Scenic Graphs II: Non–Traceable Graphs

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Abstract

A path of a graph is *maximal* if it is not a proper subpath of any other path of the graph. A graph is *scenic* if every maximal path of the graph is a maximum length path. In [4] we give a new proof of C. Thomassen's result characterizing all scenic graphs with Hamiltonian path. Using similar methods here we determine all scenic graphs with no Hamiltonian path.

1 Introduction

We employ the following notation some of which is non–standard. A path in a graph is a sequence of distinct vertices in which consecutive vertices are adjacent. The *length* of a path is the number of edges in the path. Thus a path $Q = (x_0, x_1, \ldots, x_k)$ has length k. All graphs we consider here are undirected. Therefore, although sequences have an orientation or direction, here we shall not distinguish between the sequences $(x_0, x_1, \ldots, x_k)$ and $(x_k, x_{k-1}, \ldots, x_0)$ as paths. For the path $Q = (x_0, x_1, \ldots, x_k)$ we will also use the notation (x_0, Q, x_k), and (x_i, Q, x_j) is the corresponding subpath. If (x, P, y) and (u, Q, v) are disjoint paths with y and u adjacent, then their *concatenation* is a path we denote by either $((x, P, y), (u, Q, v))$, or $(x, \ldots, y, (u, Q, v))$, or $((x, P, y), u, \ldots, v)$, or $(x, \ldots, y, u, \ldots, v)$. A similar natural extension of this notation is used for concatenations of concatenated paths. A path P is a *subpath* of Q if the sequence corresponding to P appears as a consecutive subsequence of Q. A subpath P of a path Q is *proper* if $P \neq Q$. If P is a proper

*Research supported by ONR Grant Number N00014-91-J-1098.

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subpath of Q, then we shall say that P *extends* to Q, or Q *extends* P, or Q is an *extension* of P. A path is *maximal* if it is not a proper subpath of any other path, or equivalently, if it has no extension. The *path spectrum* of a connected graph G is the set of lengths of all maximal paths in G. The concept of path spectrum was introduced by Jacobson *et al.* [3]. We say that a connected graph is *scenic* if its path spectrum is a singleton. A graph with a Hamiltonian path is called *traceable.*

The *Prism* is the graph $K_6 - C_6$ obtained from K_6 by removing the edges of a six-cycle. The *Cube* is the graph $K_{4,4} - 4K_2$ obtained by removing four disjoint edges from the complete 4×4 bipartite graph. Except for paths P_n $(n \geq 1)$, cycles C_n $(n \geq 3)$, the Prism, and the Cube, traceable scenic graphs emerge from cliques, K_n $(n \geq 1)$, and from the complete bipartite graphs $K_{p,p}$ and $K_{p,p+1}$ $(p \geq 1)$. Traceable scenic graphs were determined by C. Thomassen [9] and a different proof can be found in [4]. To present the family we need some notation. The union of t mutually disjoint edges (a matching) will be denoted by tK_2. The graph obtained from K_n by removing the edges of a copy of tK_2 $(1 \leq t \leq n/2)$ is denoted by $K_n - tK_2$. The complete $p \times p$ bipartite graph plus (resp. minus) an edge is denoted $K_{p,p} + K_2$ (resp. $K_{p,p} - K_2$). The graph obtained from the complete $p \times p$ bipartite graph by adding one edge into each partite set is denoted $K_{p,p} + 2K_2$. If $H \in \{K_3, 2K_2, K_{1,q}\}$, then $K_{p,p+1} + H$ denotes the graph obtained from the complete $p \times (p+1)$ bipartite graph by adding all the edges of H to the **largest** partite set containing $p+1$ vertices. In [4] we give a new proof of the following theorem of C. Thomassen [9]:

Theorem 1.1 *A traceable graph is scenic if and only if it belongs to one of the following families:*

$$\begin{aligned}
\Phi[K_n] &= \{K_n,\ K_n - tK_2\ (1 \leq t \leq n/2)\}, \\
\Phi[K_{p,p}] &= \{K_{p,p}, K_{p,p} - K_2, K_{p,p} + K_2, K_{p,p} + 2K_2\ \}, \\
\Phi[K_{p,p+1}] &= \{K_{p,p+1}, K_{p,p+1} + K_3, K_{p,p+1} + 2K_2, K_{p,p+1} + K_{1,q}\ (1 \leq q \leq p)\}, \\
\Psi &= \{P_n,\ C_n,\ Prism,\ Cube\}.
\end{aligned}$$

In this paper we determine all non–traceable scenic graphs[1]. In Section 2 we prove that every non–traceable scenic graph is bipartite. Let $K_{1,r}^s$ $(r \geq 3)$ be the *equi-subdivided star* obtained from a $K_{1,r}$ by subdividing each edge with $s \geq 0$ vertices. For $p \geq 2$ and $q \geq p+2$, we call $K_{p,q} - F$ a $p \times q$ *generic* graph if it is obtained from $K_{p,q}$ by removing an arbitrary star forest F with its star components centered in the q–element (i.e. largest) partite set of $K_{p,q}$. Note that a disconnected generic graph has the form $K_{p,q} - K_{p,1}$ (or equivalently $K_{p,q-1} + y$, where y is an isolated vertex in the larger partite set). We show that besides a few exceptions, every non–traceable scenic graph is either an equi-subdivided star or a connected generic graph. The main result is formulated in the following theorem.

[1]The same problem has been considered independently by M. Tarsi [8] (personal communication by editors of JGT and JCT B).

Theorem 1.2 *A non–traceable graph is scenic if and only if it is one of the graphs $G_1, \ldots, G_6$ in Fig. 1, an equi-subdivided star, or a connected generic graph.*

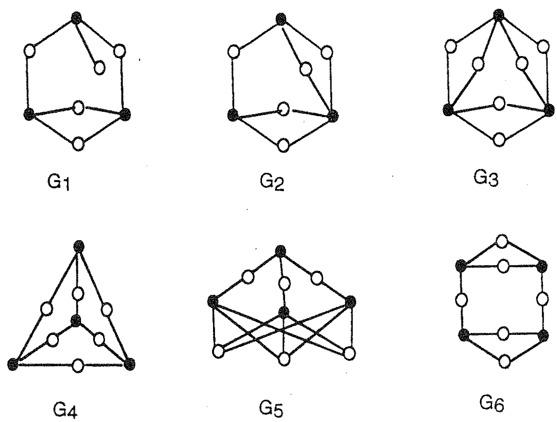

Figure 1:

It is a routine to check that the six small graphs in Fig. 1 and the equi-subdivided stars are non–traceable and scenic. To prove the same for a connected $p \times q$ generic graph one may easily show that every maximal path covers the p–element partite set of G and both of its endvertices must be in the q–element partite set. Therefore, all maximal paths in a connected generic graph have the same length, namely $2p \leq p + q - 2$. Hence connected generic graphs are scenic and non–traceable.

The next sections contain the proof of the 'only if' part of Theorem 1.2. The basic idea in the proof is the reduction of a non–traceable scenic graph G by removing a copy of $K_{2,2}$ from G together with all adjacent edges. To some extent the removal of a $K_{2,2}$ preserves the scenic property — the only exceptions are when the resulting graph is small or disconnected. Moreover, besides some exceptional cases discussed in Sections 4 and 5, both G and H must be generic graphs.

The problem of determining the maximum path length of a graph is NP–complete, and the same is true for computing the independence number (maximum number of mutually non–adjacent vertices), see [6]. R.S. Sankaranarayana and L.K. Stewart [7] have shown that deciding whether a graph is well-covered, i.e., deciding whether all maximal independent sets of a graph have the same cardinality, is a co-NP–complete problem. Concerning the analogous decision problem whether all maximal paths are maximum Theorems 1.1 and 1.2 imply that the property of being scenic can be tested in polynomial time.

2 Non–Traceable Scenic Graphs are Bipartite

Proposition 2.1 *A tree is non–traceable and scenic if and only if it is an equi-subdivided star* $K_{1,r}^{s}$ $(r \geq 3,\ s \geq 0)$.

Proof. Let G be a non–traceable scenic tree, i.e., let it be different from a path. For arbitrary $x, y \in V(G)$, we use (x, G, y) to denote the (unique) path of G with endvertices x and y. Let $P = (x, G, y)$ be a maximal path of G and let $z \in V(P) \setminus \{x, y\}$ be a vertex of degree at least three. Clearly, both x and y are leaves of G, and the subpaths (x, G, z) and (y, G, z) must have the same length. Therefore, z is the (unique) midvertex of P.

