# The $k$th Upper Generalized Exponent Set for the Class of Non–symmetric Primitive Matrices

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**Abstract**

Let $QB_n$ be the set of $n \times n$ $(n > 8)$ non–symmetric primitive matrices with at least one pair of nonzero symmetric entries. For each positive integer $2 \le k \le n-2$, we give the $k$th upper generalized exponent set for $QB_n$ by using a graph theoretical method.

## 1 Introduction

An $n \times n$ nonnegative matrix $A$ is called *primitive* if there exist some positive integer $t$ such that $A^t > 0$. The least such positive integer $t$ is called the *exponent* of $A$, denoted by $\gamma(A)$.

In [1], Brualdi and Liu defined the $k$th upper generalized exponent $F(A, k)$ as follows.

**Definition 1.1** ([1]) *Let $A$ be a primitive matrix of order $n$ and $1 \le k \le n-1$. Set*

$$F(A,k) = \min\{p \mid \text{ no set of } k \text{ rows of } A^p \text{ has a column of all zeros } \}.$$

*$F(A,k)$ is called the $k$th upper generalized exponent of $A$.*

The $k$th upper generalized exponent is a generalization of the traditional concept of the exponent. Background can be found in [1].

It is well–known that for each nonnegative matrix $A$ there exists an associated digraph $D(A)$ whose adjacency matrix has the same zero entries as $A$. A digraph $D$ is primitive iff $D$ is strongly connected and $\text{g.c.d}(r_1, r_2, \cdots, r_\lambda) = 1$, where $\{r_1, r_2, \cdots, r_\lambda\} = L(D)$ is the set of distinct lengths of the directed cycles of $D$. $A$ is primitive iff $D(A)$ is primitive.

**Definition 1.2** ([1]) *Let $X$ be the vertex subset of a primitive digraph $D$. The exponent $\exp_D(X)$ is the smallest positive integer $p$ such that for each vertex $y$ of $D$, there exists a walk of length $p$ from at least one vertex in $X$ to $y$.*

**Definition 1.3** ([1]) *Let $D$ be a primitive digraph of order $n$ and $1 \leq k \leq n-1$. Set*

$$F(D,k) = \max\{\exp_D(X) \mid X \subseteq V(D), \mid X \mid = k\}. \qquad (1.1)$$

*$F(D,k)$ is called the $k$th upper generalized exponent of $D$.*

It is obvious that

$$F(A,k) = F(D(A),k). \qquad (1.2)$$

**Definition 1.4** *Let $a_1, \cdots, a_k$ be positive integers. The Frobenius set $S(a_1, \cdots, a_k)$ of the numbers $a_1, \cdots, a_k$ is defined as*

$$S(a_1, \cdots, a_k) = \{\sum_{i=1}^{k} x_i a_i \mid x_1, \cdots, x_k \text{ are nonnegative integers }\}.$$

It is well-known, by a lemma of Schur, that if g.c.d$(a_1, \cdots, a_k) = 1$, then $S(a_1, \cdots, a_k)$ contains all sufficiently large nonnegative integers. In this case we define the *Frobenius number* $\phi(a_1, \cdots, a_k)$ to be the least integer $\phi$ such that $m \in S(a_1, \cdots, a_k)$ for all integers $m \geq \phi$.

For the case $k = 2$, it is well-known that if $a$ and $b$ are relatively prime positive integers, then the Frobenius number is

$$\phi(a,b) = (a-1)(b-1). \qquad (1.3)$$

It is easy to see the following result.

**Lemma 1.5** *Let $X$ be a set of $k$ vertices of a primitive digraph $D$ of order $n$ and $1 \leq k \leq n-1$. Let $R = \{r_{i_1}, \cdots, r_{i_t}\} \subseteq L(D)$ such that g.c.d$(r_{i_1}, \cdots, r_{i_t}) = 1$. Let $d_R(i,j)$ be the length of the shortest walk from vertex $i$ to vertex $j$ in $D$ which meets at least one cycle of each length $r_{i_1}, \cdots, r_{i_t}$. Let $d_R(X) = \max_{j \in V(D)} \min_{i \in X} d_R(i,j)$ and $\phi_R = \phi(r_{i_1}, \cdots, r_{i_t})$. Then we have*

$$\exp_D(X) \leq d_R(X) + \phi_R. \qquad (1.4)$$

Let $QB_n$ be the set of $n \times n$ $(n > 8)$ non-symmetric primitive matrices with at least one pair of nonzero symmetric entries, $QB_n^+$ the set of matrices in $QB_n$ with nonzero trace and $QB_n^0$ the set of matrices in $QB_n$ with zero trace. For each positive integer $1 \leq k \leq n-1$, let $E_{nk}$ be the set of $k$th upper generalized exponents of the matrices in $QB_n$, $E_{nk}^+$ the set of $k$th upper generalized exponents of the matrices in $QB_n^+$ and $E_{nk}^0$ the set of the $k$th upper generalized exponents of the matrices in $QB_n^0$. In this paper, we give the complete characterizations of $E_{nk}^+$ and $E_{nk}^0$, so that the $k$th upper generalized exponent set problem for $QB_n$ is settled.

