Size Ramsey Results for Paths versus Stars

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Abstract

A general upper bound for the size Ramsey number $\hat{r}(P_s, K_{1,t})$ matching the exact value for $s \leq 7$ and t sufficiently large is presented. Moreover, the remaining unknown values of the size Ramsey number for pairs of forests of order at most five are determined.

1 Introduction

For any pair of graphs G and H the size Ramsey number $\hat{r}(G, H)$ is defined to be the minimum size of a graph F such that in any 2-coloring of the edges of F, say with red and green, there is a red copy of G or a green copy of H; as usual this is denoted by $F \to (G, H)$. The size Ramsey number was introduced in [8] and has been studied for several pairs of graphs, i.e. for pairs of complete graphs and of stars ([8]) and, mostly with regard to the asymptotic behavior, for pairs of paths, of trees and of cycles ([1], [2], [12], [13]). Moreover, various results have been obtained for pairs of graphs where one of the two graphs is a matching or a star, respectively ([4], [6], [10], [14]).

Here we will consider the size Ramsey number for a path P_s of order s versus a star $K_{1,t}$. A general upper bound will be derived for $\hat{r}(P_s, K_{1,t})$, and it will be shown that this bound is best possible for $s \leq 7$ and t sufficiently large. Moreover, as an extension of the results in [9], where the size Ramsey number has been determined for almost all pairs of graphs of order at most four, we will present the still missing values of the size Ramsey number for all pairs of forests of order at most five.

The following notation will be used. As usual, the vertex-set of a graph G is denoted by $V(G)$, the edge-set by $E(G)$, and we write $p(G) = |V(G)|$ and $q(G) = |E(G)|$. A 2-coloring of a graph F always means a 2-coloring of the edges of F with red and green. A (G, H)-coloring of a graph F is a 2-coloring of F containing neither a red copy of G nor a green copy of H. Furthermore the set of red neighbors of a vertex $v \in V(F)$ in a 2-coloring of F is denoted by R_v and the set of green neighbors by G_v, respectively. Further special notation will be introduced as needed.

Notation not specifically mentioned will follow that in [11].

2 Paths versus stars

It is easy to see that

$$\hat{r}(P_2, K_{1,t}) = t, \quad \hat{r}(P_3, K_{1,t}) = t + 1.$$

The following two lemmas will determine an upper bound for $\hat{r}(P_s, K_{1,t})$ in case of $s \geq 4$ and t sufficiently large.

Lemma 1. Let $m \geq 2$. Then

$$K_{m,2t-1} \to (P_{2m+1}, K_{1,t}) \quad \text{if} \quad t \geq \max\{m+1, 2m-5\}. \tag{1}$$

Proof. We will prove (1) applying induction on m. The assertion holds for $m = 2$. Suppose now that $m \geq 3$ and that (1) holds for $m-1$ and fails for m, i.e. a $(P_{2m+1}, K_{1,t})$-coloring χ of the graph $G = K_{m,2t-1}$ exists. Consider the coloring χ. Let U be the set of the m independent vertices and $W = V(G) \setminus U$. Note that $|G_v| \leq t-1$ and $|R_v| \geq t \geq m+1$ for every $v \in U$. Pick some vertex $u \in U$. By the induction hypothesis we can find a red path P of order $2m-1$ in $G - u = K_{m-1,2t-1}$. We see that $P = w_1 u_1 w_2 u_2 \ldots w_{m-1} u_{m-1} w_m$ where $\{u_1, \ldots, u_{m-1}\} = U \setminus \{u\}$ and $w_1, \ldots, w_m \in W$. Let $W_1 = \{w_2, \ldots, w_{m-1}\}$, $W_2 = W \setminus W_1$ and, for $v \in U$, $R_v^* = R_v \cap W_2$. Note that $|R_v^*| \geq t - m + 2$ for every $v \in U$. Furthermore we see that $R_u^* \cap R_{u_1}^* = R_u^* \cap R_{u_{m-1}}^* = \emptyset$ as otherwise a red path of order $2m+1$ is unavoidable.