Assume that G has two distinct vertices of degree at least three, u and v. Consider a maximal extension P of (u, G, v). By the observation above, both u and v are midpoints of P, a contradiction. Therefore, G has exactly one vertex of degree $r \geq 3$ which is the midpoint of all paths between any two leaves. Thus G is an equi-subdivided star $K_{1,r}^{s}$, for some $s \geq 0$. □

Theorem 2.2 *Let G be a non–traceable scenic graph. If G is different from a tree, then it is a $p \times q$ bipartite graph with $p \geq 2$ and $q \geq p+2$ vertices in the partite sets. Furthermore, G has a dominating cycle on $2p$ vertices and the maximum path length in G equals $2p$.*

Proof. Let C be a cycle of G with maximum length $k = |V(C)|$. Observe that $3 < k < |V(G)|$. Indeed, C can not be a Hamiltonian cycle, because G is non–traceable. On the other hand, $k \neq 3$ holds by the following argument. Assuming that $C = (x_1, x_2, x_3)$, at least two vertices of C have degree greater than two (otherwise G would not be scenic). Let $x_1y_1, x_2y_2 \in E(G)$, for some vertices $y_1 \neq y_2$ and y_1, y_2 not in C. Because C is a maximum cycle, every maximal extension Q of the path (y_1, x_1, x_2, y_2) misses x_3. A maximal path longer than Q can be found by including x_3 into Q between x_1 and x_2, contradicting that G is scenic.

A path $T \subset G$ with $|V(T) \cap V(C)| = 1$ is called a *tail* of C. For a given vertex $z \in C$, let $T(z)$ denote the longest tail of C ending at z. Choose a maximum cycle C of G having a tail T of maximum possible length. Assume that $T = T(x)$ is a maximum tail of C at x, clearly it has length $t \geq 1$.

Let y, x, y', x' be consecutive vertices on C (they are distinct, since $k \geq 4$). Let $T(y)$ and $T(y')$ be maximum length tails of C at y and y', respectively. Because C is a maximum cycle, both $T(y)$ and $T(y')$ are disjoint from $T(x)$. Observe that $(T(x), (x, C, y), T(y))$ and $(T(x), (x, C, y'), T(y'))$ are maximal paths of G, and because G is scenic, $T(y)$ and $T(y')$ have the same length s. Clearly, $1 \leq t$, $0 \leq s \leq t$, and the maximum path length in G is $s + t + (k - 1)$.

First we show that there is no vertex $z \in V(G) \setminus V(C)$ with $yz, y'z \in E(G)$. Suppose that such a z exists. Because C has maximum length, z is not on $T(x)$. Hence z could substitute for x in C; that is $(C - x) + z$ would be a maximum cycle with longer tail $T(x) + y$ at y, a contradiction.

The previous paragraph implies that $T(y)$ and $T(y')$ are disjoint. If $s \neq 0$ then $(T(y), (y, C - x, y'), T(y'))$ is a maximal path of length $2s + (k-2)$. From $2s+k-2 = s + t + k - 1$ we obtain $s = t + 1 > t$, a contradiction. Consequently, $s = 0$. Next we show that $t = 1$. Note that this will imply that every vertex not in C is adjacent to some vertex of C (that is C is a dominating cycle in G.), and the maximum path lengths equals k.

Consider a maximum length tail $T(x')$ at x'. Because G is scenic, and $((y', C, x'), T(x'))$ is a maximal path, $T(x')$ has t edges. If $T(x)$ and $T(x')$ are disjoint, then $(T(x), (x, C - y', x'), T(x'))$ is a maximal path of length $2t + (k - 2) = t + k - 1$ implying $t = 1$. If $T(x)$ and $T(x')$ are not disjoint, then they must share a vertex $z \notin V(C)$ such that $zx, zx' \in E(G)$. In this case $(y', x, z, (x', C - \{x, y'\}, y))$ is a maximal path of length $k = t + k - 1$, implying $t = 1$.

The argument above shows that the vertices of C have a two-coloring, namely $z \in C$ is assigned color $|T(z)|$ $(= 0$ or $1)$. In particular, C is an even cycle of length $k = 2p$, for some $p \geq 2$. Let us color all vertices off of C with 0. We claim that this is a proper two–coloring of G, i.e., G is bipartite.

Any vertex off of C can only be adjacent to vertices of color 1 on C, by the definition of our coloring and because G is connected. Now assume that uv is a chord of C between vertices of the same color ϵ. Let u' and v' be neighbors of u and v, respectively, such that $V(C)$ is partitioned into two subpaths of C: C_1 going from u to v' and C_2 going from v to u' . If $\epsilon = 1$, then both u' and v' have color 0, and the concatenation of C_1 and C_2 along the edge uv would result in a maximal path Q of length $k - 1$. Therefore, $\epsilon = 0$, and both u' and v' have color 1. This implies that u' and v' have a neighbor z and w not in C, respectively. If $z = w$, then the path Q above together with z would result in a cycle of length $k + 1$. Hence $z \neq w$, and $(z, (u', Q, v'), w)$ is a maximal path of length $k + 1$, a contradiction.

Therefore G is a (connected) bipartite graph with p vertices in one partite sets and $q \geq p + 1$ in the other one. If there was just one vertex not in C then G would be traceable. This shows that $q \geq p + 2$ and the maximum path length is $2p$. □

3 Small Non–Traceable Scenic Graphs

For $p \geq 2$ and $q \geq p + 2$, denote by $\mathcal{G}_{p,q}$ the class of all $p \times q$ bipartite graphs which are non–traceable scenic graphs different from trees. Notice that members of $\mathcal{G}_{p,q}$ have all properties described in Theorem 2.2. In this section we determine $\mathcal{G}_{p,q}$ for

$p = 2$ and 3. Recall that G is $p \times q$ generic, if $G \cong K_{p,q} - F$, where F is some star forest with all star components centered at the q–element partite set.

Proposition 3.1 *If $G \in \mathcal{G}_{2,q}$, then G is a connected generic graph.*

Proof. Let $\{x, y\}$ be the smallest partite set of G and $Q = V(G) \setminus \{x, y\}$. Note that 4 is the maximum path length in G (by Theorem 2.2). Because G is connected, every vertex of Q is adjacent to either x or y. Assume that one of x and y, say y, is non–adjacent to $u, v \in Q$. In this case the path (u, x, v) would be maximal, a contradiction. This proves that $G \cong K_{2,q} - F$, where $F \cong K_2$ or $2K_2$. $\square$

Proposition 3.2 *If $G \in \mathcal{G}_{3,q}$ is not generic, then G is either G_1, G_2 or G_3.*

Proof. Suppose $P = \{x_1, x_2, x_3\}$ is the smallest partite set of G and $Q = V(G) \setminus P$. Note that G has a dominating 6–cycle C and 6 is the maximum path length in G (by Theorem 2.2). Because G is connected, every vertex not in C is adjacent to at least one vertex of P. For every $I \subseteq \{1, 2, 3\}$, define $Q(I) = \{z \in Q \setminus V(C) \ : \ zx_i \in E(G) \text{ iff } i \in I\}$. Obviously, $Q(I) \cap Q(J) = \emptyset$ holds for every $I \neq J$. Set $q(I) = |Q(I)|$. Observe that $\sum_{|I|=1} q(I) \leq 1$ and, for $|I| = 2$, $q(I) \leq 1$ must hold, because otherwise, one easily finds maximal paths of length 4 or 2. On the other hand, $q(I) \geq 2$, for some I, because G is non–traceable.

Case a: C is an induced 6–cycle of G. If $q(\{i\}) = 1$ for some $i \in \{1, 2, 3\}$, then $q(I) = 0$ must hold for every I containing i, because otherwise, one easily finds a maximal path of length 4. Therefore, $q(\{1, 2, 3\} \setminus \{i\}) = 1$ and $G \cong G_1$ follows. Assume now that $q(I) = 0$, for every $|I| = 1$. If $q(\{1, 2, 3\}) = 0$, then $\sum_{|I|=2} q(I) \geq 2$, because otherwise, G would be traceable. Therefore G is isomorphic to one of G_2 and G_3. If $q(\{1, 2, 3\}) > 0$, then $q(I) = 0$, for every $|I| = 2$. This implies that G is generic.

