SEMIREGULAR GROUP DIVISIBLE DESIGNS WHOSE DUALS ARE SEMIREGULAR GROUP DIVISIBLE

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ABSTRACT

A construction method for semiregular group divisible designs is given. This method can be applied to yield many classes of (in general, non–symmetric) semiregular group divisible designs whose duals are semiregular group divisible. In particular, the method can be used to construct many classes of transversal designs whose duals are semiregular group divisible designs, but not transversal designs. Also, some of the semiregular group divisible designs the construction method yields can be used to construct self–dual regular group divisible designs with two groups of points and blocks. Four infinite classes of such regular group divisible designs are constructed.

1. INTRODUCTION

Group divisible designs whose duals are also group divisible have received some attention (see [3] and [13]). Under the assumption that repeated points and repeated blocks are not permitted, Mitchell [13] has shown that, if $\mathcal{G}$ and its dual $\mathcal{G}^d$ are both group divisible, then $\mathcal{G}$ and $\mathcal{G}^d$ are each semiregular or $\mathcal{G}$ is symmetric and $\mathcal{G}$ and $\mathcal{G}^d$ are each regular. Some infinite classes of self–dual semiregular group divisible designs are known. Until recently, apart from the members of a class of self–dual semiregular group divisible designs given by Jungnickel ([11], p.282), the known self–dual semiregular group divisible designs were all self–dual transversal designs (such as those that arise from a generalized Hadamard matrix). On the other hand there seem to be no infinite classes of non–symmetric semiregular group divisible designs whose duals are semiregular group divisible in the literature (although two pairs of mutually dual semiregular group divisible designs appear in [5]). In a recent paper ([14]) the author has used certain substructures of symmetric BIBDs to obtain two infinite classes of, in general non–symmetric, semiregular group divisible designs whose duals are semiregular group divisible. The symmetric group divisible designs in one of these classes ([14], Theorem 3) have the parameters of the designs of Jungnickel mentioned above. The symmetric group divisible designs in the other class ([14], Theorem 2) are new. However, the method of [14] is of limited applicability. In order to obtain a better appreciation as to what semiregular group divisible designs with semiregular group divisible duals exist more powerful construction methods are needed. In Section 4 of this paper a method of constructing such group divisible designs is given. This method is shown to yield many infinite classes of semiregular group divisible designs whose duals are semiregular group divisible, some of which contain new symmetric examples. Many of the classes obtainable contain only non–symmetric examples since they contain only transversal designs whose duals are semiregular group divisible designs but not transversal designs. The method of Section 4 is, in fact, a generalization of the method introduced by Shrikhande and Raghavarao [18] in order to construct affine α–resolvable designs.

The situation regarding regular group divisible designs whose duals are group divisible is as follows: First, such regular group divisible designs are self–dual. (We give a simple proof of this result of Mitchell [13] in Section 3.) Second, a range of self–dual regular group divisible designs are known. (For some examples see Bose [3], pp 95–96, and Dembowski [7], Section 7.2). Indeed, almost all of the known symmetric regular group divisible designs are self–dual. The only known infinite classes of exceptions to this appear in Jungnickel and Vedder [12] and Rahilly [15].

Regular group divisible designs have some properties in common with BIBDs. For example, the number of blocks of a regular group divisible design is greater than or equal to its number of points. Intuitively, the closest a regular group divisible design $\mathscr{G}$ comes to being a symmetric BIBD is when $\mathscr{G}$ is self–dual and has two groups of points and two groups of blocks. (The dual of a regular group divisible design cannot be a BIBD.) Eight such regular group divisible designs are given in [5]. These designs, however, have "trivial" complements which are each simply a pair of disjoint symmetric BIBDs. It seems to the author that a construction method which yields self–dual regular group divisible designs with two groups of points and blocks and with non–trivial complements would be of interest. In Section 5 we show that the existence of a self–dual semiregular group divisible design whose parameters satisfy a certain simple condition (equation (15)) implies the existence of a self–dual regular group divisible design with two point and block classes and non–trivial complement. Four infinite classes of such designs are constructed. The self–dual semiregular group divisible designs which lead to two of these classes are obtained by a double application of the construction method of Section 4. In Section 5 an infinite class of self–dual regular group divisible designs with more than two point and block classes is also constructed.

