Decomposing block-intersection graphs of Steiner triple systems into triangles

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Abstract

The problem of decomposing the block intersection graph of a Steiner triple system into triangles is considered. In the case when the block intersection graph has even degree, this is completely solved, while when the block intersection graph has odd degree, removal of some spanning subgraph of odd degree is necessary before the rest can be decomposed into triangles. In this case, some decompositions are presented when the original Steiner triple system can be "triangulated", that is, can have its triples partitioned into sets of three so that any two of the three intersect but so that there is no point common to all three triples. The existence of a single parallel class in the Steiner triple system is also assumed for a few cases.

1 Introduction

A Steiner triple system of order n, STS(n), is a pair (V, B) where V is the vertex set of a complete graph K_n of order n and B is a set of triangles, i.e. 3-cycles, which partition the edge-set of K_n. We shall refer to the triangles in such a decomposition as *triples*. We shall also reserve the use of the word triple exclusively here for members of the block set B of a STS.

The *block intersection graph* of a Steiner triple system (V, B), denoted by BIG(STS(n)) for short, is a graph with the triples in B being the vertices of the graph, and with an edge joining two of its vertices if and only if the corresponding triples contain a common point. Since a STS(n) has replication number $(n-1)/2$, any BIG(STS(n)) is clearly regular of degree $3((n-1)/2-1)$, that is, $3(n-3)/2$. Moreover, each point in V will correspond to a unique clique in the BIG(STS(n)), and any two of these n cliques will intersect in precisely one vertex. (For graph theory terminology used here, such as *clique*, see for instance Bondy and Murty [1].)

Henceforth it is only the n cliques arising in this way that we are interested in, so we exclude the case $n = 7$ and also ignore any cliques arising in a BIG from subsystems of order 7 in the STS. Of course the BIG of the STS(7) can trivially be decomposed into triangles, because BIG(STS(7)) is K_7! Note that each of these n cliques is of size equal to the replication number of the STS(n), namely $(n-1)/2$.

When the order n of the STS is not important, we shall frequently refer to the block intersection graph of a STS(n) as merely the BIG.

In [2], various partitions of the triples in a STS(n) into small configurations are considered. In particular, one such is a "triangulation" of a STS(n), which is a partition of the triples into sets of three, any two of the three intersecting, but with no point common to all three triples. (And if the STS(n) $= (V, B)$ has $|B|$ not divisible by three, then either one or two triples are omitted from the partition, depending upon whether $|B|$ is 1 or 2 (mod 3).) Thus three triples of the form

$$\{a, b, d\}, \ \{a, c, e\}, \ \{b, c, f\} \qquad (*)$$

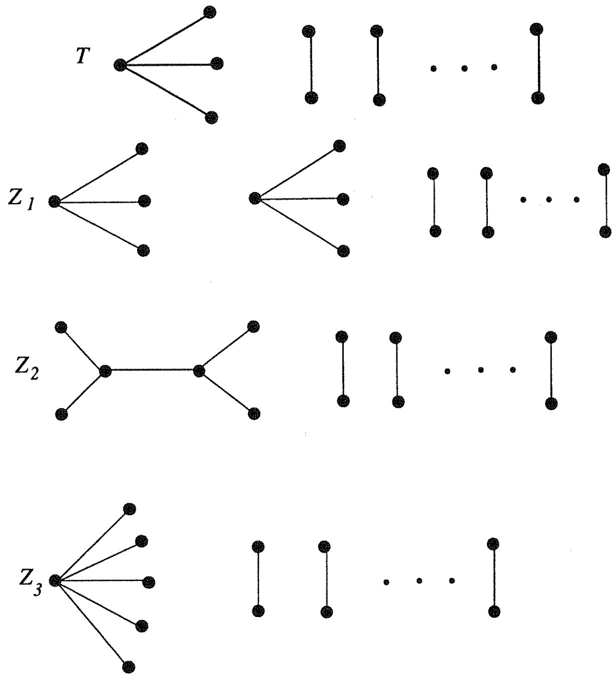
form a "triangle" in a possible triangulation, where the points a, b and c are each in two of the three triples. Henceforth when we use the term "triangle" in quotes, we shall always mean such a set of three triples. It is still an open conjecture as to whether *any* STS(n) may be triangulated; certainly there exists a STS(n) for each admissible $n \equiv 1$ or 3 (mod 6) which can be triangulated (see [2, 4]).

Such a triangulation of a STS(n) will correspond to a parallel class of triangles in the BIG(STS(n)). However, any triangle in a BIG(STS(n)) does *not* necessarily correspond to such a "triangle" consisting of three triples $(*)$. For instance, three triples of the form

$$\{a, b, c\}, \ \{a, d, e\}, \ \{a, f, g\}$$

will also correspond to a triangle in the BIG(STS(n)), although these three triples form a "3-windmill" and not a "triangle", using the terminology in [2].