Case b: G has no induced 6–cycle. Assume first that C has just one chord, say at x_3. In this case $q(\{1, 2\}) = 0$ (otherwise G would contain a C_6). Furthermore, $q(\{1, 3\}) = q(\{2, 3\}) = 0$ and $q(I) = 0$, for every $|I| = 1$, because G has no maximal paths of length less than 6. This proves that G is generic. Assume now that every 6–cycle of G induces at least two chords. A similar argument as above shows that G must be generic. This proves the proposition. $\square$

4 $K_{2,2}$–removal

In this section our goal is to prove that (to some extent) the removal of a $K_{2,2}$ preserves the scenic property — the only exceptions are when the resulting graph is small or disconnected.

Proposition 4.1 *If $p \geq 4$ and $G \in \mathcal{G}_{p,q}$ is different from G_4, then G contains a copy of $K_{2,2}$.*

Proof. By Theorem 2.2, G is bipartite and has a dominating cycle $C = (x_1, y_1, x_2, y_2, \ldots, x_p, y_p)$ of length $2p$, where $P = \{x_1, \ldots, x_p\}$ is one of the partite sets of G. Furthermore, the proof of Theorem 2.2 implies that every vertex of P has a neighbor off of C. Assume that G has no $K_{2,2}$. For every i, $1 \leq i \leq p$, there exist vertices $u, v \in V(G - C)$ with $ux_i, vx_{i+2} \in E(G)$ and $ux_{i+1}, vx_{i+1} \notin E(G)$ (because G is $K_{2,2}$–free). (Indices are reduced modulo p in this paragraph.) If $u \neq v$, then the path $(u, (x_i, C - \{y_i, x_{i+1}, y_{i+1}\}, x_{i+2}), v)$ is maximal and has length $2p - 2$. Hence $u = v$ follows, moreover, u must be adjacent to all vertices $x_i, x_{i+2}, \ldots$, and x_{i-2}. The same argument shows that there exists a vertex $w \in V(G - C)$ different from u and adjacent to all $x_{i+1}, x_{i+3}, \ldots$, and x_{i-1}. This implies that p must be even, in particular C has length $2p \geq 8$.

If $p \geq 6$, then the path $(y_1, x_1, u, x_3, y_3, x_4, w, x_2, y_2)$ of length 8 can not be maximal, hence it extends by an edge $y_\epsilon x_i$, where $\epsilon = 1$ or 2, and $4 \leq i \leq p$. Now $j = \epsilon$ or $\epsilon + 1$ has the same parity as i, hence x_i and x_j are adjacent to the same vertex $z = u$ or w. Then $\{y_\epsilon, x_i, z, x_j\}$ induces a $K_{2,2}$, a contradiction. Thus we have $p = 4$. Because any additional vertices or any further edges included to $C \cup \{u, w\}$ would complete a $K_{2,2}$, $G \cong G_4$ follows. □

For $G' \subset G$, $G - V(G')$ denotes the graph obtained from G by removing the vertices of G' together with all incident edges.

Theorem 4.2 *For $p \geq 4$, let $G \in \mathcal{G}_{p,q}$ $(q \geq p + 2)$, and let $K \cong K_{2,2}$ be a subgraph of G. If G is different from G_5, then either $G - V(K) \in \mathcal{G}_{p-2,q-2}$ or $G - V(K)$ is a scenic graph (traceable or non–traceable) plus an isolated vertex.*

Proof. Let a_1, a_2, b_1, and b_2 be the vertices of K. Let $H = G - V(K)$, let P and Q be the partite sets of H with $|P| = p - 2$ and $|Q| = q - 2$, furthermore, let $\{a_1, a_2\} \cup P$ and $\{b_1, b_2\} \cup Q$ be the partite sets of G. We know from Theorem 2.2 that

(*) *every maximal path of G has both end vertices in the larger partite set, $\{b_1, b_2\} \cup Q$, and contains all vertices from the smaller one, $\{a_1, a_2\} \cup P$.*

Our goal is to show that similar properties are satisfied by the maximal paths of H, as well. Let $M = (u, \ldots, v)$ be a maximal path in H (we may assume $u \neq v$). We shall prove that $u, v \in Q$, moreover, M contains all vertices of P. Because M has an extension in G containing a_1 and a_2, we may assume that there is an edge, say $uz \in E(G)$, for some $z \in V(K)$. Thus M can be extended in G from its endvertex u to include the four vertices of K. This new path has no extension in G from the other endvertex v (because (u, M, v) is maximal in H). Hence (*) implies $v \in Q$.

Due to the argument above we may assume that if $M = (u, \ldots, v)$ is a maximal path of H, then $v \in Q$ and u sends an edge to K. The proof of the theorem consists of two claims, each will be verified in several numbered steps.

Claim I: *Every maximal path of H has both end vertices in Q.*

Proof. We assume that $M = (u, \ldots, v)$ is a maximal path of H with $v \in Q$. Suppose to the contrary that $u \in P$, and let $ub_1 \in E(G)$. Let $M = (u, u', \ldots, v', v)$, and let $Y = V(H) \setminus V(M)$. Observe that $|Y| \geq 2$ holds, because $q \geq p + 2$.

Assume that $va_i \in E(G)$. The path (u, M, v, a_i, K, b_j) extends in G from u, by property (*). This contradicts the maximality of M in H. Similar argument shows that $va_i, u'a_i \notin E(G)$, for each $i = 1$ and 2. Note also that, by property (*), path (v, M, u, b_1, K, a_i) has an extension in G from a_i. This implies that, for each $i = 1$ and 2, there exists $y_i \in Y$ with $a_i y_i \in E(G)$.

(1) *There is an edge from Y to M.*

Suppose $b_2 x \in E(G)$, for some $x \in Y \cap P$. The path (x, b_2, K, a_i, y_i) covering K extends in G to include all vertices of P, as required by property (*). In this case there must be an edge from Y to M.

Next we suppose that $\{b_1, b_2\}$ has no neighbor in $Y \cap P$. If $y_1 \neq y_2$, the path $(y_1, a_1, b_1, a_2, y_2)$ extends to include P which requires of using some edge going from Y to M. Thus we may also assume that y_1 is the unique neighbor of $\{a_1, a_2\}$ in Y. Because $q \geq p+2$, there is some $y' \in Y \cap Q$ different from y_1. By the connectivity of G, there is an edge $zy' \in E(G)$. If $z \in Y \cap P$, then the path $((v, M, u), (b_1, a_1, y_1, a_2, b_2))$ has no extension to include z, thus $z \in V(M)$ follows.

Note that (1) implies that M has at least 4 vertices. In particular, $u' \neq v$, and $u \neq v'$.

(2) *There is a vertex $y \in Y \cap Q$ such that $yv' \in E(G)$.*

Let $x \in V(M)$ be the closest vertex to v such that $xy \in E(G)$, for some $y \in Y$. By (1), such x exists, we shall show that $x = v'$. Suppose to the contrary that $x \neq v'$.

If $x \in Q$, then no extension of the path $S = (y, (x, M, u), (b_1, K, a_i), y_i)$ ($i = 1$ or 2) can include v', by the choice of x. This contradicts (*), thus $x \in P$ follows. Note also that the path S above cannot exist, consequently, we have $y = y_i$, for $i = 1, 2$. Therefore, y is the only vertex of $Y \cap Q$ which is adjacent to a_1 and a_2. The path $(y, (x, M, u), (b_1, K, a_2)$ extends in G with some $a_2 t \in E(G)$, where $t \in Q$. By the assumption on y, we know that $t \notin Y \cap Q$, that is t is a vertex of (x, M, v) different from v.

If $Y \cap P = \emptyset$, then every vertex of Y sends an edge to M, because G is connected. Define $x' \in V(M)$ as the first vertex along the subpath (x, M, u) having some neighbor $y' \in Y \setminus \{y\}$. Because $xy' \notin E(G)$, we have $x' \neq x$. Let

x^* be the last vertex on (x, M, x') adjacent to y (possibly $x^* = x$). The path $(y', (x', M, u), (b_1, K, a_2), (t, M, x^*), y)$ is maximal and misses v', a contradiction.