In regard to the early sections of this paper, Section 2 contains basic definitions and facts concerning group divisible designs and resolutions of tactical configurations and in Section 3 some general properties of group divisible designs whose duals are group divisible are given. To a large extent, the constructions of Sections 4 and 5 rely on having available a supply of self–dual transversal designs. At the end of Section 3 constructions for self–dual transversal designs, due to various authors, which involve generalized Hadamard matrices are mentioned.

We treat designs as incidence structures in the manner of [7]. We denote the set of points of a design incident with a block B by (B).

2. GROUP DIVISIBLE DESIGNS

A tactical configuration with v points, b blocks, r blocks on each point and k points on each block is called a *(v,b,r,k)–configuration.* A (v,b,r,k)–configuration $(\mathcal{P},\mathcal{B},\mathcal{I})$ is said to be a *group divisible design* (GDD) if there is a partition of $\mathcal{P}$ into "groups" $\mathcal{P}_1,\cdots,\mathcal{P}_{m_2}$, where $m_2 \geq 2$, such that there are integers $m_1 \geq 2$ and λ_1 and λ_2 such that

(a) $|\mathcal{P}_i| = m_1$ for all $i = 1,\cdots,m_2$,

(b) any two points common to a group are on λ_1 blocks of $\mathcal{B}$,

(c) any two points in different groups are on λ_2 blocks of $\mathcal{B}$, and

(d) $\lambda_1 \neq \lambda_2$.

We say that such a GDD $\mathcal{G}$ "has parameters v,b,r,k; m_1, m_2; λ_1, λ_2". We also say that $\mathcal{P}_1,\cdots,\mathcal{P}_{m_2}$ form a *group division* of $\mathcal{G}$.

The parameters of a GDD satisfy the following equations

$$vr = bk \tag{1}$$

$$v = m_1 m_2 \tag{2}$$

and

$$(m_1 - 1)\lambda_1 + m_1(m_2 - 1)\lambda_2 = r(k - 1). \tag{3}$$

Let A be an incidence matrix of a GDD $\mathscr{G}$ with parameters v,b,r,k; m_1, m_2; λ_1, λ_2. The eigenvalues of AA^t are rk, $r - \lambda_1$ and $rk - v\lambda_2$ with respective multiplicities 1, $v - m_2$ and $m_2 - 1$. Since AA^t is a Gram matrix we must have $rk - v\lambda_2 \geq 0$. Group divisions can be exhaustively classified into the following mutually exclusive types:

1. Singular for which $r = \lambda_1$.
2. Semiregular for which $r > \lambda_1$ and

$$rk = v\lambda_2 \ . \tag{4}$$

3. Regular for which $r > \lambda_1$ and $rk > v\lambda_2$.

Since a GDD has a unique group division we can apply the terms "singular", "semiregular" and "regular" to GDDs as well as to group divisions.

Clearly, for a semiregular GDD we must have

$$r + (m_1 - 1)\lambda_1 = m_1\lambda_2 \ . \tag{5}$$

Also, a regular GDD is of rank v ([7], p.4) and so we must have $b \geq v$ for regular GDDs. It is also easy to show that a GDD $\mathscr{G}$ has repeated points if and only if $\mathscr{G}$ is singular.

These results for GDDs appear in Bose and Connor [4].

Next, let $k > 0$ and $\overline{m}_2 \geq 2$. An *α–resolution* of a (v,b,r,k)–configuration $\mathscr{C} = (\mathscr{P}, \mathscr{B}, \mathscr{I})$ is a partition $\mathscr{B}_1, \cdots, \mathscr{B}_{\overline{m}_2}$ of $\mathscr{B}$ such that each point of $\mathscr{P}$ is on precisely α blocks of each block class $\mathscr{B}_i$. If $\mathscr{B}_1, \cdots, \mathscr{B}_{\overline{m}_2}$ is an α–resolution of $\mathscr{C} = (\mathscr{P},\mathscr{B},\mathscr{I})$, then the substructure of $\mathscr{C}$ defined by $\mathscr{P}$ and $\mathscr{B}_1$ is a $(v,|\mathscr{B}_1|,\alpha,k)$–configuration and so $|\mathscr{B}_1|$ is independent of i. Clearly $|\mathscr{B}_1| = \frac{b}{\bar{m}_2}$ $(= \bar{m}_1$ say) for $i = 1,\cdots,\bar{m}_2$. It follows that $\alpha = \frac{r}{\bar{m}_2}$. A 1–resolution of a (v,b,r,k)–configuration is called a *parallelism.*

