Covering a Bipartite Graph with Cycles Passing through Given Edges

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Abstract

We propose a conjecture: for each integer $k \geq 2$, there exists $N(k)$ such that if $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = n \geq N(k)$ and $d(x)+d(y) \geq n+k$ for each pair of non-adjacent vertices x and y of G with $x \in V_1$ and $y \in V_2$, then for any k independent edges $e_1, \ldots, e_k$ of G, there exist k vertex-disjoint cycles $C_1, \ldots, C_k$ in G such that $e_i \in E(C_i)$ for all $i \in \{1, \ldots, k\}$ and $V(C_1 \cup \cdots \cup C_k) = V(G)$. If this conjecture is true, the condition on the degrees of G is sharp. We prove this conjecture for the case $k = 2$ in the paper.

1 Introduction

Let k be a positive integer and let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2$. It is well known [1, 3] that if $d(x) + d(y) \geq n + 1 + k$ for each pair of non-adjacent vertices x and y of G with $x \in V_1$ and $y \in V_2$, then for any forest F with at most k edges and consisting of vertex-disjoint paths of G, G has a hamiltonian cycle passing through all the edges of F. We propose the following conjecture.

Conjecture *A For each integer $k \geq 2$, there exists $N(k)$ such that if $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = n \geq N(k)$ and $d(x) + d(y) \geq n + k$ for each pair of non-adjacent vertices x and y of G with $x \in V_1$ and $y \in V_2$, then for any k independent edges $e_1, \ldots, e_k$ of G, there exist k vertex-disjoint cycles $C_1, \ldots, C_k$ in G such that $e_i \in E(C_i)$ for all $i \in \{1, \ldots, k\}$ and $V(C_1 \cup \cdots \cup C_k) = V(G)$.*

If this conjecture is true, the condition on the degrees of G is sharp. To see this, let $G = (X, Y; E)$ be a bipartite graph obtained from the complete bipartite graph $K_{n-1,n}$ by adding a new vertex x_0 to $K_{n-1,n}$ such that $N_G(x_0) = \{x_1, x_2, \ldots, x_k\}$ where $x_1, x_2, \ldots, x_k$ are k vertices of $K_{n-1,n}$ whose degrees in $K_{n-1,n}$ are $n-1$. Then for each pair of non-adjacent vertices x and y of G with $x \in X$ and $y \in Y$, we have $x_0 \in \{x, y\}$ and $d(x) + d(y) = n + k - 1$. Let $e_1, \ldots, e_k$ be k independent edges in G such that e_i is incident with x_i for all $i \in \{1, \ldots, k\}$ and $e_1 = x_0x_1$. Clearly, every cycle passing through e_1 must contain at least three vertices in $\{x_0, x_1, \dots, x_k\}$. Therefore G does not possess k vertex-disjoint cycles satisfying the requirement.

In this paper, we prove the conjecture for the case $k = 2$. To state the result, let F be a graph obtained from $K_{4,4}$ by removing three independent edges from $K_{4,4}$. We prove the following:

Theorem *B Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 4$. Suppose $d(x) + d(y) \geq n + 2$ for each pair of non-adjacent vertices x and y of G with $x \in V_1$ and $y \in V_2$. Then for any two independent edges e_0 and e_1 of G, G has two vertex-disjoint cycles C_0 and C_1 such that $e_i \in E(C_i)$ for each $i \in \{0, 1\}$ and $V(C_0 \cup C_1) = V(G)$, unless G is isomorphic to F.*