If $Y \cap P \neq \emptyset$, then the path $((v, M, x), y, (a_2, K, b_1), (u, M, u''))$ is not maximal in G. Therefore, there exists a vertex $z \in Y \cap P$ with $u''z \in E(G)$. No extension of the path $(z, (u'', M, u), (b_1, K, a_2), (t, M, x), y)$ may contain v', a contradiction. This proves (2).

(3) *There is a vertex $w \in Y \cap P$ such that $wu' \in E(G)$.*

By (2), there is a vertex $y \in Y \cap Q$ such that $yv' \in E(G)$. Let C be the connected component of the subgraph of H induced by Y and containing y.

Assume that $uv \notin E(G)$. First we verify that in this case C does not send any edge to K. Otherwise, let $S = (y, \ldots, y', z)$ be a shortest path from y to K (with $z \in V(K)$). Any extension of the path $((u, M, v'), (y, S, y'), (z, K, z'))$ (with z and z' in opposite partite sets) has endvertex at $u \in P$, which contradicts property (*). Hence $C \cup \{v\}$ has no neighbor in K. Let t be the last vertex on (v', M, u) that sends an edge to some $w \in C \cup \{v\}$. Either the path $(v, (t, M, v'), y)$ or the path $((v, M, t), w)$ leads to a contradiction, since no extension of these paths may include a_i ($i = 1$ or 2).

So we may assume that $uv \in E(G)$. Recall that $u'a_i \notin E(G)$, for $i = 1$ and 2, The path $(v, (u, M, v'), y)$ extends to include a_i. Let $S = (y, \ldots, z)$ be a shortest path from y to K (with $z \in V(K)$). Consider the path J obtained from the paths $((u', M, v'), (y, S, z))$ and (b_1, u, v) by joining them in K with a shortest path between b_1 and z. Because J misses either a_1 or a_2, there exists a vertex $w \in Y \cap P$ such that $u'w \in E(G)$. This proves (3).

(4) To conclude the proof of Claim I we show that the existence of the vertices $y, w \in Y$ obtained in (2) and (3) leads to a contradiction.

Let $S = (y, \ldots, y', z)$ be a shortest path from y to K as introduced in (3) above. If $z = b_i$ ($i = 1$ or 2), then any extension of $(w, (u', M, v'), (y, S, y'), (b_i, K, a_2))$ misses u. Hence we may assume that $z = a_i$ ($i = 1$ or 2). Furthermore, the path $(w, (u', M, v'), (y, S, y'), (a_i, K, b_2))$ extends with $b_2u \in E(G)$.

Let $R = (w, \ldots, w', r)$ be some path that we start adding when the path $(v, u, b_1, (a_i, S, y), (v', M, u'), w)$ is extended to include all vertices of $P \cup \{a_1, a_2\}$. In particular, the extension will include $a_{3-i} \in K$, thus R should enter K. Actually we assume that r is the first vertex from K along R. If $r = b_2$, then any extension of $(y, (v', M, u'), (w, R, w'), (b_2, K, a_2))$ would miss u. Hence $r = a_{3-i}$. From this we obtain that the path $(w, (u', M, v'), (y, S, a_i), b_2, (a_{3-i}, R, w'))$ must extend with $wb_1 \in E(G)$ to include u. Now any extension of $(y, (v', M, u'), w, (b_1, K, a_2))$ misses u, a contradiction. This concludes the proof of Claim I. □

Claim II: *Every maximal path of H with distinct endvertices contains all vertices of P.*

Proof. Suppose to the contrary that there exists a maximal path $M = (u, \dots, v', v)$ of H such that $P \setminus V(M)) \neq \emptyset$. Assume that M is the *longest* such path. By Claim I, we have $u, v \in Q$. Because M extends in G, and by the symmetry of the endvertices, we may assume that $ua_1 \in E(G)$. Let $Y = V(H) \setminus V(M)$. For $i = 1$ or 2, the path $((v, Mu), (a_1, K, b_i))$ extends in G. Hence, for every $i = 1$ and 2, there exists a vertex $y_i \in Y \cap P$ with $b_i y_i \in E(G)$.

(1) *There is an edge from Y to M.*

First assume that $a_j z \in E(G)$, for some $z \in Y \cap Q$ and $j = 1$ or 2. Any maximal extension of the path $(y_1, (b_1, K, a_j), z)$ has to cover vertices of M, thus there exists an edge between Y and M. Assume now that there is no edge from $\{a_1, a_2\}$ to Y. If $y_1 \neq y_2$, then the path $(y_1, b_1, a_1, b_2, y_2)$ extends to include a_2, hence there is an edge from Y to M. So we may suppose that $y_1 = y_2$ is the only neighbor of b_1 and b_2 in Y. In this case the path $((v, M, u), a_1, b_1, y_1, b_2, a_2)$ is maximal in G. This contradicts property (*) and concludes the proof of (1).

(2) *There is a vertex $y \in Y \cap Q$ such that $yv' \in E(G)$.*

By (1), there is an edge between M and Y. Let x be the first vertex along the path (v, M, u) which has a neighbor from Y, say $xy \in E(G)$, for some $y \in Y$. Suppose to the contrary that $x \neq v'$.

If $x \in P$ then the path $(y, (x, M, u), (a_1, K, b_i), y_i)$, where $i = 1$ or 2, has no extension including v', by the choice of x. Hence $x \in Q$. Moreover, as the path above can not exist, $y = y_1 = y_2$ is the only vertex of $Y \cap P$ adjacent to b_i ($i = 1$, 2) and x.

The path $((v, M, u), a_1, b_1, y, b_2, a_2)$ extends with $a_2 w \in E(G)$, for some $w \in Y \cap Q$. The path $((v, M, u), (a_1, K, b_1), y)$ extends at y, thus $yz \in E(G)$, for some $z \in Y \cap Q$. If $z \neq w$, then the path $(z, y, (x, M, u), a_1, b_1, a_2, w)$ misses v'. Thus we conclude that $z = w$ is the only neighbor of y from $Y \cap Q$.

For $i = 1$ or 2, the path $(w, a_2, b_{3-i}, y, (x, M, u), b_i)$ must extend at b_i to include v'. Thus there is an edge $b_i t \in E(G)$, where $t \in P$ is a vertex of (v, M, x). The path $(w, a_2, b_i, (t, M, u), a_1, b_{3-i}, y)$ misses v' unless $t = v'$. Therefore, we may assume that $b_i v' \in E(G)$ for $i = 1$ and 2. The path $(y, (x, M, u), a_1, b_1, v', b_2, a_2, w)$ has no extension at y. This contradicts property (*). Therefore, $yv' \in E(G)$ follows.

(3) *For $j = 1$ or 2, there exists a path $S = (y, \dots, x, b_j)$ such that $V(S) \setminus \{b_j\} \subset Y$.*

Let C be the connected component of the subgraph of G induced by Y and containing y. First we show that there is a vertex $x \in C$ that is adjacent to some vertex of K. Suppose this is false. In particular, we may assume that the neighbor $y_1 \in Y \cap P$ of b_1 is not in C.

If $a_2v \in E(G)$, then no extension of $(y, (v', M, u), a_1, b_1, a_2, v)$ contains y_1. Hence $a_2v \notin E(G)$. Similarly, if $a_1v \notin E(G)$, then no extension of $(y, (v', M, u), a_1, , v)$ contains y_1. Hence $a_1v \notin E(G)$. Let $t \in V(M)$ be the last vertex on (v', M, u) adjacent to v or to some vertex $x \in C$. One of the paths $((v, M, t), x)$ and $(y, (v', M, t), v)$ exists and misses y_1, a contradiction. Thus we obtain that some $x \in C$ is adjacent to some vertex of K.

The existence of x implies that there is a path $S = (y, \ldots, x, z)$, for some $z \in V(K)$, such that $V(S) \setminus \{z\} \subset Y$. Now suppose that in every such path S we have $z = a_i$ ($i = 1$ or 2). In particular, no vertex of C is adjacent to b_1 or b_2. If $z = a_2$, then any extension of $((v, M, u), a_1, b_1, (z, S, y))$ would miss y_1. Hence $z = a_1$, for every path S, and a_2 has no neighbor in C. The path $((v, M, u), (z, S, y))$ has no extension that includes a_2, a contradiction. This proves (3).