First suppose that $R_{u_1}^* \cap R_{u_{m-1}}^* \neq \emptyset$. We may assume that $w_1 \in R_{u_1}^* \cap R_{u_{m-1}}^*$. This yields a red cycle C of order $2m-2$ containing $u_1, \ldots, u_{m-1}$ and $w_1, \ldots, w_{m-1}$. A vertex $w_i \in W_1$ must be joined red to u because of $R_u^* \subset G_{u_1}$, $|G_{u_1}| \leq t-1$ and $|R_u| \geq t$. Moreover, we can find vertices $x \in R_{u_{i-1}}^* \setminus \{w_1\}$ and $y \in R_u^* \setminus \{x\}$. But then a red path of order $2m+1$ can be obtained using edges of C and the edges xu_{i-1}, $w_i u$, uy.

The remaining case is that the sets $R_{u_1}^*$, $R_{u_{m-1}}^*$ and R_u^* are pairwise disjoint and contain at least $t-m+2$ elements each. It follows that $m - 2 + 3(t-m+2) \leq |W| = 2t-1$ which implies that $t \leq 2m-5$. In view of the assumption $t \geq \max\{m+1, 2m-5\}$ we see that only $t = 2m-5$ and $m \geq 6$ is left. Here a contradiction can only be avoided if $W_2 = R_{u_1}^* \cup R_{u_{m-1}}^* \cup R_u^*$ and if $R_{u_1}^*$, $R_{u_{m-1}}^*$ and R_u^* contain $t - m + 2 = m - 3$ elements each. This implies that all edges from u_1, u_{m-1} and u to W_1 are red. Now consider the vertex u_2. There must be a red neighbor w of u_2 in W_2. If $w \in R_{u_1}^*$, we take a vertex $w' \in R_u^*$ and obtain the red $P_{2m+1} = w' u w_2 u_2 w u_1 w_3 u_3 w_4 u_4 \ldots w_{m-1} u_{m-1} w_m$. The remaining cases that

$w \in R^*_{u_{m-1}}$ or $w \in R^*_u$ similarly lead to a contradiction, and the proof of Lemma 1 is complete. ■

Lemma 2. Let $m \geq 2$. Then

$$K_m + \overline{K}_{2t-1-m} \to (P_{2m}, K_{1,t}) \quad \text{if} \quad t \geq 4m-3. \tag{2}$$

Proof. We apply induction on m. In case of $m = 2$ the assertion holds for $t \geq 4$. Suppose now that $m \geq 3$ and that (2) holds for $m-1$ and fails for m, i.e. a $(P_{2m}, K_{1,t})$-coloring χ of the graph $G = K_m + \overline{K}_{2t-1-m}$ exists. Consider the coloring χ. Let U be the vertex-set of the K_m and $W = V(G) \setminus U$. Pick some vertex $u \in U$ and remove the edges joining u to W. This yields a graph $G' = K_{m-1} + \overline{K}_{2t-1-(m-1)}$. By the induction hypothesis we can find a red path P of order $2m-2$ in G'.

Case I: P does not contain the vertex u. Then P has to contain the remaining $m-1$ vertices $u_1, \ldots, u_{m-1}$ from U, $m-1$ vertices $w_1, \ldots, w_{m-1} \in W$, and at most one edge belonging to $[U]$ can occur in P. This leads to one of the following two subcases.