We discuss only finite simple graphs and use standard terminology and notation from [2] except as indicated. Let G be a graph. For a vertex $u \in V(G)$ and a subgraph H of G, $N(u, H)$ is the set of neighbors of u contained in H, i.e., $N(u, H) = N_G(u) \cap V(H)$. We let $d(u, H) = |N(u, H)|$. Thus $d(u, G)$ is the degree of u in G. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. Let e be an edge of G. An e-subgraph of G is a subgraph H of G such that $e \in E(H)$. If P is an e-path, we define $\sigma(e, P) = \min(|E(P')|, |E(P'')|)$ where P' and P'' are two components of $P - e$. If $\sigma(e, P) = 0$, we say e is an endedge of P. We use $l(C)$ and $l(P)$ to denote the length of a cycle C and the length of a path P, respectively. For a path P of an odd length, say $P = x_1 x_2 \dots x_{2q}$, we define $E_0(P) = \{x_1x_2, x_{2q-1}x_{2q}\} \cup \{x_i x_{i+1} | i = 2, 4, \dots, 2q-2\}$ and $E_1(P) = \{x_j x_{j+1} | j = 3, 5, \dots, 2q-3\}$, and moreover, let $r(e, P) = 0$ if $e \in E_0(P)$ and $r(e, P) = 1$ if $e \in E_1(P)$.

2 Lemmas

The following lemmas are Ore-type lemmas in bipartite graphs. The proofs of them can be found in or easily deduced from [1, 3, 4]. Let $G = (V_1, V_2; E)$ be a given bipartite graph in the following.

Lemma 2.1 *Let e be an edge and $P = x_1 x_2 \dots x_{2q}$ an e-path in G. Let $y \in V(G) - V(P)$ such that $\{x_{2q}, y\} \not\subseteq V_i$ for every $i \in \{1, 2\}$. If $d(x_{2q}, P) + d(y, P) \geq q + 1 + r(e, P)$, then G has an e-path P' such that $V(P') = V(P) \cup \{y\}$. Moreover, if $e \neq x_1 x_2$, then P' is a path from y to x_1.*

Proof. Clearly, the lemma holds if $yx_{2q} \in E$. So we may assume $yx_{2q} \notin E$. As $d(y, P) > 0$, it is also easy to see that if $e = x_1x_2$ and $x_1x_{2q} \in E$, then the lemma holds. Hence we may assume that if $e = x_1 x_2$, then $x_1 x_{2q} \notin E$. Let $I = \{x_{i+1} | x_i x_{2q} \in E\}$. Then $|N(y, P) \cap I| = |N(y, P)| + |I| - |N(y, P) \cup I| \geq q + 1 + r(e, P) - q = 1 + r(e, P)$. If $r(e, P) = 0$ then there exists $x_{i+1} \in N(y, P) \cap I$. Clearly, $x_i x_{i+1} \neq e$. On the other hand if $r(e, P) = 1$ then there exist i and j with $i \neq j$ such that $\{x_{i+1}, x_{j+1}\} \subseteq N(y, P) \cap I$. We may assume w.l.o.g. that $x_i x_{i+1} \neq e$. In either case, $P' = y x_{i+1} x_{i+2} \dots x_{2q} x_i x_{i-1} \dots x_1$ is the desired path. □

Lemma 2.2 *Let e be an edge and $P = x_1 x_2 \dots x_{2q}$ an e-path with $q \geq 2$ in G. If $d(x_1, P) + d(x_{2q}, P) \geq q + 1 + r(e, P)$, then G has an e-cycle C with $V(C) = V(P)$.*

Proof. Clearly, the lemma holds if $x_1x_{2q} \in E$. So we may assume $x_1x_{2q} \notin E$. As in the proof of Lemma 2.1, the condition implies that there exist x_i and x_j for some $\{i,j\} \subseteq \{1,3,\dots,2q-1\}$ such that $\{x_1x_{i+1}, x_{2q}x_i, x_1x_{j+1}, x_{2q}x_j\} \subseteq E$ with $i \neq j$ if $r(e,P)=1$. As $x_1x_{2q} \notin E$, we see that $e \notin \{x_ix_{i+1}, x_jx_{j+1}\}$ if $r(e,P)=0$. We may assume w.l.o.g. that $e \neq x_ix_{i+1}$ if $i \neq j$. Then $C' = x_1x_2\dots x_ix_{2q}x_{2q-1}\dots x_{i+1}x_1$ is the desired cycle. □