Notice that if $k = 1$, then $F(A,k) = \gamma(A)$. In this case, the exponent sets $E_{n1}^+$ and $E_{n1}^0$ have already been determined in [3]. So we will only consider the cases $2 \leq k \leq n-2$.

We will make use of the following notations. Let $D$ be an primitive digraph with $D = (V(D), E(D))$. Let $C_r$ be a cycle of length $r$ ( called an $r$–cycle). We denote the distance from vertex $x$ to vertex $y$ of $D$ by $d(x,y)$. If $i, j \in V(D)$, then $(i,j)$ denotes an arc from vertex $i$ to vertex $j$ and $[i,j]$ denotes a edge between two vertices $i$ and $j$, i.e. a 2–cycle.

# 2 The generalized exponent set $E_{nk}^+$

In this section we will determine the generalized exponent set $E_{nk}^+$.

**Theorem 2.1** *Let $n, k$ be positive integers with $2 \le k \le n-2$ and $A \in QB_n^+$. Then*

$$F(D(A), k) \le 2n - k - 2. \tag{2.1}$$

**Proof.** Let $X$ be any $k$–vertex subset of $D(A)$, $w$ a loop of $D(A)$ and $[u, v]$ a edge of $D(A)$.

Case 1: $w \in X$. Then $\exp_{D(A)}(X) \le \max_{y \in V(D(A))} d(w, y) \le n - 1 \le 2n - k - 2$.

Case 2: $\{u, v\} \subseteq X$. Then $\exp_{D(A)}(X) \le \max_{y \in V(D(A))} \min\{d(u, y), d(v, y)\} \le n-2 < 2n - k - 2$.

Other cases: Let $l = \max_{y \in V(D(A))} d(w, y)$ and $h = \min_{x \in X} d(x, w)$. Then $l \le n - 1$ and $h \le n - k$.

(1) $l \le n - 2$ or $h \le n - k - 1$. Then $\exp_{D(A)}(X) \le h + l \le 2n - k - 2$.

(2) $l = n - 1$ and $h = n - k$. Then $\exp_{D(A)}(X) \le n \le 2n - k - 2$.

The proof of the theorem is completed. ∎

**Theorem 2.2** *Let $n, k$ be positive integers with $2 \le k \le n - 2$. Then*

$$\{k+1, k+2, \cdots, 2n-k-2\} \subseteq E_{nk}^+. \tag{2.2}$$

**Proof.** Suppose $k + 1 \le m \le n - 1$. Firstly, we consider $D_1 = D(A)$ with vertex set $V(D_1) = \{1, 2, \cdots, n\}$ and arc set $E(D_1) = \{(1,1), [1,2], (2,3), (3,4), \cdots, (m-1, m), (m, m+1), (m, m+2), \cdots, (m, n), (m+1, 1), (m+2, 1), \cdots, (n, 1)\}$.

It is obvious that $A \in QB_n^+$. Take $X_0 = \{3, 4, \cdots, k+2\}$. It is not difficult to verify that there is no walk of length $2m - k - 1$ from any vertex of $X_0$ to the vertex $m + 1$. So we have

$$F(D_1, k) \ge \exp_{D_1}(X_0) \ge 2m - k. \tag{2.3}$$

On the other hand, let $X$ be any $k$–vertex subset of $D_1$. If $\{1, 2\} \cap X \ne \emptyset$, then

$$\exp_{D_1}(X) \le m + 1 \le 2m - k. \tag{2.4}$$

If $\{1, 2\} \cap X = \emptyset$, letting $i$ be the vertex of $X$ which is closest to 1, then $d(i, 1) \le m + 1 - k - 2 + 1 = m - k$ and so

$$\exp_{D_1}(X) \le m - k + m = 2m - k. \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) we have

$$F(D_1, k) = 2m - k. \tag{2.6}$$

Next, we consider $D_2 = D(A)$ with vertex set $V(D_2) = \{1, 2, \cdots, n\}$ and arc set $E(D_2) = \{(1,1), (2,2), [1,2], (2,3), (3,4), \cdots, (m-1, m), (m, m+1), (m, m+2), \cdots, (m, n), (m+1, 1), (m+2, 1), \cdots, (n, 1), (m+1, 2), (m+2, 2), \cdots, (n, 2)\}$.

It is obvious that $A \in QB_n^+$. Take $X_0 = \{3, 4, \cdots, k+2\}$. It is not difficult to verify that there is no walk of length $2m-k-2$ from any vertex of $X_0$ to the vertex $m+1$. Then $F(D_2, k) \geq \exp_{D_2}(X_0) \geq 2m-k-1$.