(4) *For* $k = 1$ *or* 2, $\quad ua_k, va_{3-k} \in E(G)$.

Assume that $S = (y, \ldots, x, b_1)$ is a path guaranteed by (3). Let $R = (r, \ldots, y'')$ be a path (possibly empty) such that $((v, M, u), (a_1, K, b_1), (x, S, y), (r, R, y''))$ is maximal in G. The path $((v, M, u), a_1, b_1, (x, S, y), (r, R, y''))$ has an extension to include a_2. Thus either $va_2 \in E(G)$ which proves (4), or we have $y''a_2 \in E(G)$.

Assume that $va_2 \notin E(G)$. Let v'' be the neighbor of v' in M different from v. The path $(v, v', y, (r, R, y''), a_2, b_1, a_1, (u, M, v''))$ extends with $v''w \in E(G)$, for some $w \in V(S) \cap P$. Thus we obtain a path $M' = ((u, M, v''), (w, S, y), v', v)$ which is maximal in H and longer than M. By the choice of M, we have $P \subset M'$, and $w = x$. This implies that R is empty ($y = y''$), furthermore, $ya_2, v''x \in E(G)$, and $b_1x, b_2x \in E(G)$. Observe that the path $((u, M, v''), x, (b_1, K, a_2), y, v', v)$ is maximal in G, hence we have $S = (y, x, b_1)$.

The path $(b_2, a_1, (u, M, v''), x, y, a_2, b_1)$ extends to include v', the only uncovered vertex of P; therefore, $b_jv' \in E(G)$, for $j = 1$ or 2. The path $(v, v', b_j, a_1, b_{3-j}, x, (v'', M, u))$ extends to include a_2. Thus we have $ua_2 \in E(G)$ (recall that, by assumption, $va_2 \notin E(G)$). If $va_1 \in E(G)$, then we are done. Assuming that $va_1 \notin E(G)$, we obtain that $ya_1, \in E(G)$, by the symmetry of a_1 and a_2. For $i = 1$ and 2, the path $(v, v', y, x, b_1, a_i, (u, , M, v''))$ extends with $v''a_{3-i} \in E(G)$. The path $((u, M, v''), (a_1, K, b_j), v', v)$ is maximal in G and misses x, a contradiction. This concludes the proof of (4).

In the next step we use $S = (y, \ldots, x, b_j)$, $j = 1$ or 2, a path guaranteed by (3), together with further paths similar to those in the proof of (4).

(5) $\quad P \setminus V(M) = \{x\}$, *$xb_i \in E(G)$, for $i = 1, 2$, and there exists $z \in Y \cap Q$ such that $za_1, zx \in E(G)$.*

By (4), and by the symmetry of a_1 and a_2, we may assume that $va_2 \in E(G)$. Also assume that $S = (y, \ldots, x, b_2)$. The path $N = (v, (a_2, K, b_2), (x, S, y), (v', M, u))$ is maximal, hence $(P \setminus V(M)) \subset V(S)$. Observe that N has no chord induced by two non–consecutive vertices of S; for otherwise, a shorter maximal path of G would result by using that chord to skip over some vertex of $V(S) \cap P$. The same argument shows that if $b_1 y_1 \in E(G)$, for some $y_1 \in Y \cap P$, then $y_1 = x$ follows. Thus we have $b_1 x \in E(G)$.

The path $(a_1, b_1, x, b_2, a_2, (v, M, u))$ extends with $a_1 z \in E(G)$, for some $z \in Y \cap Q$. Note that $z \notin V(S)$, because otherwise, the maximal path $((u, M, v'), (y, S, z), a_1, b_1, a_2, v)$ would miss x. We show next that $zx \in E(G)$. Every extension of $(z, a_1, b_1, a_2, (v, M, u))$ contains x, thus $z \in C$, where C is the connected component containing y in the subgraph of H induced by Y. This implies that $zz' \in E(G)$, for some $z' \in V(S) \cap P$. The maximal path $((u, M, v'), (y, S, z'), z, a_1, b_1, a_2, v)$ contains x, thus $z' = x$. Observe that the path $((u, M, v), a_2, b_1, a_1, z, x, b_2)$ must contain $V(S) \cap P$, on the other hand S has no chord from b_2. Therefore, $S = (y, x, b_2)$ which concludes the proof of (5).

(6) $P = \{v', x\}$, $Q = \{u, v, y, z\}$, *and* $v'z \notin E(G)$.

The path $(z, x, b_1, a_1, (u, M, v'), y)$ extends to include a_2. Hence we have either $ya_2 \in E(G)$ or $za_2 \in E(G)$. Suppose first that $ya_2 \in E(G)$. The path $(v, v', y, a_2, b_1, a_1, (u, M, v''))$ extends to include x, thus $v''x \in E(G)$. The path $(z, a_1, (u, M, v''), x, b_1, a_2, b_2)$ extends to include v'. Hence we have either $v'b_2 \in E(G)$ or $v'z \in E(G)$. None of them is possible, because in the first case $((u, M, v'), (b_2, K, a_2), v)$, and in the second case $((u, M, v'), z, a_1, b_2, a_2, v)$ is a maximal path of G missing x. Therefore, we may assume that $ya_2 \notin E(G)$ and $za_2 \in E(G)$, that is y and z are not interchangeable. If $zv' \in E(G)$, then y and z are interchangable with respect to v'. Thus we may also assume that $zv' \notin E(G)$,

We show that $v'' = u$. Suppose that this is false, that is $u' \neq v'$, where u' is the neighbor of u in M. The path $(y, v', v, (a_2, K, b_2), x, z)$ extends to include uncovered vertices of $V(M) \cap P$. Let w be the last vertex on (v'', M, u) adjacent to y or z. In the first case $(y, (w, M, v), (a_2, K, b_2), x, z)$ and in the second case $(z, (w, M, v), (a_2, K, b_2), x, y)$ is a maximal path, therefore, $w = u'$ must hold. Observe that $u'z \notin E(G)$, for otherwise, the maximal path $((v, M, u'), z, a_2, b_1, a_1, u)$ in G would miss x. Hence we have $u'y \in E(G)$.

The path $((v'', M, u'), y, v', v, a_2, b_1, a_1, u)$ extends with $v''x \in E(G)$. The path $(z, a_1, (u, M, v''), x, b_1, a_2, b_2)$ extends with $b_2 v' \in E(G)$. Thus we obtain that $((u, M, v'), (b_2, K, a_2), v)$ is a maximal path of G missing x, a contradiction. Therefore, $u' = v'$ and (6) follows.

To conclude the proof of Claim II we show that $G \cong G_5$. By (5) and (6), G is a 4×6 bipartite graph such that its edges determined so far (explicitly or by symmetry) induce a G_5. It is easy to check that including any of the four edges ua_2, va_1, or $v'b_i$, $i = 1, 2$, would result in a non–scenic graph containing a maximal path of length less than 8. Therefore, $G \cong G_5$ follows, contradicting the assumption of the theorem. □

Claim II implies that H has at most one non–trivial connected component, and this component is scenic. If H is connected, then it is non–traceable, because $q \geq p+2$. If H is disconnected, then it has exactly one trivial component (i.e., isolated vertex). Indeed, in case of two isolated vertices $u, u' \in V(H)$, one would easily find a path $M \subset K+\{u, u'\}$ which is maximal in G and misses all vertices in the non–trivial component of H. This contradicts (*) and concludes the proof of Theorem 4.2. □

5 $K_{2,2}$–extension

In this section we consider ways that a $K_{2,2}$ can be "added" to non–traceable scenic graphs so that the property of being scenic is preserved. If G is a non–traceable scenic graph containing a copy $K \cong K_{2,2}$, then we say that G is a *scenic $K_{2,2}$–extension* of $H = G - V(K)$.

We use the following notations throughout this section. We assume that G is scenic non–traceable $K_{2,2}$–extension of $H = G - V(K)$. The vertices of K are a_1, a_2, b_1, and b_2, the partite sets of H are P and Q with $|P| \leq |Q| - 2$, and the partite sets of G are $P \cup \{a_1, a_2\}$ and $Q \cup \{b_1, b_2\}$. In the figures accompanying the proofs, black circles indicate vertices in the smaller partite set of G. Let (a_i, K, b_j) denote the Hamiltonian path of K from a_i to b_j $(1 \leq i, j \leq 2)$. For $H' \subseteq H$ and $u, v \in V(H')$, we denote by (u, H', v) a path of H' between u and v spanning as many vertices of $V(H') \cap P$ as possible.