I.1: P does not contain an edge belonging to $[U]$. Then we may assume that $P = w_1u_1w_2u_2 \ldots w_{m-1}u_{m-1}$. Let $W_1 = \{w_2, \ldots, w_{m-1}\}$, $W_2 = W \setminus W_1$ and, for $v \in U$, $R^*_v = R_v \cap W_2$. We see that $|R^*_v| \geq t - 2m + 2$ for every $v \in U$. No red P_{2m} in χ implies that $R^*_u \cap R^*_{u_1} = R^*_u \cap R^*_{u_{m-1}} = \emptyset$. Thus, $R^*_{u_1} \cap R^*_{u_{m-1}} \neq \emptyset$ since otherwise $2t - 2m + 1 = |W_2| \geq |R^*_{u_1}| + |R^*_{u_{m-1}}| + |R^*_u| \geq 3t - 6m + 6$ in contradiction to $t \geq 4m - 3$. We may assume that $w_1 \in R^*_{u_1} \cap R^*_{u_{m-1}}$. Then P and the edge w_1u_{m-1} yield a red cycle C of length $2m-2$. A red edge from u to C would yield a red path of order $2m$ because of $|R^*_u| > 1$. Thus, we have $2m-2$ green edges from u to C. Then no green $K_{1,t}$ in χ implies at most $t-2$ green edges from u_1 to W, i.e. $|R^*_{u_1} \setminus \{w_1\}| \geq t - 2m + 2$. Since $R^*_u \cap R^*_{u_1} = \emptyset$, there must be at least $t - 2m + 2$ green edges from u to $W \setminus V(C)$ and together with the $2m-2$ green edges joining u to $V(C)$ this yields a green $K_{1,t}$.

I.2: P contains an edge ab belonging to $[U]$. In this case we may assume that $P = w_1u_1w_2u_2 \ldots w_iabw_{i+1} \ldots u_{m-1}w_{m-1}$. Let $W_1 = \{w_2, \ldots, w_{m-2}\}$, $W_2 = W \setminus W_1$ and, for $v \in U$, $R^*_v = R_v \cap W_2$. If $R^*_a \cap R^*_b \neq \emptyset$, we obtain a red path P of order $2m-2$ as in case I.1. Thus, we may assume that $R^*_a \cap R^*_b = \emptyset$. Note that $|W_2| = 2t - 2m + 2$ and $|R^*_v| \geq t - 2m + 3$ for every $v \in U$. Using $t \geq 4m - 3$ we see that among any three vertices from U there are two of them with a common red neighbor in W_2. Thus, a vertex $c \in (R^*_a \cup R^*_b) \cap R^*_u$ exists. We may assume that $c \in R^*_a \cap R^*_u$. Moreover, $R^*_u \cap R^*_{u_1} = R^*_u \cap R^*_{u_{m-1}} = \emptyset$ as otherwise a red path of order $2m$ would occur. This implies $R^*_{u_1} \cap R^*_{u_{m-1}} \neq \emptyset$, and we obtain a red cycle C' of order $2m-3$. But then we obtain a red path of order $2m$ by removing the edge ab from C' and adding the edges ac, cu and ud where $d \in R^*_u \setminus \{c\}$.

Case II: P contains the vertex u. We may assume that any red path of order $2m-2$ contains u and that P contains every $v \in U$, since otherwise we would obtain a situation equivalent to case I. Thus, u cannot be an end-vertex of P since otherwise we could find a red edge joining the neighbor of u on P, which must also belong to U, to a vertex in $W \setminus V(P)$ yielding a red P_{2m-2} without u. Similarly, no red P_{2m-2} can have an end-edge belonging to $[U]$. Let $W_1 = V(P) \cap W$, $W_2 = W \setminus W_1$, and let x and y be the neighbors of u on P. Note that $|W_1| = m-2$, $|W_2| = 2t-2m+1$ and $x, y \in U$. Let $R_v^* = R_v \cap W_2$ for $v \in U$. Using that $|R_v^*| \geq t-2m+2$ for every $v \in U$ and $t \geq 4m-3$ we obtain that

$$R_{v_1}^* \cap R_{v_2}^* = R_{v_1}^* \cap R_{v_3}^* = \emptyset \quad \Longrightarrow \quad |R_{v_2}^* \cap R_{v_3}^*| \geq 2 \text{ for all } v_1, v_2, v_3 \in U. \tag{3}$$

We see that $R_x^* \cap R_y^* = \emptyset$ since otherwise we would obtain a red path P_{2m-2} not containing u. Moreover, no red P_{2m} in χ implies that $R_x^* \cap R_u^* = \emptyset$ or $R_y^* \cap R_u^* = \emptyset$. Thus, (3) yields that $|R_y^* \cap R_u^*| \geq 2$ or $|R_x^* \cap R_u^*| \geq 2$. We may assume that $w, w' \in R_y^* \cap R_u^*$. Suppose that P contains an end-vertex $z \in U$. But then we can replace uy by uw and wy and add a red edge to P joining z to a vertex in $W_2 \setminus \{w\}$ yielding a red P_{2m}. Thus, $P = w_1u_1 \ldots xuy \ldots u_2w_2$ where $w_1, w_2 \in W$ and $u_1, u_2 \in U$. It is allowed that $u_1 = x$ or $u_2 = y$.

