# Which Digraphs Are Round?

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**Abstract**

A digraph $D$ is *round* if the vertices of $D$ can be circularly ordered as $v_1, v_2, \ldots, v_n$ so that, for each vertex $v_i$, the out-neighbours of $v_i$ appear consecutively following $v_i$ and the in-neighbours of $v_i$ appear consecutively preceding $v_i$ in the ordering. We characterize round digraphs in terms of forbidden substructures. Our proof implies a polynomial algorithm to decide if a digraph is round.

## 1 The theorem

We assume that a digraph has no loops or multiple arcs but may contain a cycle of length 2. If it contains no cycle of length 2, then it is an *oriented graph.*

Let $D$ be a digraph. We say that a vertex $x$ is *adjacent to* a vertex $y$ in $D$ if there is at least one arc between $x$ and $y$. If $xy$ is an arc of $D$, then we say that $x$ *dominates* $y$ and use the notation $x \to y$ to denote this. If $x \to y$, then $y$ is an *out-neighbour* of $x$ and $x$ is an *in-neighbour* of $y$. The set $O(x)$ of all out-neighbours of $x$ is called the *outset* of $x$ and the set $I(x)$ of all in-neighbour of $x$ is called the *inset* of $x$. We shall let $d^+(x) = |O(x)|$ and $d^-(x) = |I(x)|$ and call $d^+(x)$ (resp. $d^-(x)$) the *outdegree* (resp. the *indegree*) of $x$.

A digraph $D$ is *round* if the vertices of $D$ can be circularly ordered as $v_1, v_2, \ldots, v_n$ so that, for each vertex $v_i$, the out-neighbours of $v_i$ appear consecutively following $v_i$ and the in-neighbours of $v_i$ appear consecutively preceding $v_i$ in the ordering. We shall refer to the ordering $v_1, v_2, \ldots, v_n$ as a *round enumeration* of $D$.

A digraph is *semicomplete* if there is at least one arc between any pair of vertices. A *tournament* is thus a semicomplete oriented graph. A digraph is called *locally semicomplete* if the outset as well as the inset of each vertex induces a semicomplete digraph, [1]. A locally semicomplete oriented graph is called a *local tournament*, [4].

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Suppose that $D$ is a round digraph and that $v_1, v_2, \ldots, v_n$ a round enumeration of $D$. We claim that $D$ is a locally semicomplete digraph. To see this, consider an arbitrary vertex, say $v_i$. Let $x$ and $y$ be two out-neighbours of $v_i$. Assume without loss of generality that $v_i, x, y$ appear in the circular order in the round enumeration. Since $v_i \to y$ and the in-neighbours of $y$ appear consecutively preceding $y$, we must have $x \to y$. Thus the out-neighbours of $v_i$ are pairwise adjacent. Similarly, we can show that the in-neighbours of $v_i$ are also pairwise adjacent. Hence $D$ is a locally semicomplete digraph. In the case when $D$ is a round oriented graph, $D$ is a local tournament.

There is an intimate relation between locally semicomplete digraphs and circular arc graphs. A graph $G$ is a *circular arc graph* if there is a one-to-one correspondence between the vertex set of $G$ and a family of circular arcs on a circle so that two vertices are adjacent in $G$ if and only if the corresponding circular arcs intersect. A circular arc graph is *proper* if the family can be chosen so that no arc contains any other arc. It is proved [5] that a connected graph can be oriented as a local tournament if and only if it is a proper circular arc graph. Round local tournaments are particularly useful in finding a corresponding circular arc family and in designing efficient algorithms to solve problems related to proper circular arc graphs, cf. [2, 3].

If $x \to y$ but $y \not\to x$, then the arc $xy$ is called a *simple* arc. A path (resp. cycle) consisting of simple arcs is called a *simple* path (resp cycle). For a vertex $x$ of $D$, let $B(x) = O(x) \cap I(x)$, $O'(x) = O(x) - B(x)$, and $I'(x) = I(x) - B(x)$. A digraph is *connected* if its underlying graph is connected.