In this paper we consider the problem of decomposing the BIG(STS(n)), for *any* STS(n), into triangles. The case when the BIG has even degree is completely dealt with here. When the BIG has odd degree, clearly a spanning subgraph of odd degree needs to be removed first. Depending upon the number of edges the BIG contains, this spanning subgraph is either a 1-factor, or it has one edge more than a 1-factor (and is usually denoted by T for tripole), or else two edges more than a 1-factor. Following standard terminology, we refer to such a minimal set of unused edges in our BIG decomposition into triangles as the *leave*. The leaves other than a 1-factor are given in the following figures. Note that there are three spanning subgraphs of odd degree having two edges more than a 1-factor; we denote these by Z_1, Z_2 and Z_3. In these cases with nonempty leave, our results depend upon certain properties, such that the STS(n) has a partition into "triangles", that is, is able to be triangulated. However, we conjecture that such extra requirements are not necessary.



The following table lists the expected leave in the BIG in all cases. Note that when the BIG is of odd degree, we work modulo 36 in order to take account of the clique sizes and the different expected leaves.

order n of the underlying STS	clique size, $(n-1)/2$	number of edges in BIG or in BIG $-$ F	expected leave in BIG
3 (mod 12)	1 (mod 6)	0 (mod 3)	$\emptyset$
7 (mod 12)	3 (mod 6)	0 (mod 3)	$\emptyset$
1 (mod 36)	0 (mod 6)	0 (mod 3)	F
9 (mod 36)	4 (mod 6)	0 (mod 3)	F
13 (mod 36)	0 (mod 6)	2 (mod 3)	Z_i
21 (mod 36)	4 (mod 6)	1 (mod 3)	T
25 (mod 36)	0 (mod 6)	1 (mod 3)	T
33 (mod 36)	4 (mod 6)	2 (mod 3)	Z_i

2 When the BIG has even degree

Since the BIG(STS(n)) is regular of degree $3(n-3)/2$, and since $n \equiv 1$ or $3 \pmod 6$, the BIG has even degree precisely when $n \equiv 3$ or $7 \pmod{12}$. In this case we have the following easy result:

Theorem 1 *When $n \equiv 3$ or $7 \pmod{12}$, the block intersection graph of any STS(n) can be decomposed into triangles.*

Proof Note first that the number of edges in the BIG is one half the sum of the degrees, that is,

$$\frac{1}{2}\left(\frac{n(n-1)}{6}\right)\left(\frac{3(n-3)}{2}\right)=\frac{n(n-1)(n-3)}{8}.$$

When $n \equiv 3$ or $7 \pmod{12}$, this is always a multiple of 3, and so the numerical conditions are right for a decomposition of the BIG into triangles.

Furthermore, the size of each clique is $(n-1)/2$, which is 1 or 3 (mod 6) when $n \equiv 3$ or $7 \pmod{12}$. Thus each separate clique is the correct order for a STS to be placed on it. Since any two cliques meet in only one point, we may do this, and thus obtain a decomposition of the whole BIG into triangles. □

3 When the BIG has odd degree

In this case the order n of the underlying STS is 1 or 9 (mod 12). We work modulo 36 in order to take account of the different expected leaves and the sizes of the n cliques; see the table in the previous section.

Note that a maximum packing of a complete graph (such as a clique!) of size 0 (mod 6) with triangles has leave a 1-factor, whereas a maximum packing of one of size 4 (mod 6) has leave a tripole T (a spanning subgraph of odd degree with one edge more than a 1-factor). We shall refer to the vertex of degree 3 in a tripole as the *head* of the tripole.

Lemma 1 *If a STS(n) with $n \equiv 1 \pmod{36}$ can be triangulated, then its BIG can be decomposed into triangles with leave a 1-factor.*

Proof Take a triangulation of the STS. Since $n \equiv 1 \pmod{36}$, the number of triples is 0 (mod 3), and so a triangulation uses all triples in the STS. For each "triangle" in the triangulation, such as

$$\{a,b,d\},\ \{a,c,e\},\ \{b,c,f\}, \qquad (*)$$

place the three edges between the three vertices $\{a,b,d\}$, $\{a,c,e\}$ and $\{b,c,f\}$ of the BIG, forming a triangle in the packing of the BIG. Note that since these three triples are in one "triangle" of the assumed triangulation, the resulting triangle in the BIG packing has its three edges in three *different* cliques (namely, the a-clique, the b-clique and the c-clique for the "triangle" $(*)$).

Note that the triangulation of the STS yields a single parallel class of triangles in the BIG, with each triangle having its edges lying in three different cliques. In each clique these edges induce a matching, which we extend to a 1-factor in each clique. The remaining edges of the cliques are then packed by triangles within each clique. (Each clique has size 0 (mod 6) so the leave in each clique is a 1-factor.) Naturally we do this so that all the edges from the triangulation occur in these clique 1-factors; we can do this precisely because each triangle arising from the triangulation has its edges in three different cliques. Taken together, the n clique 1-factors would now produce a 3-regular leave (because each vertex of the BIG is in three cliques). However, we may reduce this leave from a 3-factor to a 1-factor, as required, by including in our decomposition the triangles induced by the triangulation of the STS. □

Notice that a STS(n) with a triangulation is known to exist for each $n \equiv 1$ or 3 (mod 6) [4]. Thus Lemma 1 applies to a non-empty set of STS (as do Lemmata 2 and 3 which follow).