Lemma 2.3 *Let e be an edge and C an e-cycle in G. Let $y \in V(G)-V(C)$. If $d(y,C) \geq 2$, then $G[V(C)\cup\{y\}]$ contains an e-cycle C' such that $l(C') < l(C)$, unless $d(y,C)=2$, $N(y,C)=\{x',x''\}$ and C has a subpath $x'zx''$ with z not incident with e.*

Proof. Say $C = x_1x_2\dots x_{2q}x_1$ with $e = x_1x_{2q}$. Let $\{x_i,x_j\} \subseteq N(y,C)$ such that $1 \leq i < j \leq 2q$ and $xy \notin E$ for all $x \in V(C)-\{x_i,x_{i+1},\dots,x_j\}$. Clearly, $C' = x_1\dots x_iyx_j\dots x_{2q}x_1$ is an e-cycle. If $l(C') \not< l(C)$, then $j=i+2$. This proves the lemma. □

Lemma 2.4 *Let e be an edge, C an e-cycle and P a path with two endvertices $u \in V_1$ and $v \in V_2$ in G such that $V(C)\cap V(P)=\emptyset$. Let $l(C)=2q$. If $d(u,C)+d(v,C) \geq q+1$, then G has an e-cycle C' with $V(C')=V(C\cup P)$.*

Proof. Let $C = x_1x_2\dots x_{2q}x_1$ with $e=x_1x_{2q}$ and $x_1 \in V_1$. The condition implies that $\{x_iv, x_{i+1}u\} \subseteq E$ for some $i \in \{1,3,\dots,2q-1\}$. Then $x_1x_{2q}x_{2q-1}\dots x_{i+1}uPvx_i x_{i-1}\dots x_1$ is the desired cycle. □

3 Proof of the Theorem

Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=n \geq 4$ such that $d(x)+d(y) \geq n+2$ for each pair of non-adjacent vertices x and y of G with $x \in V_1$ and $y \in V_2$. Suppose that there exist two independent edges e_0 and e_1 of G such that G does not have two vertex-disjoint cycles C_0 and C_1 with $e_i \in E(C_i)$ for each $i \in \{0,1\}$ and $V(C_0 \cup C_1)=V(G)$. Then we shall prove that G is isomorphic to F.

Say $e_1 = uv$. Clearly, $d(x,G-u-v)+d(y,G-u-v) \geq n+2-2=(n-1)+1$ for each pair of non-adjacent vertices x and y of $G-u-v$. Thus by Lemma 2.2, $G-u-v$ is hamiltonian. Hence $G-u-v$ has an e_0-cycle C. Choose an e_0-cycle C in $G-u-v$ such that

$$l(C) \text{ is minimal.} \tag{1}$$

Subject to (1), we choose C such that

$$\text{The length of a longest path of } G-V(C) \text{ containing } e_1 \text{ is maximal.} \tag{2}$$

Let P be a longest e_1-path in H. Subject to (1) and (2), we further choose C and P such that

$$\sigma(e_1,P) \text{ is minimal.} \tag{3}$$

Note that C does not have a chord by (1). Let $C = x_1x_2 \dots x_{2s}x_1$ with $x_1 \in V_1$ and $e_0 = x_1x_{2s}$, and $H = G - V(C)$. By our assumption on G, H does not have a hamiltonian cycle passing through e_1. Let $P = y_1y_2 \dots y_m$. W.l.o.g., say $y_1 \in V_1$. We claim

Claim 1. $V(P) = V(H)$, i.e., $m = 2n - 2s$.

Suppose $m < 2n - 2s$. We distinguish two cases: m is even or m is odd.

Case a: m is even, say $m = 2t$.