Suppose that a vertex $v_1 \in R_{u_1}^* \cap R_{u_2}^*$ exists. We may assume that $v_1 \neq w$ since otherwise we could exchange w and w'. But then we obtain a red P_{2m} by removing the edges xu, uy, w_1u_1, u_2w_2 from P and adding the edges $u_1v_1, v_1u_2, uw, wy, xv_2, uv_3$ where $v_2 \in R_x^* \setminus \{v_1\}$ and $v_3 \in R_u^* \setminus \{w, v_1\}$. Thus, $R_{u_1}^* \cap R_{u_2}^* = \emptyset$. Moreover, $R_x^* \cap R_{u_2}^* = \emptyset$ and $R_y^* \cap R_{u_1}^* = \emptyset$ since otherwise a forbidden red P_{2m-2} with an end-edge belonging to $[U]$ would occur. Especially, $w, w' \notin R_{u_1}^*$. By (3) it follows that we can find two distinct vertices $z_1 \in R_y^* \cap R_{u_2}^*$ and $z_2 \in R_x^* \cap R_{u_1}^*$ where $z_1, z_2 \neq w$.

It can be checked that a third edge ab belonging to $[U]$ must occur on P. This leads to one of the following two subcases.

II.1: The edge ab belongs to the section of P from y to w_2. Let a be the vertex that is reached before b passing through P from y to w_2. First suppose that a vertex $s_1 \in R_a^* \cap R_{u_2}^*$ exists. We may assume that $s_1 \neq w$ since otherwise we could exchange w and w'. But then we obtain a red P_{2m} by removing the edges uy, ab, u_2w_2 from P and adding the edges $uw, wy, as_1, s_1u_2, bs_2$ where $s_2 \in R_b^* \setminus \{w, s_1\}$. Thus, $R_a^* \cap R_{u_2}^* = \emptyset$. This implies that $a \neq y$. Using that $R_{u_1}^* \cap R_{u_2}^* = \emptyset$ and (3) we see that a vertex $s_3 \in R_{u_1}^* \cap R_a^*$ with $s_3 \neq w, z_1$ must exist. But then we obtain a red P_{2m} from P by removing the edges w_1u_1, w_2u_2, ab and the two edges incident to y on P and adding the edges $uw, wy, yz_1, z_1u_2, u_1s_3, s_3a, bs_4$ where $s_4 \in R_b^* \setminus \{w, z_1, s_3\}$.

II.2: The edge ab belongs to the section of P from w_1 to x. Let a be the vertex that is reached before b passing through P from w_1 to x. First suppose that a vertex $t_1 \in R_{u_1}^* \cap R_b^*$ with $t_1 \neq w$ exists. Then we obtain a red P_{2m} by removing the edges w_1u_1, ab, uy from P and adding the edges $u_1t_1, t_1b, uw, wy, at_2$ where $t_2 \in R_a^* \setminus \{w, t_1\}$. Thus, $R_{u_1}^* \cap R_b^* = \emptyset$. Taking into account that $R_{u_1}^* \cap R_{u_2}^* = \emptyset$ and (3) the remaining case is that a vertex $t_3 \in R_b^* \cap R_{u_2}^*$ with $t_3 \neq z_2$ exists. We may assume that $t_3 \neq w$. But this yields a red P_{2m} if the edges $ab, xu, uy, w_1u_1, w_2u_2$ are removed from P and the edges $u_1z_2, z_2x, uw, wy, bt_3, t_3u_2, at_4$ where $t_4 \in R_a^* \setminus \{w, z_2, t_3\}$ are added. This completes the proof of Lemma 2. ■