Lemma 2 *If a STS(n) with $n \equiv 25$ (mod 36) can be triangulated, then its BIG can be decomposed into triangles with leave a tripole.*

Proof In this case each clique of the BIG again has size 0 (mod 6). But the number of triples in a STS(n) with $n \equiv 25$ (mod 36) is 1 (mod 3) and so in a supposed triangulation of a STS(n), one triple (say $\{x, y, z\}$) is unused and does not appear in any "triangle". Nevertheless, if we repeat the construction described in Lemma 1 in the case $n \equiv 1$ (mod 36), we obtain a suitable packing of the BIG with triangles, and with leave a tripole T; the head of the tripole is the vertex corresponding to the triple $\{x, y, z\}$ which does not appear in any "triangle" of the triangulation of the STS(n). □

Lemma 3 *If a STS(n) with $n \equiv 13$ (mod 36) can be triangulated so that the two unused triples are disjoint, then its BIG can be decomposed into triangles with leave Z_1.*

If a STS(n) with $n \equiv 13$ (mod 36) can be triangulated so that the two unused triples share a common point, then its BIG can be decomposed into triangles with leave Z_2.

Finally, if a STS(n) with $n \equiv 13$ (mod 36) can be triangulated so that the two unused triples share a common point, and also with the triangulation satisfying a certain extra condition (P), then its BIG can be decomposed into triangles with leave Z_3.

Proof The clique size of the BIG is 0 (mod 6) and so a packing of a clique with triangles has individual leave for that clique being a 1-factor. However, a triangulation of a STS(n) in this case omits two of the triples, since the number of triples in a STS(n) with $n \equiv 13$ (mod 36) is 2 (mod 3). If the two triples missed by the triangulation of the STS contain no common point, (say triples $\{x, y, z\}$ and $\{u, v, w\}$), then following the construction described in Lemma 1 above, the vertices in the BIG labelled by $\{x, y, z\}$ and $\{u, v, w\}$ will remain with degree 3, while every other vertex in the BIG will have its degree dropped to one, since every other triple of the STS is in some "triangle" of the assumed triangulation of the STS. Thus the overall leave in the BIG will be Z_1 in this case.

Now suppose that the two triples missed by the triangulation share a point; say they are $\{x, y, z\}$ and $\{x, u, v\}$. When packing the x-clique with triangles, provided we choose the edge between vertex $\{x, y, z\}$ and vertex $\{x, u, v\}$ to be an edge of the 1-factor leave for the x-clique packing, the overall leave in the BIG, after including the triangles from the triangulation in the BIG packing, will be Z_2, with vertices $\{x, y, z\}$ and $\{x, u, v\}$ each having degree 3, and with these vertices of the BIG being joined by an edge in the leave.

Finally we need to show that the leave Z_3 can also be achieved.

Suppose that the assumed triangulation of the STS has the two triples of the STS which do not occur in any "triangle" of the triangulation sharing a common point. Let these triples be $\{x, y, z\}$ and $\{x, u, v\}$. Moreover, suppose that the triangulation has property (P), that is, assume that one of the following holds:

- the triple $\{y, u, *\}$ is in a "triangle" of the triangulation in which y occurs only once, *and* the triple of the form $\{z, u, *\}$ is in a different "triangle" in which z occurs only once;
- the triple $\{y, v, *\}$ is in a "triangle" of the triangulation in which y occurs only once *and* the triple of the form $\{z, v, *\}$ is in a different "triangle" in which z occurs only once;
- the triple $\{y, u, *\}$ is in a "triangle" of the triangulation in which u occurs only once *and* the triple of the form $\{y, v, *\}$ is in a different "triangle" in which v occurs only once;
- the triple $\{z, u, *\}$ is in a "triangle" of the triangulation in which u occurs only once *and* the triple of the form $\{z, v, *\}$ is in a different "triangle" in which v occurs only once.

The above four possibilities are all isomorphic, so we only need consider one of them. We deal with the first of the above possibilities.

First suppose that a "triangle" in the triangulation of the STS consists of the triples $\{a, b, c\}$, $\{a, d, e\}$ and $\{b, d, f\}$. Then we say that the vertex in the BIG corresponding to the triple $\{a, b, c\}$ is *not free* in the a or b-cliques, but is *free* in the c-clique. This means that we have the freedom to choose which vertex (triple) is paired with $\{a, b, c\}$ in the 1-factor within the c-clique, but we have no such choice in the a-clique or the b-clique.

Now consider the y-clique. When picking the 1-factor leave in here, the triple $\{y, u, *\}$ is free to be paired with any other triple containing y that is free in the y-clique. Similarly, in the z-clique, the triple $\{z, u, *\}$ can be paired with any other free triple in the z-clique when we choose the 1-factor leave in the z-clique.

Moreover, note that the triples $\{x, y, z\}$ and $\{x, u, v\}$ are totally free in all three of their cliques, since they are not in any "triangles" of the assumed triangulation. So, in the x-clique, we pair $\{x, y, z\}$ with $\{x, u, v\}$, in the y-clique we pair $\{x, y, z\}$ with $\{y, u, *\}$, and in the z-clique we pair $\{x, y, z\}$ with $\{z, u, *\}$.