Let $\mathscr{G} = (\mathscr{P},\mathscr{B},\mathscr{I})$ be a semiregular GDD with parameters v,b,r,k; m_1, m_2; λ_1, λ_2 and with groups $\mathscr{P}_1,\cdots,\mathscr{P}_{m_2}$. Bose and Connor [4] have shown that each block of $\mathscr{B}$ meets each $\mathscr{P}_i$ in $\frac{k}{m_2}$ points. A necessary condition for such a $\mathscr{G}$ to exist is that $m_2|k$. Clearly the groups of $\mathscr{G}$ form a $\left(\frac{k}{m_2}\right)$–resolution of $\mathscr{G}^d$ (the dual of $\mathscr{G}$). From (2), (3) and (4) we readily obtain $\lambda_1(m_1 - 1) = r\left(\frac{k}{m_2} - 1\right)$. The substructure of $\mathscr{G}$ defined by $\mathscr{P}_i$ and $\mathscr{B}$ is an $(m_1, b, r, \frac{k}{m_2}, \lambda_1)$–design, provided $m_2 \neq k$.

It is not difficult to show that the groups of a GDD $\mathscr{G}$ form an α–resolution of $\mathscr{G}^d$ only if $\mathscr{G}$ is semiregular. An α–resolution $\mathscr{B}_1,\cdots,\mathscr{B}_{\bar{m}_2}$ of a (v,b,r,k)–configuration is said to be an *affine α–resolution* if $\mathscr{B}_1,\cdots,\mathscr{B}_{\bar{m}_2}$ is a (necessarily semiregular) group division of $\mathscr{G}^d$.

Suppose $\mathscr{G}$ is a semiregular GDD with parameters v,b,r,k; m_1, m_2; λ_1, λ_2. Solving (1), (2), (3) and (4) for v,b,r and λ_1 in terms of k, m_1, m_2 and λ_2 yields that $\mathscr{G}$ has parameters

$$m_1m_2, \left(\frac{m_1m_2}{k}\right)^2 \lambda_2, \left(\frac{m_1m_2}{k}\right)\lambda_2, k; m_1, m_2; \frac{m_1\lambda_2(k - m_2)}{k(m_1 - 1)}, \lambda_2 . \qquad (6)$$

If $\lambda_1 = 0$, then $k = m_2$ and $\mathscr{G}$ has parameters

$$m_1m_2, \lambda_2 m_1^2, \lambda_2 m_1, m_2; m_1, m_2; 0, \lambda_2 . \qquad (7)$$

A GDD with parameters (7) is called a *transversal design.*

3. DUAL PROPERTIES

Consider a GDD $\mathscr{G}$ whose dual $\mathscr{G}^d$ is also a GDD. Under the assumption that $\mathscr{G}$ has no repeated points or repeated blocks, Mitchell [13] has shown that

(a) $\mathscr{G}$ and $\mathscr{G}^d$ are each semiregular, or
(b) $\mathscr{G}$ and $\mathscr{G}^d$ are each regular and $\mathscr{G}$ is symmetric.

REMARK. It is possible for a semiregular GDD $\mathscr{G}$ to be such that $\mathscr{G}^d$ is a singular GDD. Let $\mathscr{D}$ be an affine α–resolvable BIBD and $\mathscr{G}$ be a multiple of $\mathscr{D}^d$. Then $\mathscr{G}$ is a semiregular GDD and $\mathscr{G}^d$ is a singular GDD.