By Theorem 4.2, one may assume that H is either a non–traceable scenic graph or a (traceable or non–traceable) scenic graph plus an isolated vertex. We need the following easy corollaries of Theorem 2.2.

Lemma 5.1 *Let G be a scenic $K_{2,2}$–extension of H.*
(i) If there is a maximal path of H between $y, y' \in Q$, then there is an edge from $\{y, y'\}$ to $\{a_1, a_2\}$.
(ii) If at least two vertices of Q are adjacent to $\{a_1, a_2\}$, then there exist two independent edges $y_1a_1, y_2a_2 \in E(G)$, for some $y_1, y_2 \in Q$.

Proof. Because G is scenic, every maximal extension of the path between y and y' contains a_1 and a_2 which proves (i). The maximum path length in G is $2|P| + 2$, thus no maximal extensions of the path (a_1, K, b_2) or (a_2, K, b_2) may start at a_1 or at a_2. Therefore, both a_1 and a_2 are adjacent to Q. This observation together with the condition in (ii) imply that the edges between Q and $\{a_1, a_2\}$ can not be covered with one vertex. Hence there exist two independent edges, and (ii) follows. □

Proposition 5.2 *The equi-subdivided star $K^s_{1,r}$ ($r \geq 3$, $s \geq 1$) and the graphs $G_1, \ldots, G_6$ have no scenic $K_{2,2}$–extensions.*

Proof. Suppose on the contrary that G is a scenic $K_{2,2}$–extension of H, where H is one of the seven graphs in the proposition.

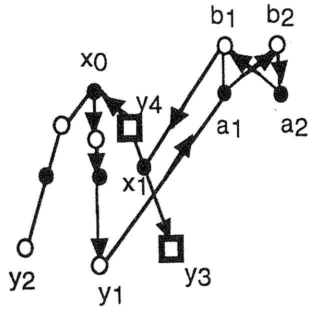

Figure 2:

Case 1: $H = K_{1,r}^{s}$ ($r \geq 3$, $s \geq 1$). Because $|P| \leq |Q|-2$, the center of H is a vertex $x_0 \in P$, and all leaves of H are in Q. Let $y_1, y_2, y_3 \in Q$ be distinct leaves of H. By Lemma 5.1 (i), one may assume that $y_1 a_1 \in E(G)$. The path $((y_2, H, y_1), (a_1, K, b_1))$ is not maximal, thus $b_1 x_1 \in E(G)$ holds, for some $x_1 \in (x_0, H, y_3)$ (see Fig. 2). If $y_4 \in Q$ is an arbitrary vertex on (x_0, H, x_1), then no extension of the path $((y_4, H, y_1), (a_1, K, b_1), (x_1, H, y_3))$ contains the vertices of P on the path (x_0, H, y_2), a contradiction.

Case 2: $H = G_1, G_2$ or G_3. Let $y, y' \in Q$ be any pair of vertices such that their removal does not disconnect H (note that all pairs satisfy this in $H = G_2$ or G_3, and just one pair fails it in $H = G_1$). It is easy to check that between y and y' there exists a maximal path in H (actually, covering all vertices in P). Hence, by Lemma 5.1 (i) and (ii), there exist $y_1 a_1, y_2 a_2 \in E(G)$, with distinct $y_1, y_2 \in Q$. Consider a maximal path $(b_1, a_1, (y_1, H - x_1, y_2), a_2, b_2)$ in H which does not cover a vertex $x_1 \in P$. This path has an extension $b_1 x_1 \in E(G)$ to include x_1. Fig. 3 (a) shows a particular case, where $H = G_1$. The argument works for any other choice of H, and for other positions of y_1 and y_2, as well. Thus we always have $x_1 b_1 \in E(G)$, for some vertex $x_1 \in P$.

Let x_2 and x_3 be the other two vertices in P. If x_2 and x_3 have two common neighbors in H, then, by Lemma 5.1 (i), one of them is adjacent to K, say $y_2 a_2 \in E(G)$. The maximal path $(y_1, x_1, (b_1, K, a_2), y_2, x_3, y_3)$ shown in Fig. 3 (b) misses x_2, a contradiction. Assume now that the previous argument does not apply (even if we relabel the vertices of P), because there is no edge from $\{x_2, x_3, y_2\}$ to K. In this case any path of H between x_2 and y_2 not containing edge $x_2 y_2$ is maximal in G and misses K, a contradiction.

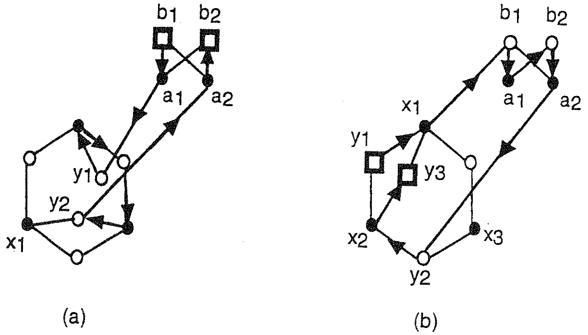

Figure 3:

Case 3: $H = G_4$. Since G is connected, either $x_1 b_\epsilon \in E(G)$ or $y_1 a_\epsilon \in E(G)$ holds, for some $x_1 \in P$ or $y_1 \in Q$, and $\epsilon = 1$ or 2. Assume that $y_1 a_1 \in E(G)$ and let x_1 be a neighbor of y_1. The path $((b_1, K, a_1), (y_1, H - x_1, y_3))$ extends to include x_1 (see Fig. 4 (a)). Thus $x_1 b_1 \in E(G)$ follows. Because there is a path of H between y_2 and y_3 that covers all vertices of P, say $y_2 a_2 \in E(G)$. The path $(y_1, x_1, (b_1, K, a_2), (y_2, H - \{x_1 x_4\}, y_4))$ in Fig. 4 (b) is maximal and misses x_4, a contradiction.

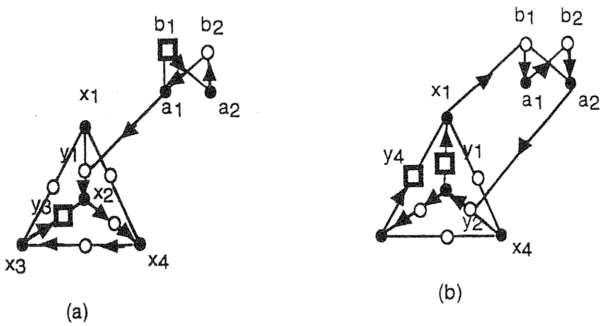

Figure 4:

Case 4: $H = G_5$. It is easy to verify that between any pair $y, y' \in Q$ there exists a maximal path in H. Hence by Lemma 5.1, $y_1 a_1, y_2 a_2 \in E(G)$, for some $y_1, y_2 \in Q$. The path $(b_1, a_1, (y_1, H - x_1, y_2), a_2, b_2)$ as shown in Fig. 5 (a) extends to include x_1. Thus one may assume that $x_1 b_1 \in E(G)$, so $(y_1, x_1, (b_1, K, a_2), (y_2, H - \{x_1, x_4\}, y_4))$ in Fig. 5 (b) is a maximal path missing x_4, a contradiction.

Case 5: $H = G_6$. Label the vertices of H as shown in Fig. 6. An easy argument using Lemma 5.1 shows the existence of $x_1b_1, y_2a_2 \in E(G)$. The maximal path $(y_3, x_3, y_1, x_1, (b_1, K, a_2), y_2, x_2, y_4)$ misses x_4, a contradiction.