If we now proceed as described above, we would obtain a Z_2 leave with $\{x, y, z\}$ and $\{x, u, v\}$ being the vertices of degree 3, joined to each other, and with $\{x, y, z\}$ also joined to $\{y, u, *\}$ and $\{z, u, *\}$ (see Figure 1). But note that the triples $\{y, u, *\}$,

$\{z, u, *\}$ and $\{x, u, v\}$ all lie in the u-clique, and so might form a triangle in that clique, in which case we could trade the edges of this triangle with some of the edges of the Z_2 leave and obtain a Z_3 leave with vertex (triple) $\{x, u, v\}$ of degree 5. (See Figures 2 and 3.)

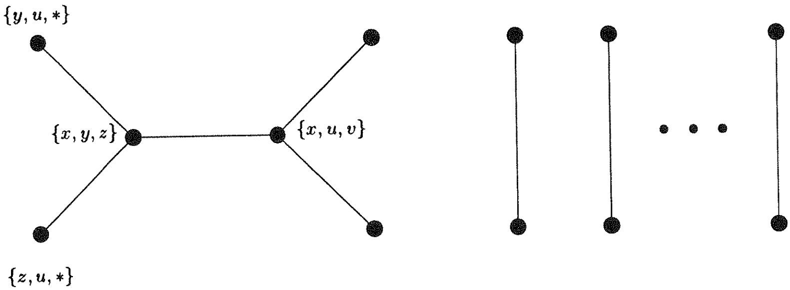


Figure 1

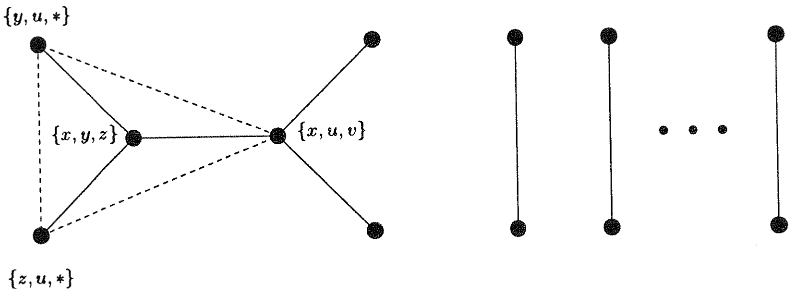


Figure 2

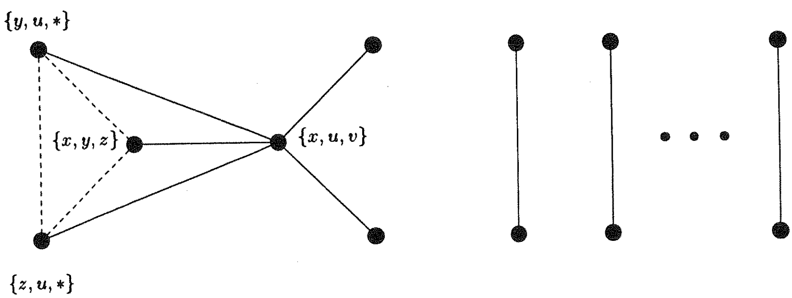


Figure 3

We now consider when the triples $\{y, u, *\}$, $\{z, u, *\}$ and $\{x, u, v\}$ can indeed form a triangle, which would thus permit the trade used to obtain the Z_3 leave. The triple $\{x, u, v\}$ is free in the u-clique, while $\{y, u, *\}$ and $\{z, u, *\}$ appear in different "triangles" of the triangulation. So these three triples can be paired, in the 1-factor leave within the u-clique, with three other triples. When we pack the u-clique with triangles, we are free to choose the first triangle in the packing, provided the three triples forming this triangle are on distinct edges of the 1-factor. With the triples $\{y, u, *\}$, $\{z, u, *\}$ and $\{x, u, v\}$ being on separate edges of the 1-factor, we may choose the first triangle of the packing so that it contains these three triples.

Thus the trade from Z_2 to Z_3 is indeed possible when the triangulation possesses property (P). □

Example We now present an example which shows that each of the three possible leaves can indeed be achieved. Consider the cyclic STS(13) formed by the starter set $\{\{0,1,4\},\{0,2,7\}\}$. One possible triangulation of this STS is yielded by the following "triangles":

$$\{\{0,1,4\},\{1,2,5\},\{11,0,5\}\} \qquad \{\{1,3,8\},\{3,5,10\},\{4,5,8\}\}$$
$$\{\{5,7,12\},\{6,7,10\},\{10,12,4\}\} \qquad \{\{7,8,11\},\{7,9,1\},\{9,11,3\}\}$$
$$\{\{2,3,6\},\{3,4,7\},\{0,2,7\}\} \qquad \{\{2,4,9\},\{4,6,11\},\{5,6,9\}\}$$
$$\{\{6,8,0\},\{8,9,12\},\{9,10,0\}\} \qquad \{\{12,1,6\},\{10,11,1\},\{11,12,2\}\}$$

The unused triples in the above triangulation are $\{12,0,3\}$ and $\{8,10,2\}$, and so this triangulation will yield a Z_1 leave.