Bang-Jensen [1] showed that a connected local tournament $D$ is round if and only if for each vertex $x$, $O(x)$ and $I(x)$ induce transitive tournaments, i.e., tournaments which contain no cycles. The main theorem of this paper is a generalization of this result.

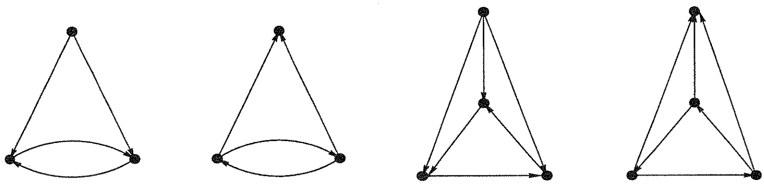
**Theorem 1.1** *Let $D$ be a connected locally semicompleted digraph. Then $D$ is round if and only if for each vertex $x$, $O'(x)$ and $I'(x)$ induce transitive tournaments and $B(x)$ induces a (semicomplete) subdigraph containing no simple cycles.*

# 2 The proof

Let $D$ be a round digraph and let $v_1, v_2, \ldots, v_n$ be a round enumeration of $D$. For each vertex $v_i$, the definition of a round enumeration implies that the vertices in $I'(v_i)$ appear consecutively preceding $v_i$ and the vertices in $O'(v_i)$ appear consecutively following $v_i$. Thus the vertices in $B(v_i)$ also appear consecutively between vertices of $I'(v_i)$ and the vertices of $O'(v_i)$. So, when $B(v_i) \neq \emptyset$, if we traverse beginning at $v_i$ in the circular order of the round enumeration, we encounter first the vertices in $O'(v_i)$, then the vertices in $B(v_i)$, and finally the vertices in $I'(v_i)$. In this section, we shall prove Theorem 1.1. But first we have some lemmas.

**Lemma 2.1** *Let $D$ be a digraph and let $D'$ be a induced subdigraph of $D$. If $D$ is round, then $D'$ is round.*

**Proof:** Let $v_1, v_2, \ldots, v_n$ be a round enumeration of $D$. Suppose that $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$ $(j_1 < j_2 < \ldots < j_k)$ are the vertices of $D'$. Then $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$ is a round enumeration of $D'$. □



**(a)** **(b)** **(c)** **(d)**

Figure 1: Some forbidden substructures for round digraphs.

**Lemma 2.2** *If $D$ is a round digraph, then none of the digraphs in Fig. 1 is an induced subdigraph of $D$.*

**Proof:** The statement follows from Lemma 2.1 and the fact that none of digraphs in Fig. 1 is round. □

**Lemma 2.3** *Let $D$ be a round digraph. Then, for each vertex $x$ of $D$, the subdigraphs induced by $I'(x)$ and $O'(x)$ contain no cycle.*

**Proof:** The statement follows from Lemma 2.2 and the fact that if the subdigraph induced by $I'(x)$ or $O'(x)$ contains a cycle then $D$ would contain one of the digraphs in Fig. 1 as an induced subdigraph. □

**Lemma 2.4** *Let $D$ be a round digraph. Then, for each vertex $x$ of $D$, the subdigraph induced by $B(x)$ contains no simple cycle.*

**Proof:** Suppose the subdigraph induced by some $B(x)$ contains a simple cycle $C$. Let $v_1, v_2, \ldots, v_n$ be a round enumeration of $D$. Without loss of generality, assume that $x = v_1$. Then the simple cycle $C$ must contains a simple arc $v_a v_b$ with $a > b$. Now $v_1 \in I(v_a)$ but $v_b \notin I(v_a)$, contradicting the assumption that $v_1, v_2, \ldots, v_n$ is a round enumeration of $D$. □

**Proof of Theorem 1.1:**

The necessity follows from lemmas 2.3 and 2.4. For sufficiency, we first consider the case when $D$ contains a simple cycle. We claim that $O'(x) \neq \emptyset$ for each vertex $x$ of $D$. To prove this, it suffices to show that there is a simple cycle containing all vertices of $D$. Let $C : x_1 \to x_2 \to \ldots \to x_l \to x_1$ be a longest simple cycle in $D$. Suppose that $C$ does not contain all vertices of $D$. Then there is a vertex $v$ which is not in $C$ and $v$ is adjacent to some vertex of $C$.