On the other hand, let $X$ be any $k$–vertex subset of $D_2$. If $\{1,2\} \cap X \neq \emptyset$, then $\exp_{D_2}(X) \leq m \leq 2m-k-1$. If $\{1,2\} \cap X = \emptyset$, letting $j$ be the vertex of $X$ which is closest to 2, then $d(j, 2) \leq m+1-k-2+1 = m-k$ and $\exp_{D_2}(X) \leq m-k+m-1 = 2m-k-1$.

So we have

$$F(D_2, k) = 2m - k - 1. \tag{2.7}$$

Notice that $k+1 \leq m \leq n-1$. Combining (2.6) and (2.7) we obtain (2.2). ∎

**Theorem 2.3** *Let $n, k$ be positive integers with $2 \leq k \leq n-2$. Then*

$$\{2, 3, \cdots, k\} \subseteq E_{nk}^+. \tag{2.8}$$

**Proof.** Suppose $2 \leq m \leq k$. We consider $D_2 = D(A)$ in theorem 2.2.

Take $X_0 = \{n, n-1, \cdots, n-k+1\}$. Then $|X_0| = k$. Since $n-k+1 \geq 3$, it is not difficult to verify that there is no walk of length $m-1$ from any vertex of $X_0$ to the vertex $m+1$. Then $F(D_2, k) \geq \exp_{D_2}(X_0) \geq m$.

On the other hand, let $X$ be any $k$–vertex subset of $D_2$. If $1 \in X$, then $\exp_{D_2}(X) \leq m$. If $1 \notin X$, then $X \cap \{m+1, m+2, \cdots, n\} \neq \emptyset$ and so $\exp_{D_2}(X) \leq m$.

So we have $F(D_2, k) = m$. Noticing that $2 \leq m \leq k$, we obtain (2.8). ∎

**Theorem 2.4** *Let $n, k$ be positive integers with $2 \leq k \leq n-2$. Then*

$$E_{nk}^+ = \{1, 2, 3, \cdots, 2n-k-2\}. \tag{2.9}$$

**Proof.** We consider $D = D(A)$ with vertex set $V(D) = \{1, 2, \cdots, n\}$ and arc set $E(D) = \{(i,j) \mid i, j = 1, 2, \cdots, n\} \setminus \{(2,1)\}$.

It is obvious that $A \in QB_n^+$ and $F(D, k) = 1$. So $1 \in E_{nk}^+$.

Combining (2.1), (2.2) and (2.8) we obtain (2.9). ∎

# 3 The generalized exponent set $E_{nk}^0$

In this section we will determine the generalized exponent set $E_{nk}^0$.

**Lemma 3.1** ([2]) *Suppose $\Gamma$ is primitive digraph of order $n$ and $s$ is the length of the shortest directed cycles of $\Gamma$. Then*

$$F(\Gamma, k) \leq (n-k)s + (n-s), \quad (1 \leq k \leq n-1). \tag{3.1}$$

**Theorem 3.2** *Let $n, k$ be positive integers with $2 \leq k \leq n-2$.*

(1) *If $n$ is even, then*

$$\{11, 12, \cdots, 3n-2k-3\} \subseteq E_{nk}^0. \tag{3.2}$$

(2) *If $n$ is odd, then*

$$\{11, 12, \cdots, 3n-2k-5, 3n-2k-4, 3n-2k-2\} \subseteq E_{nk}^0. \tag{3.3}$$

**Proof.** Firstly, let $4 \le s \le m \le n-1$ and $m - s = 0 \pmod 2$. We consider $D_1(m) = D(A)$ with vertex set $V(D_1(m)) = \{1, 2, \cdots, n\}$ and arc set $E(D_1(m)) = \{[1,2], (2,3), (2,4), \cdots, (2, s-1), (3, s), (4, s), \cdots, (s-1, s), (s, s+1), (s+1, s+2), \cdots, (m-1, m), (m, m+1), (m, m+2), \cdots, (m, n), (m+1, 1), (m+2, 1), \cdots, (n, 1)\}$.

It is obvious that $A \in QB_n^0$. Let $R = \{2, m-s+5\}$. We consider two cases.

Case 1: $k \le n-4$ and $\max\{4, 2k-m+4\} \le s \le k+3 \le m \le n-1$. In this case, we will prove that

$$F(D_1(m), k) = 3m - 2k - s + 5. \tag{3.4}$$

Take $X_0 = \{3, 4, \cdots, s-1, s+1, s+3, \cdots, 2k-s+5\}$. Then $|X_0| = k$ and $2k - s + 5 \le m + 1$. It is not difficult to verify that there is no walk of even length $3m - 2k - s + 4$ from any vertex of $X_0$ to the vertex $m+1$. So we have $F(D_1(m), k) \ge \exp_{D_1(m)}(X_0) \ge 3m - 2k - s + 5$.