A second possible triangulation is yielded by the following "triangles":

$$\{\{0,1,4\},\{1,2,5\},\{11,0,5\}\} \qquad \{\{1,3,8\},\{3,5,10\},\{4,5,8\}\}$$
$$\{\{5,7,12\},\{6,7,10\},\{10,12,4\}\} \qquad \{\{7,8,11\},\{7,9,1\},\{9,11,3\}\}$$
$$\{\{2,3,6\},\{3,4,7\},\{0,2,7\}\} \qquad \{\{2,4,9\},\{4,6,11\},\{5,6,9\}\}$$
$$\{\{6,8,0\},\{8,9,12\},\{9,10,0\}\} \qquad \{\{8,10,2\},\{10,11,1\},\{11,12,2\}\}$$

The unused triples in the above triangulation are $\{12,0,3\}$ and $\{12,1,6\}$. Since the two unused triples have a point in common, we obtain a Z_2 leave when the edge between these two triples comprises part of the 1-factor leave obtained by extending the matching induced in the 12-clique.

This second triangulation we now use to also obtain a Z_3 leave. Notice that the triple $\{1,3,8\}$ is in a "triangle" in which the point 1 occurs exactly once and that the triple $\{2,3,6\}$ is in a "triangle" in which the point 6 occurs exactly once. Hence the first condition listed for property P is satisfied, where $u = 3$, $v = 0$, $x = 12$, $y = 1$ and $z = 6$, and thus we can obtain a Z_3 in which the triple $\{12,3,0\}$ has degree 5.

For the remaining cases, in which $n \equiv 9$, 21, 33 (mod 36), we note that $n \equiv 0$ (mod 3) and so parallel classes of triples *might* exist for the STS in question. (However, it is known that for instance a parallel-class-free STS of order 21 exists [3].) We obtain decompositions of the BIG of those STS which have both a single parallel class of triples and also an accompanying set $\mathcal{T}$ of "triangles" which satisfy the following criteria:

- when $n \equiv 9 \pmod{36}$, each triple not in the parallel class is contained in exactly one "triangle" of $\mathcal{T}$;
- when $n \equiv 21 \pmod{36}$, all but one triple not in the parallel class are contained in exactly one "triangle" of $\mathcal{T}$;
- when $n \equiv 33 \pmod{36}$, all but two triples not in the parallel class are contained in exactly one "triangle" of $\mathcal{T}$; and
- each triple in the parallel class is contained in exactly four "triangles" of $\mathcal{T}$, in such a way that among the four corresponding triangles in the BIG, the eight edges incident with the triple of the parallel class are partitioned such that three are in one of the triple's three cliques, three are in a second clique and two are in the third clique.

We call such a parallel class and accompanying set of "triangles" a *pseudo-triangulation.*

Claim *If a STS(n) possesses a pseudo-triangulation, then $n \geq 21$.*

Proof Since the STS has a parallel class, necessarily $n \equiv 0 \pmod 3$. The number of triples in the STS is $n(n-1)/6$, of which $n/3$ are in the assumed parallel class of the pseudo-triangulation. Each of these $n/3$ triples of the parallel class forms four "triangles" with eight other triples, for a total of $3n$ triples. It is thus necessary that $3n \leq n(n-1)/6$, which can be easily solved to show that either $n \leq 0$ or $n \geq 19$. We may ignore values of $n \leq 0$, and so we have $n \geq 19$. However, noting that $n \equiv 0 \pmod 3$, we conclude that $n \geq 21$. □

So, for the remaining cases in which $n \equiv 9, 21, 33 \pmod{36}$, only the case $n = 9$ is too small to permit a pseudo-triangulation. We thus present a decomposition of the BIG of the STS(9) into triangles, with leave a 1-factor, in the Appendix.

Lemma 4 *If a STS(n) with $n \equiv 9 \pmod{36}$ possesses a pseudo-triangulation, then its BIG can be decomposed into triangles with leave a 1-factor.*

Proof Each clique has size 4 (mod 6), and a packing of any complete graph of order 4 (mod 6) has leave a tripole. Thus naively packing each clique of the BIG with triangles will result in a leave having several more edges than the required 1-factor.

Consider, however, the edges in the BIG which are contained within the triangles induced by the "triangles" of the pseudo-triangulation of the STS. In each clique of the BIG, these edges appear as a matching plus either one copy of $K_{1,3}$ or one copy of $K_{1,2}$. We now choose additional edges in each clique in order to extend these edges into a tripole. (It will be these added edges which will form the final leave in the BIG.) The non-tripole edges in each clique correspond to the edges of a triangle packing within each clique; we begin our triangle decomposition of the BIG with the triangles in these clique packings.

Our goal now is to combine several of the edges of the tripoles into triangles, so that the only remaining edges in the BIG constitute a 1-factor. To do this, we first observe that each triple of the parallel class of the pseudo-triangulation corresponds to a vertex in the BIG which is the head of the tripole in each of its three cliques. We also observe that each "triangle" of the pseudo-triangulation induces a triangle in the BIG, and that the edges of these triangles have not yet been used by any triangles of our decomposition. So we now add to our decomposition the triangles induced by the "triangles" of the pseudo-triangulation.