As an immediate consequence of Lemma 1 and Lemma 2 we obtain

Theorem 1. Let $s \geq 4$ be fixed, $m = \lfloor s/2 \rfloor$, $t_1 = \max\{m+1, 2m-5\}$ and $t_2 = 4m-3$. Then

$$\hat{r}(P_s, K_{1,t}) \leq \begin{cases} 2mt - m & \text{if } s \text{ is odd and } t \geq t_1, \\ 2mt - \frac{1}{2}m^2 - \frac{3}{2}m & \text{if } s \text{ is even and } t \geq t_2. \end{cases} \quad (4)$$

In the following we will prove that the upper bound given in Theorem 1 is attained for $s \leq 7$ if t is sufficiently large. Some additional notation will be useful. A graph is called an (n,t)-brush if it contains exactly n vertices of degree at least t and no two vertices of degree at most $t-1$ are adjacent. We will always use $u_1, \ldots, u_n$ to denote the vertices of degree at least t in an (n,t)-brush G and let $U = \{u_1, \ldots, u_n\}$ and $W = V(G) \setminus U$. The number of vertices in W adjacent to u_i will be denoted by l_i. Moreover, we define $\hat{r}_n(P_s, K_{1,t})$ to be the minimum size of an (n,t)-brush G with $G \to (P_s, K_{1,t})$.

Lemma 3. Any graph G satisfying $G \to (P_s, K_{1,t})$ and $G - e \not\to (P_s, K_{1,t})$ for every edge $e \in E(G)$ is an (n,t)-brush with $n \geq m = \lfloor s/2 \rfloor$. Moreover,

$$\hat{r}(P_s, K_{1,t}) = \min\{\hat{r}_n(P_s, K_{1,t}) : n \geq m\}. \quad (5)$$

Proof. Suppose that G contains an edge $e = uv$ where $d(u), d(v) \leq t-1$. Take a $(P_s, K_{1,t})$-coloring of $G - e$ and add the edge e in green. This yields a $(P_s, K_{1,t})$-coloring of G, a contradiction. Additionally, if $n < \lfloor s/2 \rfloor$, no subgraph P_s occurs in G and a $(P_s, K_{1,t})$-coloring of G trivially exists. This obviously implies (5). ∎

Lemma 4. Let $s \geq 4$, $m = \lfloor s/2 \rfloor$ and $t \geq m$. Then

$$\hat{r}_m(P_s, K_{1,t}) \geq \begin{cases} 2mt - m & \text{if } s \text{ is odd}, \\ 2mt - \frac{1}{2}m^2 - \frac{3}{2}m & \text{if } s \text{ is even}. \end{cases} \quad (6)$$

Proof. First let s be odd. Consider an (m,t)-brush G where $q(G) \leq 2mt - m - 1$. Then $l_i \leq 2t-2$ for some $i \in \{1, \ldots, m\}$. Let $w_1, \ldots, w_{l_i}$ be the vertices in W adjacent to u_i. Then color the edges from u_i to $w_1, \ldots, w_{\lfloor l_i/2 \rfloor}$ and the edges between $U \setminus \{u_i\}$ and $w_{\lfloor l_i/2 \rfloor + 1}, \ldots, w_{l_i}$ green and all remaining edges red. This gives a $(P_s, K_{1,t})$-coloring of G implying (6) for s odd.

Now let s be even. Consider an (m,t)-brush G where $q(G) \leq 2mt - m^2/2 - 3m/2 - 1$. Then $d(u_i) \leq 2t - 3$ for some $i \in \{1, \ldots, m\}$. Color $\min\{t-2, l_i\}$ edges from u_i to W red and all other edges incident to u_i green. Then color the edges between $U \setminus \{u_i\}$ and the red neighbors of u_i green and all remaining edges red. This gives a $(P_s, K_{1,t})$-coloring of G implying (6) for s even, and the proof of Lemma 4 is complete. ■

A direct consequence of Lemma 1, Lemma 2 and Lemma 4 is

Theorem 2. Let $s \geq 4$, $m = \lfloor s/2 \rfloor$, and let $t \geq \max\{m+1, 2m-5\}$ for s odd and $t \geq 4m-3$ for s even. Then

$$\hat{r}_m(P_s, K_{1,t}) = \begin{cases} 2mt - m & \text{if } s \text{ is odd,} \\ 2mt - \frac{1}{2}m^2 - \frac{3}{2}m & \text{if } s \text{ is even.} \end{cases} \tag{7}$$