On the other hand, let $X$ be any $k$–vertex subset of $D_1(m)$. If $\{1, 2\} \cap X \ne \emptyset$, then by (1.4) we have $\exp_{D_1(m)}(X) \le d(1, m+1) + \phi(2, m-s+5) \le 3m-2k-s+5$. If there are vertices $i, j \in X$ such that $(i,j) \in E(D_1(m))$, then $\exp_{D_1(m)}(X) \le \max\limits_{y \in V(D_1(m))} d(j, y) \le 3m - 2k - s + 5$. In addition, letting $l$ be the vertex of $X$ which is closest to 1, we have $1 \le d(l, 1) \le m + 1 - 2k + s - 5 + 1 = m + s - 2k - 3$ and $\exp_{D_1(m)}(X) \le d(l, 1) + m - s + 4 + \phi(2, m - s + 5) \le 3m - 2k - s + 5$.

So we obtain (3.4). By hypotheses we also have the following.

(i) If $3 \le k \le \frac{n-1}{2}$, then

$$\{i \mid i \text{ is odd and } 3m-3k+2 \le i \le 4m-4k+1\} \subseteq E_{nk}^0, \quad (k+3 \le m \le 2k). \tag{3.5}$$

(ii) If $\frac{n-1}{2} \le k \le n-4$, then

$$\{i \mid i \text{ is odd and } 3m-3k+2 \le i \le 4m-4k+1\} \subseteq E_{nk}^0, \quad (k+3 \le m \le n-1). \tag{3.6}$$

(iii) If $2 \le k \le \frac{n-1}{2}$, then

$$\{i \mid i \text{ is odd and } 3m-3k+2 \le i \le 3m-2k+1\} \subseteq E_{nk}^0, \quad (2k \le m \le n-1). \tag{3.7}$$

Case 2: $m = n-1$, $\frac{n+1}{2} \le k \le n-2$ and $4 \le s \le 2k - n + 3$. In this case, we will prove that

$$F(D_1(n-1), k) = 3n - 2k - s + 2. \tag{3.8}$$

Take $X_0 = \{2, 3, 4, \cdots, 2k-n+1, 2k-n+2, 2k-n+4, \cdots, n\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of even length $3n-2k-s+1$ from any vertex of $X_0$ to the vertex $n$. So we have $F(D_1(n-1), k) \ge \exp_{D_1(n-1)}(X_0) \ge 3n - 2k - s + 2$.

On the other hand, let $X$ be any $k$–vertex subset of $D_1(n-1)$. There are adjacent vertices of $D_1(n-1)$ in $X$. Let $l = \min\{d(j, 1) \mid j \in X$ and there exist $i \in X$ such that $(i,j) \in E(D_1(n-1))\}$, which implies that $l \le 2n - 2k - 1$. Then $\exp_{D_1(n-1)}(X) \le l + n - s + 3 \le 3n - 2k - s + 2$.

We obtain (3.8). Noticing that $4 \le s \le 2k-n+3$, we also have

$$\{i \mid i \text{ is odd and } 4n-4k-1 \le i \le 3n-2k-2\} \subseteq E^0_{nk}, \quad (\frac{n+1}{2} \le k \le n-2). \tag{3.9}$$

Next, let $4 \le s < m \le n-1$ and $m-s=1 \pmod 2$. We consider $D_2(m)=D(A)$ with vertex set $V(D_2(m))=\{1,2,\cdots,n\}$ and arc set $E(D_2(m))=\{[1,2],(2,3),(2,4),\cdots,(2,s-1),(3,s),(4,s),\cdots,(s-1,s),(s,s+1),(s+1,s+2),\cdots,(m-1,m),(m,m+1),(m,m+2),\cdots,(m,n),(m+1,2),(m+2,2),\cdots,(n,2),(m,1)\}$.

It is obvious that $A \in QB^0_n$. Let $R=\{2,m-s+4\}$. We consider two cases.

Case 1: $k \le n-5$ and $\max\{4,2k-m+5\} \le s \le k+3 < m \le n-1$. In this case, we will prove that

$$F(D_2(m),k)=3m-2k-s+3. \tag{3.10}$$

Take $X_0=\{3,4,\cdots,s-1,s+1,s+3,\cdots,2k-s+5\}$. Then $|X_0|=k$ and $2k-s+5 \le m$. It is not difficult to verify that there is no walk of odd length $3m-2k-s+2$ from any vertex of $X_0$ to the vertex $m+1$. So we have $F(D_2(m),k) \ge \exp_{D_2(m)}(X_0) \ge 3m-2k-s+3$.