Choose a vertex y_0 from $H - V(P)$ such that $y_0 \in V_1$. By Lemma 2.1 and (2), $d(y_0, P) + d(y_{2t}, P) \leq t + r(e_1, P)$. Then we have $d(y_0, H) + d(y_{2t}, H) \leq \frac{1}{2}|V(H)| + r(e_1, P)$. It follows that $d(y_0, C) + d(y_{2t}, C) \geq s + 2 - r(e_1, P)$. Suppose first that $d(y_0, C) + d(y_{2t}, C) \geq s + 2$. Then we have $d(y_0, C) \geq 2$. By Lemma 2.3 and (1), we must have $d(y_0, C) = 2$, and consequently, $d(y_{2t}, C) = s$. Furthermore, $N(y_0, C) = \{x_i, x_{i+2}\}$ for some $i \in \{2, 4, \dots, 2s - 2\}$. Then $C' = C - x_{i+1} + y_0x_i + y_0x_{i+2}$ is an e_0-cycle with $l(C') = l(C)$ and $P' = P + y_{2t}x_{i+1}$ is an e_1-path with $l(P') = l(P) + 1$, contradicting (2). Hence we must have $r(e_1, P) = 1$ and $d(y_0, C) + d(y_{2t}, C) = s + 1$. It follows that $t \geq 3$ and $d(y_0, P) + d(y_{2t}, P) = t + 1$. In particular, $d(y_0, P) > 0$. If G has an e_1-cycle C' with $V(C') = V(C)$, then $C' + y_0$ has an e_1-path P' with $V(P') = V(P) \cup \{y_0\}$, contradicting (2). Therefore by Lemm 2.2, we have $d(y_1, P) + d(y_{2t}, P) \leq t + 1$. It follows that $d(y_1, C) + d(y_{2t}, C) \geq n + 2 - t - 1 \geq s + 2$. By Lemma 2.3 and (1), $d(y_1, C) \leq 2$ and $d(y_{2t}, C) \leq 2$. We conclude that $d(y_1, C) = d(y_{2t}, C) = s = 2$. W.l.o.g., say $|V(P_1)| \leq |V(P_2)|$ where P_1 and P_2 are two components of $P - e_1$. Then $C'' = C - x_3 + y_1$ is an e_0-cycle with $l(C'') = l(C)$ and $P'' = P - y_1 + y_{2t}x_3$ is an e_1-path with $l(P'') = l(P)$ and $\sigma(e_1, P'') = \sigma(e_1, P) - 1$, contradicting (3).

Case b: m is odd, say $m = 2t + 1$.

We have $y_{2t+1} \in V_1$. Then either $e_1 = y_{2i-1}y_{2i}$ or $e_1 = y_{2i+1}y_{2i}$ for some $i \in \{1, 2, \dots, t\}$. W.l.o.g., say the former holds. Then $r(e_1, P - y_1) = 0$ and $\sigma(e_1, P - y_1) > 0$ if e_1 is on $P - y_1$. Choose y_0 from $H - V(P)$ such that $y_0 \in V_2$. By Lemma 2.1 and (2), if $d(y_0, P - y_1) + d(y_{2t+1}, P - y_1) \geq t + 1$, then G has a path P' from y_0 to y_2 such that $V(P') = V(P - y_1) \cup \{y_0\}$, and moreover, P' is an e_1-path when e_1 is on $P - y_1$. Thus $P' + y_2y_1$ is an e_1-path, contradicting (2). Hence $d(y_0, P) + d(y_{2t+1}, P) = d(y_0, P - y_1) + d(y_{2t+1}, P - y_1) \leq t$. It follows that $d(y_0, C) + d(y_{2t+1}, C) \geq n + 2 - t - d(y_0, H - V(P)) \geq s + 3$. Thus $d(y_0, C) \geq 3$. By Lemma 2.3, this is in contradiction with (1). So the claim is true. □

Let $t = n - s$. Then $m = 2t$ by Claim 1. We divide our proof into the following two cases: $r(e_1, P) = 0$ or $r(e_1, P) = 1$.

Case 1: $r(e_1, P) = 0$.