Let $\mathscr{G}$ be a semiregular GDD with parameters v,b,r,k; m_1, m_2; λ_1, λ_2 and $\mathscr{G}^d$ be a semiregular GDD with parameters b,v,k,r; $\overline{m}_1$, $\overline{m}_2$; ρ_1, ρ_2. Then, as well as equations (1) to (4), we must have

$$b = \overline{m}_1 \overline{m}_2 \tag{8}$$

$$(\overline{m}_1 - 1)\rho_1 + \overline{m}_1(\overline{m}_2 - 1)\rho_2 = k(r-1) \tag{9}$$

and

$$rk = b\rho_2 \ (= v\lambda_2)\ . \tag{10}$$

Let A be an incidence matrix of $\mathscr{G}$. Then AA^t has one non–zero eigenvalue distinct from rk (namely $r - \lambda_1$) of multiplicity $v - m_2$ and A^tA has one such eigenvalue (namely $k - \rho_1$) of multiplicity $b - \overline{m}_2$. Since the non–zero eigenvalues of AA^t and A^tA are the same with the same multiplicities, we also must have

$$r - \lambda_1 = k - \rho_1 \tag{11}$$

and

$$v - m_2 = b - \overline{m}_2\ . \tag{12}$$

REMARKS. The groups of $\mathscr{G}$ and $\mathscr{G}^d$ form a tactical decomposition ([7], p.7) of $\mathscr{G}$.

Since $v - m_2 = b - \overline{m}_2$ this tactical decomposition is (in the terminology of [1]) a "strong" tactical decomposition of $\mathcal{G}$, a result of Mitchell [13] which also applies when $\mathcal{G}$ and $\mathcal{G}^d$ are regular GDDs.

A GDD is said to be *self–dual* if $\mathcal{G}^d$ is a GDD with the same parameters as $\mathcal{G}$.

PROPOSITION. Suppose $\mathcal{G}$ is a GDD with parameters $v,b,r,k; m_1, m_2; \lambda_1, \lambda_2$ whose dual is a GDD with parameters $b,v,k,r; \overline{m}_1, \overline{m}_2, \rho_1, \rho_2$.

(a) If $\mathcal{G}$ and $\mathcal{G}^d$ are semiregular, then the following conditions are equivalent:

(i) $\mathcal{G}$ is self–dual,

(ii) $\mathcal{G}$ is symmetric, and

(iii) $m_1 = \overline{m}_1$.

(b) If $\mathcal{G}$ and $\mathcal{G}^d$ are regular, then $\mathcal{G}$ is self–dual.

Proof. (a) Suppose $\mathcal{G}$ and $\mathcal{G}^d$ are semiregular GDDs with parameters as given.

(i) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (ii)

Equations (2), (8), (12) and $m_1 = \overline{m}_1$ imply $m_2 = \overline{m}_2$. That $v = b$ then follows from (2) and (8).

(ii) $\Longrightarrow$ (i)

From (1) we have $r = k$. Equation (12) and $v = b$ yield $m_2 = \overline{m}_2$. Then (2), (8), $v = b$ and $m_2 = \overline{m}_2$ yield $m_1 = \overline{m}_1$. That $\lambda_2 = \rho_2$ follows from (10) and $v = b$, and that $\lambda_1 = \rho_1$ follows from (11) and $r = k$.

(b) Suppose $\mathcal{G}$ and $\mathcal{G}^d$ are regular GDDs. From an earlier observation we have $b \geq v$ and $v \geq b$. So $v = b$ whence $r = k$. Let A be an incidence matrix of $\mathcal{G}$. We use the fact that the non–zero eigenvalues of AA^t and A^tA are the same with the same multiplicities.

The eigenvalues of AA^t (resp. A^tA) not equal to k^2 are $k - \lambda_1$ and $k^2 - v\lambda_2$ ($k - \rho_1$ and $k^2 - v\rho_2$) with respective multiplicities $v - m_2$ and $m_2 - 1$ ($v - \overline{m}_2$ and $\overline{m}_2 - 1$). Since $\mathscr{G}$ and $\mathscr{G}^d$ are regular these eigenvalues are all non–zero. Now, if $k - \lambda_1 = k^2 - v\rho_2$, then $v + 1 = m_2 + \overline{m}_2$. But m_2 and $\overline{m}_2$ divide v and so m_2 divides $\overline{m}_2 - 1$ and $\overline{m}_2$ divides $m_2 - 1$ which is impossible since $m_2 \geq 2$ and $\overline{m}_2 \geq 2$. So $k - \lambda_1 = k - \rho_1$, $k^2 - v\lambda_2 = k^2 - v\rho_2$ and $v - m_2 = v - \overline{m}_2$. Thus $\lambda_1 = \rho_1$, $\lambda_2 = \rho_2$ and $m_2 = \overline{m}_2$. But then $m_1 = \frac{v}{m_2} = \frac{v}{\overline{m}_2} = \overline{m}_1$.