On the other hand, let $X$ be any $k$–vertex subset of $D_2(m)$. If $\{1,2\} \cap X \ne \emptyset$, then by (1.4) we have $\exp_{D_2(m)}(X) \le d(1,m+1)+\phi(2,m-s+5) \le 3m-2k-s+3$. If there are vertices $i,j \in X$ such that $(i,j) \in E(D_2(m))$, then $\exp_{D_2(m)}(X) \le \max\limits_{y \in V(D_2(m))} d(j,y) < 3m-2k-s+3$. In addition, letting $l$ be the vertex of $X$ which is closest to 2, we have $1 \le d(l,2) \le m+1-2k+s-5+1=m+s-2k-3$ and $\exp_{D_2(m)}(X) \le d(l,2)+m-s+3+\phi(2,m-s+4) \le 3m-2k-s+3$.

So we obtain (3.10). By hypotheses we also have the following.

(i) If $3 \le k \le \frac{n-2}{2}$, then

$$\{i \mid i \text{ is even and } 3m-3k \le i \le 4m-4k-2\} \subseteq E^0_{nk}, \quad (k+4 \le m \le 2k+1). \tag{3.11}$$

(ii) If $\frac{n-2}{2} \le k \le n-5$, then

$$\{i \mid i \text{ is even and } 3m-3k \le i \le 4m-4k-2\} \subseteq E^0_{nk}, \quad (k+4 \le m \le n-1). \tag{3.12}$$

(iii) If $2 \le k \le \frac{n-2}{2}$, then

$$\{i \mid i \text{ is even and } 3m-3k \le i \le 3m-2k-1\} \subseteq E^0_{nk}, \quad (2k+1 \le m \le n-1). \tag{3.13}$$

Case 2: $m=n-1$, $\frac{n}{2} \le k \le n-2$ and $4 \le s \le 2k-n+4$. In this case, we will prove that

$$F(D_2(n-1),k)=3n-2k-s. \tag{3.14}$$

Take $X_0=\{2,3,4,\cdots,2k-n+2,2k-n+3,2k-n+5,\cdots,n-1\}$. Then $|X_0|=k$ and it is not difficult to verify that there is no walk of odd length $3n-2k-s-1$ from any vertex of $X_0$ to the vertex $n$. So we have $F(D_2(n-1),k) \ge \exp_{D_2(n-1)}(X_0) \ge 3n-2k-s$.

On the other hand, let $X$ be any $k$–vertex subset of $D_2(n-1)$. There are adjacent vertices of $D_2(n-1)$ in $X$. Let $l=\min\{d(j,2) \mid j \in X$ and there exist $i \in$

$X$ such that $(i,j) \in E(D_2(n-1))\}$, which implies that $l \le 2n-2k-2$. Then $\exp_{D_2(n-1)}(X) \le l+n-s+2 \le 3n-2k-s$.

So we obtain (3.14). Noticing that $4 \le s \le 2k-n+4$ we also have

$$\{i \mid i \text{ is even and } 4n-4k-4 \le i \le 3n-2k-4\} \subseteq E^0_{nk}, \quad (\frac{n}{2} \le k \le n-2). \quad (3.15)$$

The theorem now follows from (3.5)–(3.7), (3.9) and (3.11)–(3.13), (3.15). ∎

**Theorem 3.3** *Let $n$ be odd and $2 \le k \le n-2$. Then*

$$3n-2k-3 \in E^0_{nk}. \quad (3.16)$$

**Proof.** We consider $D = D(A)$ with vertex set $V(D) = \{1,2,\cdots,n\}$ and arc set $E(D) = \{[1,2],[2,3],(3,4),(4,5),\cdots,(n-1,n),\cdots,(n,1)\}$.

It is obvious that $A \in QB^0_n$. Let $R = \{2,n\}$. We will prove that

$$F(D,k) = 3n-2k-3. \quad (3.17)$$

Case 1: $2 \le k \le \frac{n-1}{2}$. Take $X_0 = \{4,6,\cdots,2k+2\}$ (if $k = \frac{n-1}{2}$, then $X_0 = \{4,6,\cdots,n-1,1\}$). Then $|X_0| = k$ and there is no walk of odd length $3n-2k-4$ from any vertex of $X_0$ to the vertex $n$. So $F(D,k) \ge 3n-2k-3$.

On the other hand, let $X$ be any $k$–vertex subset of $D$. If $\{1,2,3\} \cap X \ne \emptyset$, then by (1.4) we have $\exp_D(X) \le n-1+n-1 \le 3n-2k-3$. If $\{1,2,3\} \cap X = \emptyset$ and there are adjacent vertices of $D$ in $X$, then $\exp_D(X) \le n-5+n < 3n-2k-3$. If $\{1,2,3\} \cap X = \emptyset$ and there are not adjacent vertices of $D$ in $X$, then $k \le \frac{n-3}{2}$. By (1.4) we have $\exp_D(X) \le n-2k-2+n+n-1 = 3n-2k-3$.

So we obtain (3.17) for $2 \le k \le \frac{n-1}{2}$.