Lemma 5. Let $s \geq 4$ be fixed and $m = \lfloor s/2 \rfloor$. Then for t sufficiently large (especially for $t \geq 2m^2 + 2m + 1$ in case of s odd and for $t \geq \frac{3}{2}m^2 + \frac{3}{2}m - 2$ in case of s even)

$$\hat{r}(P_s, K_{1,t}) = \min\{\hat{r}_n(P_s, K_{1,t}) : m \leq n \leq 2m-2\}. \tag{8}$$

Proof. Let $\hat{r}_n = \hat{r}_n(P_s, K_{1,t})$. Note that for the t in question $\hat{r}_m$ is determined by Theorem 2. We will show that $\hat{r}_n \geq \hat{r}_m$ for $n \geq 2m-1$. Then the assertion follows by Lemma 3.

First let G be an (n,t)-brush where $n \geq 2m+1$. Using Theorem 2 we obtain $q(G) \geq (2m+1)t - \binom{2m+1}{2} \geq \hat{r}_m$ for the t in question. This trivially implies $\hat{r}_n \geq \hat{r}_m$.

It remains that $2m-1 \leq n \leq 2m$. Let $r_i = d(u_i) - t$ for $i = 1, \ldots, n$. Without loss of generality we may assume that $r_1 \geq r_2 \geq \ldots \geq r_n$. Moreover, we will use that

$$\sum_{i=1}^{n} r_i = q(G) - nt + q([U]) \tag{9}$$

for any (n,t)-brush G.

First consider a $(2m,t)$-brush G with $q(G) < \hat{r}_m$. Using Theorem 2 and $q([U]) \leq \binom{2m}{2}$ we obtain by (9) that $\sum_{i=1}^{2m}(r_i + 1) \leq t - (2m-1)$ for the t in question. Note that $l_i \geq t - (2m-1)$ for $i = 1, \ldots, n$. Thus, for $i = 1, \ldots, 2m$ we can color $r_i + 1$ edges from u_i to W red such that no vertex in W is incident to more than one red edge. Coloring all remaining edges green we obtain a $(P_s, K_{1,t})$-coloring of G. This implies $\hat{r}_{2m} \geq \hat{r}_m$.

Finally consider a $(2m-1,t)$-brush G with $q(G) < \hat{r}_m$. Let $U^{(1)} = \{u_1, \ldots, u_{m-1}\}$, $U^{(2)} = \{u_m, \ldots, u_{2m-2}\}$ and $U^{(3)} = \{u_{2m-1}\}$. Color the edges in $[U^{(1)}]$ and $[U^{(2)}]$ red and all other edges in $[U]$ green. Now let n_i be the number of red edges incident to u_i in $[U]$. It can be checked by considering (9) that for the t in question

$$\begin{array}{llll} l_{2m-1} & \geq t + r_{2m-1} - (2m-2) & \geq r_{2m-1} + 1, & \\ l_i & \geq t + r_i - (n_i + m) & \geq r_i + 1 - n_i + r_{2m-1} + 1 & \text{if } m \leq i \leq 2m-2, \\ l_i & \geq t + r_i - (n_i + m) & \geq r_i + 1 - n_i + \sum_{i=m}^{2m-1}(r_i + 1) & \text{if } 1 \leq i \leq m-1. \end{array}$$

Thus, for $i = 2m-1, \ldots, 1$ we can color $r_i + 1 - n_i$ edges from u_i to W red such that no vertex in W is adjacent in red to two of the sets $U^{(1)}$, $U^{(2)}$ and $U^{(3)}$. If all remaining edges are colored green, a $(P_s, K_{1,t})$-coloring of G is obtained. This implies $\hat{r}_{2m-1} \geq \hat{r}_m$, and the proof of Lemma 5 is complete. ■