Each triple of the parallel class, being contained in exactly four "triangles", will thus have its corresponding vertex go from degree 9 to degree 1 in the BIG. Likewise, each triple not in the parallel class, being contained in exactly 1 "triangle" will have its corresponding vertex go from degree 3 to degree 1 in the BIG. We thus have a triangle decomposition with a 1-factor leave. □

Example To illustrate that Lemma 4 applies to a non-empty set of STS, we now present a pseudo-triangulation of a STS(45).

Consider the cyclic STS(45) formed with starter set $\{\{0,1,3\}, \{0,4,9\}, \{0,6,17\}, \{0,7,25\}, \{0,8,24\}, \{0,10,23\}, \{0,12,26\}, \{0,15,30\}\}$. One possible pseudo-triangulation of this STS is yielded by the parallel class induced by the triple $\{0,15,30\}$ and the following "triangles":

$$\sigma^j\{\{18,19,21\},\{19,20,22\},\{20,21,23\}\}$$

$$\begin{array}{cc}
\sigma^i\{\{0,6,17\},\{7,17,30\},\{0,15,30\}\} & \sigma^i\{\{15,21,32\},\{22,32,0\},\{0,15,30\}\} \\
\sigma^i\{\{30,36,2\},\{37,2,15\},\{0,15,30\}\} & \sigma^i\{\{30,34,39\},\{15,23,39\},\{0,15,30\}\} \\
\sigma^i\{\{0,4,9\},\{38,0,18\},\{30,38,9\}\} & \sigma^i\{\{15,19,24\},\{0,8,24\},\{19,31,0\}\} \\
\sigma^i\{\{33,34,36\},\{8,15,33\},\{34,1,15\}\} & \sigma^i\{\{3,4,6\},\{23,30,3\},\{4,16,30\}\}
\end{array}$$

where σ represents the permutation $(0,1,\ldots,44)$, for $i \in \{0,1,\ldots,14\}$ and for $j \in \{0,3,6,9,12\}$.

Lemma 5 *If a STS(n) with $n \equiv 21 \pmod{36}$ possesses a pseudo-triangulation, then its BIG can be decomposed into triangles with leave a tripole.*

Proof This case is exactly the same as Lemma 4 above, except that one triple of the STS(n) does not occur in any "triangle" of the assumed pseudo-triangulation. Thus in the BIG, the degree of this triple (vertex) remains at 3, and it becomes the head of the tripole leave in the BIG. □

Example To illustrate that Lemma 5 applies to a non-empty set of STS, we now present a pseudo-triangulation of a STS(21).

Consider the cyclic STS(21) formed with starter set $\{\{0,1,10\}, \{0,3,8\}, \{0,2,6\}, \{0,7,14\}\}$. One possible pseudo-triangulation of this STS is yielded by the parallel class induced by the triple $\{0,7,14\}$ and the following "triangles":

$$\{\{15,17,0\},\{17,19,2\},\{19,0,4\}\}$$
$$\{\{16,18,1\},\{18,20,3\},\{20,1,5\}\}$$

$$\sigma^i\{\{0,3,8\},\{1,3,7\},\{0,7,14\}\} \quad \sigma^i\{\{14,15,3\},\{7,10,15\},\{0,7,14\}\}$$
$$\sigma^i\{\{0,1,10\},\{8,10,14\},\{0,7,14\}\} \quad \sigma^i\{\{7,8,17\},\{14,17,1\},\{0,7,14\}\}$$

where σ represents the permutation $(0,1,\ldots,20)$, for $i \in \{0,1,\ldots,6\}$. The unused triple in this pseudo-triangulation is $\{0,2,6\}$.

Lemma 6 *If a STS(n) with $n \equiv 33 \pmod{36}$ possesses a pseudo-triangulation in which the two unused triples are disjoint, then its BIG can be decomposed into triangles with leave Z_1.*

If a STS(n) with $n \equiv 33 \pmod{36}$ possesses a pseudo-triangulation in which the two unused triples share a common point, then its BIG can be decomposed into triangles with leave Z_2.

Finally, if a STS(n) with $n \equiv 33 \pmod{36}$ possesses a pseudo-triangulation in which the two unused triples share a common point, and if also the pseudo-triangulation satisfies a certain extra condition (P'), then its BIG can be decomposed into triangles with leave Z_3.

Proof The clique size of the BIG is 4 (mod 6) and so a packing of a clique with triangles has individual leave for that clique being a tripole. However, a pseudo-triangulation of a STS(n) in this case omits two of the triples. If the two triples missed by the pseudo-triangulation of the STS contain no common point, (say triples $\{x,y,z\}$ and $\{u,v,w\}$), then following the construction described in Lemma 4 above, the vertices in the BIG labelled by $\{x,y,z\}$ and $\{u,v,w\}$ will remain with degree 3, while every other vertex in the BIG will have its degree dropped to one, since every other triple of the STS is in some "triangle" of the assumed pseudo-triangulation of the STS. Thus the overall leave in the BIG will be Z_1 in this case.