This concludes the proof of the proposition. □

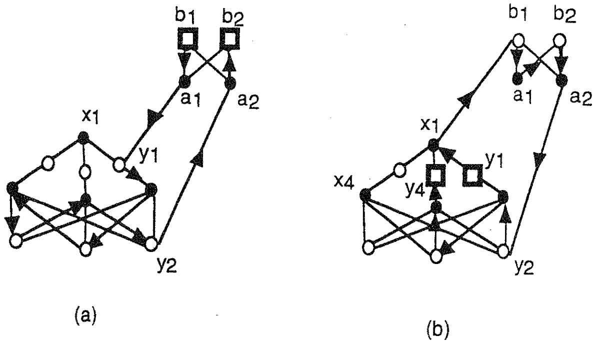

Figure 5:

The following technical lemma will be used when proving that a $K_{2,2}$–extension of a generic graph is generic. We note in advance that the only exception will be the generic graph $K_{2,4} - 2K_2$ which has a non–generic $K_{2,2}$–extension, namely G_6. Recall that a $p \times q$ generic graph has the form $K_{p,q} - F$, where the partite sets P and Q contain $p \geq 2$ and $q \geq p + 2$ vertices, respectively, and F is a star forest with its star components centered in Q.

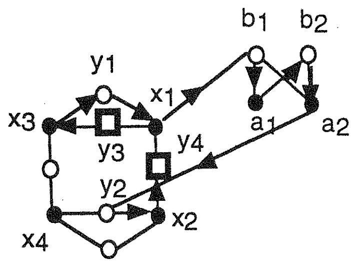

Figure 6:

Lemma 5.3 *Let H be a $p \times q$ generic graph with partite sets P and Q. If $H \neq K_{2,4} - 2K_2$, then*
(A) H has a maximal path between any two non–isolated vertices $y, y' \in Q$;
(B) for every $x \in P$ and for every $y, y' \in Q$ which are distinct non–isolated vertices of $H - x$, there is a path in $H - x$ between y and y' that contains all vertices in $P \setminus \{x\}$.

Proof. (A) Let $M = (y, x, \ldots, x', y')$ be a maximum length path of H from y to y'. We shall prove that M contains P. Suppose on the contrary that $x_1 \in P \setminus V(M)$. First assume that there are vertices $y_1, y_2, y_3 \in Q \setminus V(M)$. Because H is generic, x_1 is adjacent to y or y', say $x_1 y' \in E(H)$. Moreover, by the pigeon hole principle, some y_i is adjacent to both x' and x_1, for $i = 1, 2$ or 3. The path $((y, M, x'), y_i, x_1, y')$ would be longer than M, a contradiction.

Thus we may assume that $Q \setminus V(M) = \{y_1, y_2\}$, $P \setminus V(M) = \{x_1\}$. Furthermore, x_1 is non–adjacent to one of y_1 and y_2, say $x_1 y_2 \notin E(G)$. We have $x_1 y_1, x_1 y', x_1 y \in E(G)$, and by the argument above, $x y_1, x' y_1 \notin E(G)$. Hence $x y_2, x' y_2 \in E(G)$. Also $H \neq K_{2,4} - 2K_2$, thus $p \geq 3$. In particular, $x \neq x'$, and $M = (y, x, \ldots, y'', x', y')$. We shall prove by induction on p that in the particular generic graph H described above there exists a path from y to y' that covers P. This will contradict our assumption and will prove (A).

For $p = 3$, the path $(y, x, y_2, x', y'', x_1, y')$ covers P. Thus (A) is true for $p = 3$. Assume that $p \geq 4$ and (A) is true for $p - 1$. Because our graph H is generic, x is adjacent to every vertex of $Q \setminus \{y_1\}$ and x_1 is adjacent to every vertex of $Q \setminus \{y_2\}$. Hence y and y'' are not isolated vertices in $H' = H - \{x', y'\}$. By the induction hypothesis, H' has a path $M' = (y, \ldots, y'')$ that contains $P \setminus \{x'\}$. The path $((y, M', y''), x', y')$ covers P, a contradiction. Thus (A) follows.

(B) If H or $H' = H - x$ has an isolated vertex $u \in Q$, then $H - \{x, u\}$ is a complete bipartite graph and (B) obviously holds. Assume that H and H' are both connected, in particular, $H' \neq K_{2,4} - 2K_2$. Now (B) follows by applying (A) for the generic graph H'. □

Proposition 5.4 *If H is the union of an isolated vertex and one of the following graphs: $G_1, \ldots, G_6$, an equi–subdivided star $K^s_{1,r}$ ($r \geq 2$, $s \geq 1$), or a connected $p \times q$ generic graph ($p \geq 2$, $q \geq p + 2$) different from a complete bipartite graph, then H has no scenic $K_{2,2}$–extension.*

Proof. Let u be the isolated vertex of H and let $H' = H - u$ be one of the graphs in the proposition. Suppose on the contrary that G is a scenic $K_{2,2}$–extension of H. Observe that $u \in Q$, for otherwise, G would have a path $(u, (b_1, K, a_1))$ and a maximal extension of it with a black end vertex $u \in P$. One may assume that $u a_1 \in E(G)$. The path $S = (u, (a_1, K, b_2))$ extends in G with an edge $b_2 z$, for some

$z \in V(H') \cap P$. All maximal extensions of S are obtained by concatenating a maximal path of H' starting at z. Hence all maximal paths of H' starting at z have the same length. This is obviously not true, for any black vertex z of H', if H' is one of the graphs $G_1, \ldots, G_6$ in Fig. 1 or an equi–subdivided star $K^s_{1,r}$ with $r \geq 2$, $s \geq 1$.

Now suppose that H' is a connected generic graph different from a complete bipartite graph. The previous argument shows that $H' \neq K_{2,4} - 2K_2$. Let $xy \notin E(H')$, for some $x \in P$ and $y \in Q \setminus \{u\}$. By the connectivity of H', y is non–isolated in $H' - x$. In addition, because $H' \neq K_{2,4} - 2K_2$, we may choose x and y such that every $y' \in Q \setminus \{u\}$ is a non–isolated vertex of $H' - x$.

By Lemma 5.3 (A), there is a maximal path S_1 of H' between any two distinct vertices $y', y'' \in Q \setminus \{u, y\}$. This path covers P and extends in G, say from end vertex y' with an edge to $\{a_1, a_2\}$. If $y'a_2 \in E(G)$, then $M_1 = ((y'', S_1, y'), a_2, b_1, a_1, u)$ is a maximal path of G. If $y'a_2, y''a_2 \notin E(G)$, then one may assume that $y'a_1, ua_2 \in E(G)$, and hence $M_2 = ((y'', S_1, y'), a_1, b_1, a_2, u)$ is a maximal path of G.

By Lemma 5.3 (B), $H' - x$ has a path S_2 between y' and y covering all vertices in $P \setminus \{x\}$. By a similar argument as above, we obtain that either $M'_1 = ((y, S_2, y'), a_2, b_1, a_1, u)$ or $M'_2 = ((y, S_2, y'), a_1, b_1, a_2, u)$ exists and is a maximal path of G. The lengths of the maximal paths M_i and M'_i are different, for $i = 1$ or 2, hence G is not scenic. This contradiction concludes the proof of the proposition. □

Proposition 5.5 *If G is a scenic $K_{2,2}$–extension of a $p \times q$ generic graph H, then either $G \cong G_6$ or G is generic.*

Proof. By definition, G is generic if and only if at most one edge is missing at any vertex of $P \cup \{a_1, a_2\}$.

Case 1: *H is connected and different from $K_{2,4} - 2K_2$*. Suppose that $xy \notin E(G)$, for some $x \in P$ and $y \in Q$. By Lemma 5.3 (A), there is a maximal path of H between any two distinct vertices $y_1, y_2 \in Q \setminus \{y\}$. This path extends in G, say $y_1a_i \in E(G)$ ($i = 1$ or 2). Obviously, y and y_1 are non–isolated vertices in $H - x$, thus by Lemma 5.3 (B), there is a path $S = (y, \ldots, y_1)$ in $H - x$ containing $P \setminus \{x\}$. The path $((y, S, y_1), (a_i, K, b_j))$ extends, hence $b_jx \in E(G)$ holds, for $j = 1$ and 2.

Let $x \in P$ be a vertex such that $xb_i \notin E(G)$, $i = 1$ or 2. We shall prove that $xb_{3-i} \in E(G)$. By the argument above, $xy \in E(G)$, for every $y \in Q$. Lemmas 5.3 (A) and 5.1 imply the existence of independent edges $ya_1, y'a_2 \in E(G)$, $y, y' \in Q$. If y and y' are non–isolated in $H - x$, then by Lemma 5.3 (B), $H - x$ has a path S between y and y' which contains $P \setminus \{x\}$. The path $(b_i, a_1, (y, S, y'), a_2, b_{3-i})$ extends with $b_{3-i}x \in E(G)$.