Theorem 3. Let $m = \lfloor s/2 \rfloor$. Then for $s = 4$ if $t \geq 4$, for $s = 5$ if $t \geq 2$, for $s = 6$ if $t \geq 16$ and for $s = 7$ if $t \geq 25$

$$\hat{r}(P_s, K_{1,t}) = \begin{cases} 2mt - m & \text{if } s = 5 \text{ or } s = 7, \\ 2mt - \frac{1}{2}m^2 - \frac{3}{2}m & \text{if } s = 4 \text{ or } s = 6. \end{cases} \tag{10}$$

Proof. Using Lemma 5 and Theorem 2 we obtain (10) in case of $s = 4$ for $t \geq 7$ and in case of $s = 5$ for $t \geq 13$. It can be checked by some case analysis that (10) already holds for $t \geq 4$ and $t \geq 2$, respectively.

Now consider $6 \leq s \leq 7$. Theorem 1 establishes the upper bound for $t \geq 9$ and $t \geq 4$, respectively. Let G be a $(4, t)$-brush with $q(G) \leq 6t-4$ and let $t \geq 15$. Considering Lemma 5 and Theorem 2 the proof is complete if it can be shown that a $(P_s, K_{1,t})$-coloring of G exists. We define $N(u_i)$ to be the set of vertices in W adjacent to u_i. For any permutation i, j, k, l of $1, 2, 3, 4$ let $N_{i,j,k,l} = (N(u_i) \cup N(u_j)) \cap (N(u_k) \cup N(u_l))$, and let n_d be the number of vertices of degree d in W. Note that

$$|N_{1,2,3,4}| + |N_{1,3,2,4}| + |N_{1,4,2,3}| = 2n_2 + 3n_3 + 3n_4 \leq q(G) \leq 6t - 4. \tag{11}$$

Consider some fixed $N_{i,j,k,l}$. Let $A = N_{i,j,k,l} \setminus N(u_j)$, $B = N_{i,j,k,l} \setminus N(u_i)$ and $C = N_{i,j,k,l} \setminus (A \cup B)$.

First suppose that $\min\{|A|, |B|\} \leq t-3$ and $\max\{|A|+|C|, |B|+|C|\} \leq 2t-6$. We may assume that $|B| \leq |A|$. If $|B|+|C| \geq t-3$, choose a $(t-3-|B|)$-element subset $C' \subset C$ and a $|B|$-element subset $A' \subset A$. Then color the edges between $\{u_i, u_j\}$ and $A' \cup B \cup C' \cup \{u_k, u_l\}$ and the edges between $\{u_k, u_l\}$ and $(C \setminus C') \cup (A \setminus A')$ green and all remaining edges in G red. If $|B| + |C| < t-3$, choose a $(t-3-|C|)$-element subset $A' \subset A$ in case of $|A| \geq t-3-|C|$, otherwise put $A' = A$. Then color the edges between $\{u_i, u_j\}$ and $A' \cup B \cup C \cup \{u_k, u_l\}$ and the edges between $\{u_k, u_l\}$ and $A \setminus A'$ green and all remaining edges in G red. In both cases a $(P_s, K_{1,t})$-coloring of G is obtained.

The remaining case is that $|A|, |B| \geq t-2$ or $\max\{|A|+|C|, |B|+|C|\} \geq 2t-5$ for every permutation i, j, k, l of $1, 2, 3, 4$, which implies that $|N_{i,j,k,l}| \geq 2t-5$.

First let $|A|, |B| \geq t-2$ for some $N_{i,j,k,l}$, we may assume for $N_{1,2,3,4}$. Then (11) and $|N_{1,3,2,4}|, |N_{1,4,2,3}| \geq 2t-5$ imply that $|N_{1,2,3,4}| \leq 3t-9$ for $t \geq 15$. Choose $(t-3)$-element subsets $A' \subset A$ and $B' \subset B$. Then color the edges between $\{u_1, u_2\}$ and $A' \cup B' \cup \{u_3, u_4\}$ and the edges between $\{u_3, u_4\}$ and $(A \setminus A') \cup (B \setminus B') \cup C$ green and all remaining edges in G red. Again a $(P_s, K_{1,t})$-coloring of G is obtained.