Case 2: $\frac{n+1}{2} \le k \le n-2$. Take $X_0 = \{1,3,4,5,\cdots,2k-n+3,2k-n+5,\cdots,n-1\}$. Then $|X_0| = k$ and there is no walk of odd length $3n-2k-4$ from any vertex of $X_0$ to the vertex $n$. So $F(D,k) \ge 3n-2k-3$.

On the other hand, let $X$ be any $k$–vertex subset of $D$. There are adjacent vertices of $D$ in $X$. Let $l = \min\{d(j,1) \mid j \in X$ and there exist $i \in X$ such that $(i,j) \in E(D)\}$, which implies that $l \le 2n-2k-2$. Then $\exp_D(X) \le l+n-1 \le 3n-2k-s$.

So we obtain (3.17) for $\frac{n+1}{2} \le k \le n-2$.

Now it is straight forward to obtain (3.16) from Case 1 and Case 2. ∎

**Lemma 3.4** ([4]) *Let digraph $D^t$ be the digraph with the same vertex set as $D$ in which there is an arc from $x$ to $y$ iff there is a walk of length $t$ from $x$ to $y$ in $D$. If $D$ is a primitive digraph, then for any positive integer $t$, $D^t$ is a primitive digraph.*

**Theorem 3.5** *Let $n$ be even.*

(1) *If $\frac{n+4}{2} \le k \le n-2$, then*

$$3n-2k-2 \in E^0_{nk}. \quad (3.18)$$

(2) *If $2 \le k \le \frac{n+2}{2}$ and $A \in QB^0_n$, then*

$$F(A,k) \le 3n-2k-3. \quad (3.19)$$

**Proof.** (1) $\frac{n+4}{2} \le k \le n-2$. We consider $D = D(A)$ with vertex set $V(D) = \{1, 2, \cdots, n\}$ and arc set $E(D) = \{(n-2, n-1), (n-1, n), (n, n-4), (n-3, n-4), (n-4, n-5), \cdots, (4, 3), (3, 2), (3, n-2), [2, 1], (1, n-3)\}$.

It is obvious that $A \in QB_n^0$. Take $X_0 = V(D)\backslash\{2, 4, \cdots, 2(n-k)\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of length $3n-2k-3$ from any vertex of $X_0$ to the vertex $n$. By (3.1) we have $F(D, k) = 3n-2k-2$. This implies that $3n-2k-2 \in E_{nk}^0$.

(2) $2 \le k \le \frac{n+2}{2}$ and $A \in QB_n^0$. Let $D$ be the associated digraph of $A$ whose shortest odd cycle length is $r$ $(3 \le r \le n-1)$ and $C_2 = [u, v]$ the 2–cycle of $D$. Let $X$ be any $k$–vertex subset of $D$ and $y$ any vertex of $D$. In the following we only need to prove that there is a vertex $x \in X$ and a walk of length $3n-2k-3$ from $x$ to $y$.

Let $q = \min\{d(u, y), d(v, y)\}$. If $q \le n-3$, then we can take a vertex $v$ of $C_2$ such that there is a walk of length $n-3$ from $v$ to $y$. Consider that digraph $D^2$. Since $v$ is a loop of $D^2$, there is a vertex $x$ in $X$ such that there exists a walk of length $n-k$ from $x$ to $v$ in $D^2$. Hence there is a walk of length $2(n-k)$ from $x$ to $v$ in $D$. According to above arguments, there is a walk of length $2(n-k)+n-3 = 3n-2k-3$ from $x$ to $y$.

If $q = n-2$. Let $d(v, y) = n-2$. We consider two cases.

Case 1: There are not adjacent vertices of $D$ in $X$. Let $x_0$ be the vertex of $X$ which is closest to $v$. Then for each positive integer $p$ with $p \ge d(x_0, v) + n - 2 + \phi(2, r)$, there exists a walk of length $p$ from $x_0$ to $y$.

Subcase 1: $\{u, v\} \cap X \ne \emptyset$. If $u \in X$, then for each positive odd integer $p \ge n-1$, there is a walk of length $p$ from $u$ to $y$. This implies that there is a walk of length $3n-2k-3$ from $u$ to $y$. If $v \in X$, noticing that $n-2+\phi(2, r) \le n-2+2(n-k)-2 = 3n-2k-4$, then there is a walk of length $3n-2k-3$ from $v$ to $y$.

Subcase 2: $\{u, v\} \cap X = \emptyset$ and there exists $C_r$ such that $V(C_r) \cap X = \emptyset$. Then $r \le n-k$. Since $d(x_0, v) + n - 2 + \phi(2, r) \le n-k+n-2+n-k-1 = 3n-2k-3$, there is a walk of length $3n-2k-3$ from $x_0$ to $y$.