We show that the previous argument applies even if one of y and y', say y', is an isolated vertex of $H - x$. Note that no $y'' \in Q \setminus \{y'\}$ is isolated in $H - x$. Thus if we can not replace y' with some $y'' \in Q \setminus \{y, y'\}$, and proceed as above, this is because $y''a_2 \notin E(G)$, for every $y'' \in Q \setminus \{y\}$. We prove that this can not happen. Because $y'x' \notin E(G)$ holds for each $x' \in P \setminus \{x\}$, there exists $ux' \in E(G)$ with $x' \in P \setminus \{x\}$ and $u \in Q \setminus \{y'\}$ such that $H - \{x', u\}$ is a connected generic graph. By Lemma 5.3 (A), the generic graph $H - \{x', u\}$ has a path S between any two vertices $y_1, y_2 \in Q \setminus \{u, y'\}$ which contains $P \setminus \{x'\}$. We know that there is an edge between $\{y_1, y_2\}$ and $\{a_1, a_2\}$. By our assumption, y_1 or y_2 is adjacent to a_1, say $a_1y_1 \in E(G)$. Thus the path $(u, x', b_1, a_1, (y_1, S, y_2))$ misses a_2, a contradiction. We conclude that at every $x \in P$ at most one edge is missing in G.

Next assume that $a_iy_1, a_iy_2 \notin E(G)$, for some $y_1, y_2 \in Q$ and $i = 1$ or 2. By Lemma 5.3 (A), H has a path $S = (y_1, x, \ldots, y_2)$ containing P. Furthermore, we know that one of y_1 and y_2 sends an edge to K, say $y_1a_{3-i} \in E(G)$. The path $(y_1, a_{3-i}, b_1, (x, S, y_2))$ is maximal in G and misses a_i, a contradiction. Therefore G is scenic.

Case 2: *H is disconnected.* By Proposition 5.4, $H = H' + u$, where $u \in Q$ is an isolated vertex of H, and H' is a complete bipartite graph. We may assume that $ua_j \in E(G)$ ($j = 1$ or 2). For every $i = 1, 2$, the path $(u, (a_j, K, b_i))$ extends with an edge, say $b_ix_i \in E(G)$, where $x_i \in P$.

First we show that $ya_{3-j} \in E(G)$, for some $y \in Q \setminus \{u\}$. This is obvious if $ua_{3-j} \notin E(G)$, because the path $(u, a_j, b_1, (x_1, H', y))$ extends with $ya_{3-j} \in E(G)$. If $ua_{3-j} \in E(G)$, then any maximal path $(y, \ldots, y')$ of H' extends with an edge, say $ya_k \in E(G)$. Now the claim follows by choosing $j = 3 - k$, because $ua_j \in E(G)$ holds for every $i = 1, 2$, by assumption.

Our next claim is that $b_ix \in E(G)$, for every $i = 1, 2$ and $x \in P$. For any $x \in P$, $H' - x$ is a complete bipartite graph, hence it has a path $S = (x_i, \ldots, y)$ containing all vertices in $P \setminus \{x\}$. The path $(u, a_j, b_{3-i}, a_{3-j}, (y, S, x_i), b_i)$ extends with $b_ix \in E(G)$.

Suppose now that $ua_i, ya_i \notin E(G)$, for some $y \in Q \setminus \{u\}$ and $i = 1$ or 2. Let $S = (x_1, \ldots, y)$ be a path of H' containing P. The path $(u, a_{3-i}, b_1, (x_1, S, y))$ is maximal in G and misses a_i, a contradiction. Therefore, it remains to show that if $ya_i \notin E(G)$, for some $y \in Q \setminus \{u\}$ and $i = 1$ or 2, then $y'a_i \in E(G)$, for every $y' \in Q \setminus \{u, y\}$. Let $x, x' \in P$, and let $S = (x', \ldots, y')$ be a path of $H' - \{x, y\}$ covering all vertices in $P \setminus \{x\}$. The path $(y, x, b_1, a_{3-i}, b_2, (x', S, y'))$ extends with $y'a_i \in E(G)$. This proves that G is generic and concludes the proof of the proposition.

Case 3: $H = K_{2,4} - 2K_2$. We show that if G is not generic, then $G \cong G_6$. Let $P = \{x_1, x_2\}$, $Q = \{y_1, \ldots, y_4\}$, and assume that the two missing edges are $x_1y_3, x_2y_4 \notin E(H)$. Suppose that $G \cong K_{4,q} - F$, and G is not generic.

First we assume that one of x_1 or x_2 has degree more than 1 in F, say $x_2b_2 \notin E(G)$. If $y_ia_j \in E(G)$ holds, for some $1 \leq i, j \leq 2$, then the maximal path $((b_2, K, a_j), y_i, x_1, y_4)$ would miss x_2. Hence there are no edges between the sets $\{a_1, a_2\}$ and $\{y_1, y_2\}$. This observation together with Lemma 5.1 imply the existence of two independent edges between the sets $\{a_1, a_2\}$ and $\{y_3, y_4\}$. Assume that

$a_1y_3, a_2y_4 \in E(G)$. We shall verify that there are no further edges between H and K.

If $x_2b_1 \in E(G)$, then the maximal path $(y_1, x_1, y_4, a_2, b_1, x_2, y_2)$ misses a_1, a contradiction. If $x_1b_j \in E(G)$ ($j = 1$ or 2), then the maximal path $(y_1, x_1, b_j, a_1, y_3, x_2, y_2)$ misses a_2, a contradiction. Assume now that one of a_1y_4 and a_2y_3 is an edge, say $a_1y_4 \in E(G)$. The maximal path $(y_1, x_1, y_4, a_1, y_3, x_2, y_2)$ misses a_2, a contradiction. Thus we obtain that $G \cong G_6$.

Second we assume that one of a_1 and a_2 has degree more than one in F. Because G is not generic, and $\{x_1, y_1, x_2, y_2\}$ induces a $K_{2,2}$ in G, we have $G - \{x_1, x_2, y_1, y_2\} \cong G - \{a_1, a_2, b_1, b_2\} \cong K_{2,4} - 2K_2$. By the symmetry of the sets $\{x_1, x_2\}$ and $\{a_1, a_2\}$ in G, the previous argument applies, and $G \cong G_6$ follows. □

Proof of Theorem 1.2. Let G be a scenic non–traceable graph. If G has no cycle, then it is an equi-subdivided star by Proposition 2.1. Otherwise, by Theorem 2.2, G is a $p \times q$ bipartite graph with $p \geq 2$ and $q \geq p + 2$. If $p = 2$ or 3 then, by Propositions 3.2 and 3.1, G is either G_1, G_2, G_3, or a connected generic graph.

From now on assume that $p \geq 4$. If $G \neq G_4$ then, by Proposition 4.1, there exists a subgraph $K \cong K_{2,2}$ of G, so that G is a scenic $K_{2,2}$–extension of $H = G - V(K)$. If $G \neq G_5$, then by Theorem 4.2, either H is a scenic non–traceable graph or H is disconnected.

If H is a scenic non–traceable graph, then H must be generic. This follows by Proposition 3.1, for $p = 4$, and by Proposition 5.2, for $p > 4$. If H is disconnected, then by Theorem 4.2, $H = H' + u$, where H' is scenic and u is an isolated vertex. If H' is traceable, then $H' \cong K_{p,p+1}$, by Theorem 1.1. If H' is non–traceable, then by definition, $H' \in \mathcal{G}_{p-2,q-3}$. By Proposition 5.4, $H' + u$ might have a scenic $K_{2,2}$–extension only if H' is a complete bipartite graph. In these cases H is a disconnected $(p-2) \times (q-2)$ generic graph.

The previous paragraph shows that, whether or not H is connected, it must be generic. Proposition 5.5 implies that G is a connected generic graph or $G \cong G_6$. Consequently, every $G \in \mathcal{G}_{4,q}$ is either G_4, G_5, G_6, or a connected generic graph. Furthermore, each graph in $\mathcal{G}_{5,q}$ and $\mathcal{G}_{6,q}$ is generic. Proposition 5.5 implies that the same is true for every $\mathcal{G}_{p,q}$, $p \geq 7$. This concludes the proof of Theorem 1.2.

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(Received 13/3/96; revised 13/3/98)