By Lemma 2.2, we have $d(y_1, P) + d(y_{2t}, P) \leq t$. Hence

$$d(y_1, C) + d(y_{2t}, C) \geq s + 2. \tag{4}$$

If $e_1 \neq y_1y_2$ and $e_1 \neq y_{2t-1}y_{2t}$, then by Lemma 2.3 and (1), $d(y_1, C) \leq 2$ and $d(y_{2t}, C) \leq 2$, and consequently, we obtain $d(y_1, C) = d(y_{2t}, C) = s = 2$ by (4).

Then we may assume w.l.o.g. that $|V(P_1)| \leq |V(P_2)|$ where P_1 and P_2 are two components of $P - e_1$. Replacing C and P by $C - x_3 + y_1$ and $P - y_1 + y_{2t}x_3$, we obtain a contradiction with (3). Hence either $e_1 = y_1y_2$ or $e_1 = y_{2t-1}y_{2t}$. W.l.o.g., say $e_1 = y_{2t-1}y_{2t}$.

If $t = 1$, then $s \geq 3$ as $n \geq 4$. Clearly, for any two vertices $x \in V(C) \cap V_1$ and $y \in V(C) \cap V_2$ with $xy \notin E$, we have $n + 2 \leq d(x) + d(y) \leq 6$, and consequently, this implies that $s = 3$ and $\{xy_2, yy_1\} \subseteq E$. Thus G is isomorphic to F. Hence we may assume that $t \geq 2$.

We claim that $s = 2$. If this is not true, i.e., $s \geq 3$, then $d(y_1, C) = 2$ and $d(y_{2t}, C) = s$ by (1), (4) and Lemma 2.3. Moreover, $N(y_1, C) = \{x_i, x_{i+2}\}$ for some $i \in \{2, 4, \ldots, 2s-2\}$. Then $C' = C - x_{i+1} + y_1x_i + y_1x_{i+2}$ is an e_0-cycle with $l(C') = l(C)$ and $P' = y_2y_3 \ldots y_{2t}x_{i+1}$ is an e_1-path with $r(e_1, P') = 0$. Thus $y_2x_{i+1} \notin E$. By Lemma 2.3 and (1), $d(y_2, C') \leq 2$ and $d(x_{i+1}, C') \leq 2$. It follows that $d(y_2, P') + d(x_{i+1}, P') \geq t + 1$. By Lemma 2.2, $G[V(P')]$ has an e_1-cycle containing all the vertices of P', a contradiction. This shows $s = 2$.

By (4), we have $d(y_1, C) = 2$ and $d(y_{2t}, C) = 2$. Clearly, the theorem holds if $x_3y_2 \in E$. Hence we may assume $x_3y_2 \notin E$. If $x_1y_2 \notin E$, then we obtain $d(y_2, P') + d(x_3, P') \geq t + 1$ with $P' = y_2y_3 \ldots y_{2t}x_3$ and $r(e_1, P') = 0$, and by Lemma 2.2, a contradiction follows. Hence we have $x_1y_2 \in E$.