Subcase 3: $\{u, v\} \cap X = \emptyset$ and there exists $C_r$ such that $V(C_r) \cap X \ne \emptyset$. Let $|V(C_r) \cap X| = m$ $(2 \le m \le k)$. Then $d(x_0, v) \le n-k-(m-1)$. When $m < k$ we have $n-k-(r-m) \ge k-m-1$, namely, $r \le n-2k+2m+1$.

If $m \le k-2$, then $d(x_0, v)+n-2+\phi(2, r) \le 3n-3k+m-1 \le 3n-2k-3$. If $m = k$, then $d(x_0, v)+n-2+\phi(2, r) \le n-k-(k-1)+n-2+n-2 = 3n-2k-3$. If $m = k-1$, noticing $r \ne n-1$, then $d(x_0, v)+n-2+\phi(2, r) \le n-k-(k-2)+n-2+n-4 < 3n-2k-3$. Hence, there is a walk of length $3n-2k-3$ from $x_0$ to $y$.

Case 2: There are adjacent vertices of $D$ in $X$. Let $l = \min\{d(j, v) \mid j \in X$ and there exist $i \in X$ such that $(i, j) \in E(D)\}$.

Subcase 1: $l \le 2(n-k)-1$. Since $v \in V(C_2)$, there is a vertex $x$ in $X$ such that there exists a walk of length $2(n-k)-1$ from $x$ to $v$. Therefore there is a walk of length $3n-2k-3$ from $x$ to $y$.

Subcase 2: $l = 2(n-k)$ and $r \le 2(n-k)-1$. Then $v \in X$ and there is a walk of length $p$ from $v$ to $y$ for each positive integer $p$ with $p \ge n-2+\phi(2, r)$. Therefore there is a walk of length $3n-2k-3$ from $v$ to $y$.

Subcase 3: $l = 2(n-k)$ and $r \ge 2(n-k)+1$. Then $v \in X, u \notin X$, $k = \frac{n+2}{2}, r = n-1$ and $3n-2k-3 = 2n-5$. It is obvious that at least one of $u$ and $v$ is on

$C_{n-1}$ for each odd cycle $C_{n-1}$. If there exists a vertex $x$ in $X$ such that $d(x,y)$ is even and $2 \le d(x,y) \le n-4$, since $x$ is in $V(C_{n-1})$, then there is a walk of length $p$ from $x$ to $y$ for each positive odd integer $p \ge d(x,y)+n-1$. This implies that there is a walk of length $2n-5$ from $x$ to $y$. Otherwise, it is obvious that $y \in X$ and $X = V(D)\backslash\{u, i \mid d(i,y) \text{ is even and } 2 \le d(i,y) \le n-4\}$. We consider two cases.

(a) If there exists $C_{n-1}$, such that $y \in V(C_{n-1})$. Noticing that $y \in X$, there is a walk of length $p$ from $y$ to $y$ for each positive odd integer $p \ge n-1$. Therefore there is a walk of length $2n-5$ from $y$ to $y$.

(b) If $y \notin V(C_{n-1})$ for each odd cycle $C_{n-1}$. Since $D$ is a strongly connected digraph, there exists $C_m$ $(4 \le m \le n)$, such that $y \in V(C_m)$. If $m = n$, letting $x$ be vertex such that $d(x,y) = n-5$, then $x \in X$ and there is a walk of length $2n-5$ from $x$ to $y$. If $m = n-2$, letting $x$ be vertex such that $d(x,y) = n-3$, then $x \in X$ and there is a walk of length $2n-5$ from $x$ to $y$. If $m \le n-4$, then there is a walk of length $p$ from $y$ to $y$ for each positive odd integer $p \ge m+n-1$. Therefore there is a walk of length $2n-5$ from $y$ to $y$.

This completes the proof of the theorem. ∎

**Theorem 3.6** *Let $n, k$ be positive integers with $2 \le k \le n-2$. Then*

$$\{4, 5, \cdots, 2n-k-2\} \subseteq E^0_{nk}. \qquad (3.20)$$

**Proof.** Suppose $4 \le m \le n$. Let $D_3(m), D_4(m)$ be the digraphs of order $n$ with vertex sets $V(D_3(m)) = V(D_4(m)) = \{1, 2, \cdots, n\}$ and arc sets $E(D_3(m)) = \{[1,2],[1,3],[2,3],(3,4),(4,5),\cdots,(m-1,m),\cdots,(m,m+1),(m,m+2),\cdots,(m,n),(m+1,1),(m+2,1),\cdots,(n,1)\}$, $E(D_4(m)) = \{[1,2],[1,3],[2,3],(3,4),(4,5),\cdots,(m-1,m),\cdots,(m,m+1),(m,m+2),\cdots,(m,n),(m+1,1),(m+2,1),\cdots,(n,1),(m+1,3),(m+2,3),\cdots,(n,3)\}$.