Assume that there is a simple arc between $v$ and some vertex, say $x_1$, of $C$. Assume further that the simple arc is from $x_1$ to $v$. (A similar discussion applies if the simple arc is from $v$ to $x_1$.) Thus $v$ and $x_2$ are in $O'(x_1)$ and hence $v$ and $x_2$

are adjacent. The arc between $v$ and $x_2$ must be simple as $D$ contains no Fig. 1(a). However the choice of $x_2$ implies that $v \in O'(x_2)$. Now both $v$ and $x_3$ are in $O'(x_2)$, implying that $v$ and $x_3$ are adjacent by a simple arc. Again we must have $v \in O'(x_3)$. Continuing this way, we see that $v$ is in $O'(x_i)$ for each $i = 1, 2, \ldots, l$. Hence $I'(v)$ contains all vertices of $C$, which contradicts the assumption that $I'(v)$ induces a transitive tournament. So we may assume that $x_1$ is in $B(v)$ and further that there is no simple arc between $v$ and $C$. Vertices $v$ and $x_2$ are adjacent because both are out-neighbours of $x_1$. Thus $x_2 \in B(v)$. Continuing this way, we see that $B(v)$ contains all vertices of $C$, contradicting the assumption that the subdigraph induced by $B(v)$ contains no simple cycle. Therefore the cycle $C$ contains all vertices of $D$, which implies that $O'(x) \neq \emptyset$ for each vertex $x$ of $D$.

We apply the following algorithm to find a round enumeration of $D$: Begin with an arbitrary vertex, say $y_1$, and, for each $i = 1, 2, \ldots$, let $y_{i+1}$ be the vertex of indegree 0 in the (transitive) tournament induced by $O'(y_i)$. Let $y_1, y_2, \ldots, y_r$ be distinct vertices produced by the algorithm such that the vertex $w$ of indegree 0 in the tournament induced by $O'(y_r)$ is in $\{y_1, y_2, \ldots, y_{r-2}\}$. We first show that $w = y_1$. If $w = y_j$ with $j > 1$, then $y_{j-1}$ and $y_r$ are both in $I'(y_j)$ and hence adjacent by a simple arc. But either $y_r \in O'(y_{j-1})$ or $y_r \in I'(y_{j-1})$ would contradict the fact that $y_j$ is the vertex of indegree 0 in the (transitive) tournaments induced by $O'(y_{j-1})$ and $O'(y_r)$. So $w = y_1$ and $C' : y_1 \to y_2 \to \ldots \to y_r \to y_1$ is a simple cycle. We next show that $r = |V(D)|$. Suppose not. Then there is a vertex $u$ which is not in $C'$ and is adjacent to some $y_i$ of $C'$.

Suppose that $u \in O'(y_i)$. Then $u$ and $y_{i+1}$ are adjacent as both are in $O'(y_i)$. Since $D$ contains no Fig. 1(a) and $y_{i+1}$ is the vertex of indegree 0 in the subdigraph induced by $O'(y_i)$, we must have $u \in O'(y_{i+1})$. Now $u$ and $y_{i+2}$ are adjacent. Similarly, we must have $u \in O'(y_{i+2})$. Continuing this way, we see that $u \in O'(y_k)$ for each $k = 1, 2, \ldots, r$. That is, $C'$ is contained in the subdigraph induced by $I'(u)$, a contradiction. A similar argument applies for the case when $u \in I'(y_i)$. So we may assume $u \in B(y_i)$ and there is no simple arc between $u$ and $C'$. Using this assumption and the definition of a locally semicomplete digraph, we can show that $C'$ is contained in the subdigraph induced by $B(u)$, which is again a contradiction. Therefore $r = |V(D)|$, i.e., the algorithm enumerates all vertices of $D$.