Now suppose that the two triples missed by the pseudo-triangulation share a point; say they are $\{x,y,z\}$ and $\{x,u,v\}$. When packing the x-clique with triangles, provided we choose the edge between vertex $\{x,y,z\}$ and vertex $\{x,u,v\}$ to be an edge of the tripole leave for the x-clique packing, the overall leave in the BIG, after including the triangles from the pseudo-triangulation in the BIG packing, will be Z_2, with vertices $\{x,y,z\}$ and $\{x,u,v\}$ each having degree 3, and with these vertices of the BIG being joined by an edge in the leave.

Finally we need to show that the leave Z_3 can also be achieved.

Suppose that the assumed pseudo-triangulation of the STS has the two triples of the STS which do not occur in any "triangle" of the pseudo-triangulation sharing a common point. Let these triples be $\{x,y,z\}$ and $\{x,u,v\}$. Moreover, suppose that the pseudo-triangulation has property (P'), that is, assume that one of the following holds:

- the triple $\{y,u,*\}$ is in a "triangle" of the pseudo-triangulation in which y occurs only once, *and* the triple of the form $\{z,u,*\}$ is in a different "triangle" in which z occurs only once, *and* neither $\{y,u,*\}$ nor $\{z,u,*\}$ is in the parallel class of the pseudo-triangulation;

- the triple $\{y, v, *\}$ is in a "triangle" of the pseudo-triangulation in which y occurs only once *and* the triple of the form $\{z, v, *\}$ is in a different "triangle" in which z occurs only once, *and* neither $\{y, v, *\}$ nor $\{z, v, *\}$ is in the parallel class of the pseudo-triangulation;

- the triple $\{y, u, *\}$ is in a "triangle" of the pseudo-triangulation in which u occurs only once *and* the triple of the form $\{y, v, *\}$ is in a different "triangle" in which v occurs only once, *and* neither $\{y, u, *\}$ nor $\{y, v, *\}$ is in the parallel class of the pseudo-triangulation;

- the triple $\{z, u, *\}$ is in a "triangle" of the pseudo-triangulation in which u occurs only once *and* the triple of the form $\{z, v, *\}$ is in a different "triangle" in which v occurs only once, *and* neither $\{z, u, *\}$ nor $\{z, v, *\}$ is in the parallel class of the pseudo-triangulation.

The above four possibilities are all isomorphic, so we only need consider one of them. We deal with the first of the above possibilities.

Now consider the y-clique. When picking the tripole leave in here, the triple $\{y, u, *\}$ is free to be paired with any other triple containing y that is free in the y-clique. Similarly, in the z-clique, the triple $\{z, u, *\}$ can be paired with any other free triple in the z-clique when we choose the tripole leave in the z-clique.

Moreover, note that the triples $\{x, y, z\}$ and $\{x, u, v\}$ are totally free in all three of their cliques, since they are not in any "triangles" of the assumed pseudo-triangulation. So, in the x-clique, we pair $\{x, y, z\}$ with $\{x, u, v\}$, in the y-clique we pair $\{x, y, z\}$ with $\{y, u, *\}$, and in the z-clique we pair $\{x, y, z\}$ with $\{z, u, *\}$.

If we now proceed as described above, we would obtain a Z_2 leave with $\{x, y, z\}$ and $\{x, u, v\}$ being the vertices of degree 3, joined to each other, and with $\{x, y, z\}$ also joined to $\{y, u, *\}$ and $\{z, u, *\}$ (see Figure 1). But note that the triples $\{y, u, *\}$, $\{z, u, *\}$ and $\{x, u, v\}$ all lie in the u-clique, and so might form a triangle in that clique, in which case we could trade the edges of this triangle with some of the edges of the Z_2 leave and obtain a Z_3 leave with vertex (triple) $\{x, u, v\}$ of degree 5. (See Figures 2 and 3.)

We now consider when the triples $\{y, u, *\}$, $\{z, u, *\}$ and $\{x, u, v\}$ can indeed form a triangle, which would thus permit the trade used to obtain the Z_3 leave. The triple $\{x, u, v\}$ is free in the u-clique, while $\{y, u, *\}$ and $\{z, u, *\}$ appear in different "triangles" of the pseudo-triangulation. So these three triples can be paired, in the tripole leave within the u-clique, with three other triples to form three edges of the matching portion of the tripole. When we pack the u-clique with triangles, we are free to choose the first triangle in the packing, provided the three triples forming this triangle are on distinct edges of the matching portion of the tripole. With the triples $\{y, u, *\}$, $\{z, u, *\}$ and $\{x, u, v\}$ being on separate edges of the matching portion of the tripole, we may choose the first triangle of the packing so that it contains these three triples.

Thus the trade from Z_2 to Z_3 is indeed possible when the pseudo-triangulation possesses property (P'). □

Example To illustrate that Lemma 6 applies to a non-empty set of STS, we now present a pseudo-triangulation of a STS(33). Moreover, we show that each of the three possible leaves can indeed be achieved.