It is obvious that the adjacency matrices of $D_3(m)$ and $D_4(m)$ belong to $QB^0_n$.

(1) Firstly, we will prove that if $4 \le m \le k+2$ then

$$F(D_3(m), k) = m. \qquad (3.21)$$

Take $X_0 = \{3, 4, 5, \cdots, k+2\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of length $m-1$ from any vertex of $X_0$ to the vertex $n$. So we have $F(D_3(m), k) \ge m$.

On the other hand, let $X$ be any $k$–vertex subset of $D_3(m)$. If $\{1,2,3\} \cap X \ne \emptyset$, then $\exp_{D_3(m)}(X) \le m$. If $\{1,2,3\} \cap X = \emptyset$, then $\{m+1, m+2, \cdots, n\} \cap X \ne \emptyset$ and $\exp_{D_3(m)}(X) \le m$.

Hence (3.21) holds.

(2) Secondly, we will prove that if $k \le n-3$ and $k+3 \le m \le n$ then

$$F(D_3(m), k) = 2m-k-2. \qquad (3.22)$$

Take $X_0 = \{4, 5, \cdots, k+3\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of length $2m-k-3$ from any vertex of $X_0$ to the vertex $n$. So we have $F(D_3(m), k) \ge 2m-k-2$.

On the other hand, let $X$ be any $k$–vertex subset of $D_3(m)$. If $\{1,2,3\} \cap X \neq \emptyset$, then $\exp_{D_3(m)}(X) \leq m$. If $\{1,2,3\} \cap X = \emptyset$, then $\exp_{D_3(m)}(X) \leq m+1-k-3+m = 2m-k-2$.

So (3.22) holds.

(3) Thirdly, we will prove that if $k \leq n-3$ and $k+3 \leq m \leq n$ then

$$F(D_4(m), k) = 2m - k - 3. \tag{3.23}$$

Take $X_0 = \{4, 5, \cdots, k+3\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of length $2m-k-4$ from any vertex of $X_0$ to the vertex $n$. So we have $F(D_4(m), k) \geq 2m - k - 3$.

On the other hand, let $X$ be any $k$–vertex subset of $D_4(m)$. If $\{1,2,3\} \cap X \neq \emptyset$, then $\exp_{D_4(m)}(X) \leq m$. If $\{1,2,3\} \cap X = \emptyset$, then $\exp_{D_4(m)}(X) \leq m+1-k-3+m-1 = 2m-k-3$.

So (3.23) holds.

The theorem now follows from (3.21), (3.22) and (3.23). ∎

**Theorem 3.7** *If $k = 2$, then $\{2,3\} \subseteq E^0_{nk}$. If $3 \leq k \leq n-2$, then $\{1,2,3\} \subseteq E^0_{nk}$.*

**Proof.** (1) Suppose $2 \leq k \leq n-2$. Let $D(A)$ be the digraph of order $n$ with vertex set $V(D(A)) = \{1, 2, \cdots, n\}$ and arc set $E(D(A)) = \{[1,2], [2,3], [2,4], \cdots, [2,n], (3,1), (4,1), \cdots, (n,1)\}$.

It is obvious that $A \in QB^0_n$ and $F(D(A), k) = 2$. So $2 \in E^0_{nk}$.

(2) Suppose $2 \leq k \leq n-2$. Let $D(A)$ be the digraph of order $n$ with vertex set $V(D(A)) = \{1, 2, \cdots, n\}$ and arc set $E(D(A)) = \{[1,2], (2,3), (2,4), \cdots, (2,n), [3,1], [4,1], \cdots, [n,1]\}$.

It is obvious that $A \in QB^0_n$ and $F(D(A), k) = 3$. So $3 \in E^0_{nk}$.

(3) Suppose $3 \leq k \leq n-2$. Let $D(A)$ be the digraph of order $n$ with vertex set $V(D(A)) = \{1, 2, \cdots, n\}$ and arc set $E(D(A)) = \{(i,j) \mid i,j = 1, 2, \cdots, n$ and $i \neq j\} \setminus \{(2,1)\}$.

It is obvious that $A \in QB^0_n$ and $F(D(A), k) = 1$. So $1 \in E^0_{nk}$.

This completes the proof of the theorem. ∎

**Theorem 3.8** *Let $n, k$ be positive integers with $2 \leq k \leq n-2$.*

(1) *If $n$ is even and $2 \leq k \leq \frac{n+2}{2}$, then*

$$E^0_{nk} = \{1, 2, \cdots, 3n-2k-3\} \backslash S. \tag{3.24}$$

(2) *If $n$ is even and $\frac{n+4}{2} \leq k \leq n-2$ or $n$ is odd, then*

$$E^0_{nk} = \{1, 2, \cdots, 3n-2k-2\} \backslash S. \tag{3.25}$$

*where $S = \{1\}$ when $k = 2$, otherwise $S = \emptyset$.* ∎

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