \leq (t-1)^2+1$. In this case, we have $2t-1 = \sqrt{4(n-r)+1}-2 < \sqrt{4n+5}-2$ and $(t-1)^2+1 = \frac{1}{4}(t(t+1)+r+3(t-1)(t-2)-r+2) \geq \frac{1}{4}(n+3)$.

Case 4. $n = t(t+2)$ with $t \geq 2$. By Lemma 4.3, $\mathcal{G}(n,p) \neq \emptyset$ for any p with $2t-1 \leq p \leq t(t-1)+1$. Note that in this case we have $2t-1 = 2\sqrt{n+1}-3 < \sqrt{4n+5}-2$ and $t(t-1)+1 = \frac{1}{4}(t^2+2t+3(t-1)^2+1) \geq \frac{1}{4}(n+3)$. □

Combining propositions 4.1, 4.2, and 4.6, we have the following:

Theorem 4.7 *For any integers n and p with $n \geq 7$ and $min\{\sqrt{4n+5}-2, \frac{1}{4}(n+3)\} \leq p \leq \Delta_n$ except $n = 9$ and $p = 3$, $\mathcal{G}(n,p) \neq \emptyset$.* □

We remark that there is no $(9,3)$-critical graph, i.e., $\mathcal{G}(9,3) = \emptyset$. Indeed, if G is such a graph, then G must contain a vertex of even degree and hence a vertex of degree 2. This implies that G contains at most 7 vertices, a contradiction.

Corollary 4.8 *For each $p \geq 5$, $\mathcal{G}(\lfloor \frac{1}{4}(p^2+4p-1) \rfloor, p) \neq \emptyset$.* □

Acknowledgement

This work has been supported by Australian Research Council Grant No. A49532661.

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(Received 4/12/96)