We now complete our claim by showing that $y_1, y_2, \ldots, y_r$ is a round enumeration. Suppose not. Then there are three vertices $y_a, y_b, y_c$ listed in the circular order in the enumeration such that one of the following two cases occurs:

1. $y_c \in O(y_a)$ and $y_b \notin O(y_a)$;
2. $y_b \in I(y_a)$ and $y_c \notin I(y_a)$.

Assume that case 1 occurs. Assume that the three vertices were chosen so that the number of vertices from $y_b$ to $y_c$ in the circular order is as small as possible. This implies that $c = b + 1$, i.e., $y_c$ is next to $y_b$ in the circular order. Now $y_a$ and $y_b$ are adjacent as both are in $I(y_c)$. Thus $y_a \in O'(y_b)$. Since we also have $y_c \in O'(y_b)$

and $D$ contains no Fig. 1(a), $y_c \in O'(y_a)$. So $y_c$ is not the vertex of indegree 0 in the (transitive) tournament induced by $O'(y_{c-1})$, contradicting the choice of $y_c$. A similar argument applies when case 2 occurs.

It remains to consider the case when $D$ contains no simple cycle. If $D$ contains no simple arcs, then it is easy to see that $D$ is semicomplete. This means that there is a cycle of length two between any pair of vertices. Thus any vertex ordering is a round enumeration of $D$. So assume that $D$ has at least one simple arc. Let $z_1$ be a vertex with $I'(z_1) = \emptyset$ and $O'(z_1) \neq \emptyset$. Such a vertex exists because $D$ contains a simple arc but no simple cycle. We apply the following algorithm to find a path in $D$: begin with $z_1$ and, for each $i = 1, 2, \ldots$, let $z_{i+1}$ be the vertex of indegree 0 in the (transitive) tournament induced by $O'(z_i)$ unless $O'(z_i) = \emptyset$. Clearly, this produces a path $P : z_1 \rightarrow z_2 \rightarrow \ldots \rightarrow z_s$ with $O'(z_s) = \emptyset$.

Using a similar argument as above, we can show that $z_1, z_2, \ldots, z_s$ is a round enumeration of the subdigraph induced by $V(P)$. Thus if $P$ contains all vertices of $D$ then $z_1, z_2, \ldots, z_s$ is a round enumeration of $D$. So assume that there is a vertex $v$ which is not in $P$ and is adjacent to some vertex of $P$. It is easy to see that there is no simple arc between $v$ and $P$. This implies that $v \in B(z_i)$ each $i = 1, 2, \ldots, s$. In fact, it is not hard to see this is so for each vertex $v \in V(D) - V(P)$.

Therefore if we repeat the above algorithm for $D - P$ we can find another path consisting of simple arcs (if any). We can continue this process in the remaining digraph until no simple arc left. Let $P_k : z_1^k \rightarrow z_2^k \rightarrow \ldots \rightarrow z_{j_k}^k$, $k = 1, 2, \ldots$, be the paths produced by the algorithm. Let $z_1^0, z_2^0, \ldots, z_{j_0}^0$ be the remaining vertices. Then it is easy to verify that

$$z_1^1, z_2^1 \ldots, z_{j_1}^1, z_1^2, z_2^2, \ldots, z_{j_2}^2, \ldots, z_1^0, z_2^0, \ldots, z_{j_0}^0$$

is a round enumeration of $D$. This completes the proof. □

It is not difficult to see that the above proof implies a polynomial algorithm to decide if a digraph is round and to to find a round enumeration of it if one exists.

**Corollary 2.5** *There is a polynomial algorithm to decide if a digraph is round and to find a round enumeration of it if one exits.* □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