Let $2a - 1$ be the greatest integer in $\{1, 3, \ldots, 2t-3\}$ such that $G[\{y_1, y_2, \ldots, y_{2a}\}]$ is isomorphic to $K_{a,a}$, $N(y_i, C) = \{x_2, x_4\}$ and $N(y_{i+1}, C) = \{x_1\}$ for all $i \in \{1, 3, \ldots, 2a-1\}$. The above argument shows that $a \geq 1$. We claim $a = t - 1$. On the contrary, assume $a < t - 1$. Let $L = y_{2a+1}y_{2a+2} \cdots y_{2t}$. Clearly, $x_1y_{2i}y_{2i-1} \cdots y_2y_1x_2x_3x_4x_1$ is an e_0-cycle in G for all $i \in \{1, 2, \ldots, a\}$. Therefore $y_{2t}y_{2i-1} \notin E$ for all $i \in \{1, 2, \ldots, a+1\}$. In particular, $G[V(L)]$ does not have a hamiltonian cycle passing through e_1. By Lemma 2.2, $d(y_{2a+1}, L) + d(y_{2t}, L) \leq t - a$. As $d(y_{2a+1}) + d(y_{2t}) \geq t + 4$, we see that $N(y_{2a+1}, C) \supseteq \{x_2, x_4\} \cup \{y_2, y_4, \ldots, y_{2a+2}\}$. Clearly, $C'' = x_1x_2y_1 \ldots y_{2a+1}x_4x_1$ is an e_0-cycle in G. Let $P'' = y_{2a+2}y_{2a+3} \cdots y_{2t}x_3$. Then $G[V(P'')]$ does not have a hamiltonian cycle passing through e_1. In particular, $x_3y_{2a+2} \notin E$. Since $r(e_1, P'') = 0$, we obtain $d(y_{2a+2}, P'') + d(x_3, P'') \leq t - a$ by Lemma 2.2. As $x_3y_{2i} \notin E$ for all $i \in \{1, 2, \ldots, a\}$, we see that $d(y_{2a+2}, P) + d(x_3, P) \leq t + 1$, and consequently, $d(x_3, C) + d(y_{2a+2}, C) \geq 3$. However, it is clear that $d(x_3, C) + d(y_{2a+2}, C) \leq 3$. It follows that $d(y_{2a+2}, P) + d(x_3, P) = t + 1$ and $d(x_3, C) + d(y_{2a+2}, C) = 3$, and consequently, $N(y_{2a+2}) \supseteq \{x_1, y_1, y_3, \ldots, y_{2a+1}\}$. This is a contradiction to the maximality of a. This shows that $a = t - 1$. If $t \geq 3$, then $x_1x_4y_1y_2x_1$ and $x_3x_2y_3y_4 \ldots y_{2t}x_3$ are the two desired cyles. Hence $t = 2$. Clearly, we have two desired cycles if $x_2y_3 \in E$. So $x_2y_3 \notin E$. As $d(x_2) + d(y_3) \geq 6$, we see that $x_4y_3 \in E$ and therefore G is isomorphic to F.

Case 2: $r(e_1, P) = 1$.

Say $e_1 = y_{2a+1}y_{2a+2}$ for some $2a + 1 \in \{3, 5, \ldots, 2t-3\}$. Then either $\sigma(e_1, P) = 2a$ or $\sigma(e_1, P) = 2t - 2a - 2$. W.l.o.g., say $\sigma(e_1, P) = 2t - 2a - 2$. Let $C' = y_{2a+1}y_{2a+2} \cdots y_{2t}y_{2a+1}$ and $H' = H - V(C')$. Then $G[V(C \cup H')]$ does not have a hamiltonian cycle passing through e_0. It is also easy to see that for every endvertex u of a hamiltonian path of H', u is not adjacent to a vertex of $C' - \{y_{2a+1}, y_{2a+2}\}$ for otherwise we would have an e_1-path Q with $V(P) = V(Q)$ and $\sigma(e_1, Q) < \sigma(e_1, P)$, contradicting (3).