Only the case that $\max\{|A|+|C|, |B|+|C|\} \geq 2t-5$ for every $N_{i,j,k,l}$ is left. This implies that at least three vertices in U are adjacent to at least $2t-5$ vertices in W

each. It follows that $q(G) \geq t + 3(2t - 5)$. But this contradicts $q(G) \leq 6t - 4$ for $t \geq 12$, and the proof of Theorem 3 is complete. ■

Remark. Considering Theorem 3 we conjecture that the upper bound given in Theorem 1 for $\hat{r}(P_s, K_{1,t})$ is also attained for $s \geq 8$ if t is sufficiently large (depending on s).

In case of $s \geq 4$, $m = \lfloor s/2 \rfloor$ and $t \geq m$ the best lower bound for $\hat{r}(P_s, K_{1,t})$ currently known to us is

$$\hat{r}(P_s, K_{1,t}) \geq (m+1)t - \binom{m+1}{2}.$$

This follows immediately from Lemma 3 since $\hat{r}_n(P_s, K_{1,t}) \geq (m+1)t - \binom{m+1}{2}$ for $n \geq m+1$ (consider the edges incident to $m+1$ of the n vertices of degree at least t of an (n,t)-brush) and $\hat{r}_m(P_s, K_{1,t}) \geq (m+1)t - \binom{m+1}{2}$ by Lemma 4.

3 Forests of order at most five

Table 1 gives the size Ramsey number for all pairs of isolate-free forests of order at most five. Additionally, Table 2 gives the corresponding *restricted size Ramsey number* $\hat{r}^*(G, H)$ which is defined as the minimum size of a graph F with $r(G, H)$ vertices and $F \to (G, H)$, where $r(G, H)$ denotes the minimum order of a graph F' satisfying $F' \to (G, H)$.

Note that the trivial results $\hat{r}(P_2, H) = \hat{r}^*(P_2, H) = q(H)$ for arbitrary H have been omitted in both tables.

Following the notation in [3] we use $S_{1,3}$ to denote the graph obtained from a star $K_{1,3}$ by joining an additional fifth vertex to one of the outer vertices of the star.

The footnotes indicate where the corresponding values of $\hat{r}(G, H)$ and $\hat{r}^*(G, H)$ have been obtained from. The remaining values can be checked by some tedious and lengthy case analysis which is omitted here.

	$2P_2$	P_3	$P_3 \cup P_2$	$K_{1,3}$	P_4	$K_{1,4}$	$S_{1,3}$	P_5
$2P_2$	**3** [5]	**4** [5]	**5**	**6** [5]	**5** [6]	**8** [5]	**8**	**6** [6]
P_3		**3** [5]	**5**	**4** [5]	**5** [9]	**5** [5]	**7**	**6** Th. 3
$P_3 \cup P_2$			**6**	**7**	**8**	**9**	**9**	**9**
$K_{1,3}$				**5** [5]	**8** [9]	**6** [5]	**9**	**10** Th. 3
P_4					**7** [9]	**11** Th. 3	**9**	**10**
$K_{1,4}$						**7** [5]	**12**	**14** Th. 3
$S_{1,3}$							**10** [3]	**11**
P_5								**10**

Table 1: $\hat{r}(G, H)$ for all pairs of isolate-free forests of order at most five

	$2P_2$	P_3	$P_3 \cup P_2$	$K_{1,3}$	P_4	$K_{1,4}$	$S_{1,3}$	P_5
$2P_2$	**5**	**4** [9]	**6**	**8**	**5** [9]	**12**	**8**	**6**
P_3		**3** [7]	**5**	**4** [7]	**5** [9]	**10** [7]	**7**	**6** Th. 3
$P_3 \cup P_2$			**8**	**9**	**8**	**12**	**9**	**10**
$K_{1,3}$				**5** [7]	**9**	**6** [7]	**10**	**10**
P_4					**7** [9]	**11** Th. 3	**9**	**10**
$K_{1,4}$						**21** [7]	**17**	**17**
$S_{1,3}$							**11**	**11**
P_5								**10**

Table 2: $\hat{r}^*(G, H)$ for all pairs of isolate-free forests of order at most five

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