Consider the cyclic STS(33) formed with starter set $\{\{0,2,6\}, \{0,5,15\}, \{0,1,9\}, \{0,3,16\}, \{0,7,19\}, \{0,11,22\}\}$. One possible pseudo-triangulation of this STS is yielded by the parallel class induced by the triple $\{0,11,22\}$ and the following "triangles":

$$\{\{20,22,26\},\{22,24,28\},\{,24,26,30\}\}$$
$$\{\{23,25,29\},\{25,27,31\},\{27,29,0\}\}$$
$$\{\{26,28,32\},\{28,30,1\},\{30,32,3\}\}$$

$$\sigma^i\{\{0,11,22\},\{0,1,9\},\{6,9,22\}\} \qquad \sigma^i\{\{0,11,22\},\{11,12,20\},\{17,20,0\}\}$$
$$\sigma^i\{\{0,11,22\},\{22,23,31\},\{28,31,11\}\} \qquad \sigma^i\{\{0,11,22\},\{7,12,22\},\{0,7,19\}\}$$
$$\sigma^i\{\{31,0,4\},\{18,23,0\},\{4,11,23\}\} \qquad \sigma^i\{\{9,11,15\},\{29,1,11\},\{15,22,1\}\}$$

where σ represents the permutation $(0,1,\dots,32)$, for $i \in \{0,1,\dots,10\}$. The unused triples in this pseudo-triangulation are $\{21,23,27\}$ and $\{29,31,2\}$ and thus a Z_1 leave is obtained.

A second pseudo-triangulation can be formed from the above pseudo-triangulation by replacing the "triangle" $\{\{23,25,29\}, \{25,27,31\}, \{27,29,0\}\}$ with the "triangle" $\{\{21,23,27\}, \{23,25,29\}, \{25,27,31\}\}$. As a result, the unused triples now are $\{27,29,0\}$ and $\{29,31,2\}$. Since these two unused triples have a point in common, we obtain a Z_2 leave when the edge between these two triples comprises part of the 1-factor leave obtained by extending the matching induced in the 29-clique.

This second triangulation we now use to also obtain a Z_3 leave. Notice that the triple $\{27,2,26\}$ is in a "triangle" in which the point 27 occurs exactly once and that the triple $\{0,2,6\}$ is in a "triangle" in which the point 0 occurs exactly once. Hence the first condition listed for property P' is satisfied, where $u = 2$, $v = 31$, $x = 29$, $y = 27$ and $z = 0$, and thus we can obtain a Z_3 in which the triple $\{29,31,2\}$ has degree 5.

We summarise our results with the following theorem.

Theorem 2 *If $n \equiv 3$ or $7 \pmod{12}$, then the block-intersection graph of any STS(n) can be decomposed into triangles*

If $n \equiv 1$, 13, or 25 (mod 36), then we obtain a BIG decomposition whenever the STS possesses a triangulation (with a leave of either a 1-factor, a tripole, Z_1, Z_2, or, when property P is satisfied, Z_3).

If $n \equiv 9$, 21, or 33 (mod 36), then we obtain a BIG decomposition whenever the STS possesses a pseudo-triangulation (with a leave of either a 1-factor, a tripole, Z_1, Z_2, or, when property P' is satisfied, Z_3).

To possess a pseudo-triangulation, a STS necessarily must possess a parallel class of triples. For $n \equiv 9$, 21, or 33 (mod 36), we note that not all STS(n) have such a parallel class [3]. We therefore conjecture that some other method of decomposition exists, in which the existence of a parallel class is not necessary.

4 Appendix

Here we present a decomposition of BIG(STS(9)), with leave a 1-factor. Let the vertices of the BIG be labelled $\{A, B, C, D, E, F, G, H, I, J, K, L\}$ where each vertex corresponds to a triple of the STS(9) as indicated below:

$$\begin{array}{llll} A : \{1,2,3\} & D : \{1,4,7\} & G : \{1,5,9\} & J : \{1,6,8\} \\ B : \{4,5,6\} & E : \{2,5,8\} & H : \{2,6,7\} & K : \{2,4,9\} \\ C : \{7,8,9\} & F : \{3,6,9\} & I : \{3,4,8\} & L : \{3,5,7\} \end{array}$$

Then the triangles in a decomposition of the BIG are as follows.

$$\begin{array}{llll} \{B,F,L\}, & \{C,F,H\}, & \{E,H,L\}, & \{F,I,K\}, \\ \{C,E,K\}, & \{B,I,J\}, & \{C,G,L\}, & \{B,E,G\}, \\ \{F,G,J\}, & \{C,D,I\}, & \{B,D,K\}, & \{D,H,J\}, \\ \{A,I,L\}, & \{A,H,K\}, & \{A,E,J\}, & \{A,D,G\}. \end{array}$$

The resulting leave in the BIG is the 1-factor

$$\{A,F\}, \{B,H\}, \{C,J\}, \{D,L\}, \{E,I\}, \{G,K\}.$$

It is interesting to note that cliques 1, 2, 4 and 5 contain no edges of the leave, while cliques 3, 6, 7 and 9 contain one edge each, and clique 8 contains two edges.

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