Let $L = y_1y_2 \ldots y_{2a}$. We have $d(y_1, C') \leq 1$ and $d(y_{2a}, C') \leq 1$. By Lemma 2.4, we have $d(y_1, C) + d(y_{2a}, C) \leq s$. We claim that H' is hamiltonian. This is obvious if $y_1y_{2a} \in E$. If $y_1y_{2a} \notin E$, then $d(y_1, L) + d(y_{2a}, L) \geq t + s + 2 - s - 2 = t$, and therefore by Lemma 2.2, H' is hamiltonian. So the claim is true. Thus $d(y, H') = 0$ for all $y \in V(C') - \{y_{2a+1}, y_{2a+2}\}$. If $d(y_1, L) + d(y_{2t}, L) \geq a + 1$, then there exists $i \in \{1, 3, \ldots, 2a-1\}$ such that $\{y_1y_{i+1}, y_iy_{2t}\} \subseteq E$, and consequently, $P' = y_{2a}y_{2a-1} \cdots y_{i+1}y_1y_2 \cdots y_iy_{2t}y_{2t-1} \cdots y_{2a+2}y_{2a+1}$ is an e_1-path with $V(P') = V(P)$ and $0 = \sigma(e_1, P') < \sigma(e_1, P)$, a contradiction. This shows $d(y_1, L) + d(y_{2t}, L) \leq a$. It follows that $d(y_1, P) + d(y_{2t}, P) \leq t+1$, and consequently, $d(y_1, C) + d(y_{2t}, C) \geq s+1$. Similarly, we can show that $d(y_{2a}, P) + d(y_{2t-1}, P) \leq t+1$ and $d(y_{2a}, C) + d(y_{2t-1}, C) \geq s + 1$. In particular, we have obtained $d(y_1, C) > 0$ and $d(y_{2a}, C) > 0$. By Lemma 2.3 and (1), $d(y_{2t-1}, C) + d(y_{2t}, C) \leq 4$. We obtain

$$\begin{aligned}
2a &\geq d(y_1, H') + d(y_{2a}, H') \\
&\geq 2(s + t + 2) - [d(y_{2t-1}) + d(y_{2t})] - [d(y_1, C \cup C') + d(y_{2a}, C \cup C')] \\
&\geq 2(s + t + 2) - (2(t - a) + 4) - (s + 2) \\
&= 2a + s - 2.
\end{aligned}$$

It follows that $s = 2$, $d(y_{2t-1}, C) + d(y_{2t}, C) = 4$ and $d(y_1, C) + d(y_{2a}, C) = 2$. Since $d(y_1, C) > 0$ and $d(y_{2a}, C) > 0$, it is clear that if $y_1x_4 \notin E$ or $y_{2a}x_1 \notin E$, then $G[V(C \cup L)$ has a hamiltonian cycle containing e_0, a contradiction. If $\{y_1x_4, y_{2a}x_1\} \subseteq E$, then $x_1x_4y_1Ly_{2a}x_1$ and $C' - y_{2t-1}y_{2t} + x_3y_{2t} + x_2y_{2t-1}$ are the two desired cycles. This proves the theorem. □

Remarks. The following example shows $N(3) \geq 7$ if $N(3)$ exists. Let G be a bipartite graph obtained from $K_{6,6}$ with a bipartition $(\{x_1, \ldots, x_6\}, \{y_1, \ldots, y_6\})$ by removing $x_3y_5, x_3y_6, y_3x_5, y_3x_6$ and x_4y_4 from $K_{6,6}$. Clearly, $d(x) + d(y) \geq 9$ for each pair of non-adjacent vertices x and y of G with $x \in \{x_1, \ldots, x_6\}$ and $y \in \{y_1, \ldots, y_6\}$. But G does not contain three vertex-disjoint cycles passing through x_1y_1, x_2y_2 and x_3y_3, respectively. Hence $N(3) \geq 7$.

As for general finite simple graphs, we proposed a conjecture in [5] and proved it for the case $k = 2$.

Conjecture C [5] *For each integer $k \geq 2$, there exists $N(k)$ such that if G is a graph of order $n \geq N(k)$ and $d(x) + d(y) \geq n + 2k - 2$ for each pair of non-adjacent vertices x and y of G, then for any k independent edges $e_1, \ldots, e_k$ of G, there exist k vertex-disjoint cycles $C_1, \ldots, C_k$ in G such that $e_i \in E(C_i)$ for all $i \in \{1, \ldots, k\}$ and $V(C_1 \cup \cdots \cup C_k) = V(G)$.*

Moreover, we know that if this conjecture is true, then the condition on the degrees of G is sharp.

Note added in the proof: Conjectures A and C were verified recently for $k = 3$. However, the verification is more tedious than the above proof.

4 References

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