This proposition has the following well–known corollary.

COROLLARY. If $\mathscr{G}$ and $\mathscr{G}^d$ are each transversal designs, then $\mathscr{G}$ is self–dual.

Proof. If $\mathscr{G}$ and $\mathscr{G}^d$ are each transversal designs, then (from (7)) $b = \lambda_2 m_1^2$ and $r = \lambda_2 m_1$. But $b = \overline{m}_1\overline{m}_2$ and $r = \overline{m}_2$. So we have $m_1 = \overline{m}_1$. The corollary then follows from Part (a) of the proposition.

If $\mathscr{G}$ is a symmetric transversal design, then the parameters of $\mathscr{G}$ and $\mathscr{G}^d$ must be of the form $\lambda_2 m_1^2, \lambda_2 m_1^2, \lambda_2 m_1, \lambda_2 m_1; m_1, \lambda_2 m_1; 0, \lambda_2$. We refer to m_1 and λ_2 as "the fundamental parameters" of $\mathscr{G}$.

A transversal design $\mathscr{G}$ is said to be *class regular* if there is an automorphism group of $\mathscr{G}$ which acts regularly on each of the groups of $\mathscr{G}$. Let $\mathscr{G}$ be a class regular transversal design with automorphism group G acting regularly on the groups of $\mathscr{G}$. Then G acts semiregularly on the block set of $\mathscr{G}$ and the block orbits of G form a parallelism of $\mathscr{G}$ ([2], p.360). If $\mathscr{G}$ is symmetric, then the block orbits of $\mathscr{G}$ constitute a group division of $\mathscr{G}^d$ ([2],pp.134–5). It follows that a symmetric class regular transversal design is a self–dual transversal design.

A *generalized Hadamard matrix* over a group G of order m_1 is a $\lambda_2 m_1 \times \lambda_2 m_1$ matrix (h_{ij}) such that

(a) $h_{ij} \in G$ for all $i,j = 1,\cdots,\lambda_2 m_1$, and

(b) $$\sum_{\ell=1}^{\lambda_2 m_1} h_{i\ell} h_{j\ell}^{-1} = \sum_{g\in G} \lambda_2 g$$

whenever $i \neq j$ and where the summations are in the group ring Z[G] of G.

We refer to such a matrix as a $GH(m_1, \lambda_2)$. The existence of a $GH(m_1, \lambda_2)$ is equivalent to the existence of a class regular symmetric (indeed self–dual) transversal design with fundamental parameters m_1 and λ_2 ([2], pp.361–2). A Hadamard matrix of order 4n is a GH(2, 2n) over the multiplicative group {1, –1}. Some direct constructions are known for (non–Hadamard) generalized Hadamard matrices. Of these we mention

1. GH(q,1), q a prime power, G elementary abelian (Drake [8]).
2. GH(q,2), q a prime power, G elementary abelian (Jungnickel [10]).
3. GH(q, q–1), q and q–1 a prime power, G elementary abelian (Seberry [17]).

Let H be a $GH(m_1, \lambda_2)$ over a group G and $\overline{H}$ be a $GH(m_1, \overline{\lambda}_2)$ over G. The direct product of H and $\overline{H}$ is a $GH(m_1, \lambda_2 \overline{\lambda}_2 m_1)$ over G([8],p.619). We note in particular that a direct product of n GH(q, 1)s over an elementary abelian group G of order q is a $GH(q, q^{n-1})$ over G. Thus self–dual transversal designs with parameters $q^{n+1}, q^{n+1}, q^n, q^n; q, q^n; 0, q^{n-1}$ exist for all prime powers q and integers $n \geq 1$.

Also, if there is a GH(m_1, λ_2) over a group G and G has a normal subgroup S of order s, then there is a GH$\left(\frac{m_1}{s}, \lambda_2 s\right)$ over the factor group G/S ([2], p.363). These constructions provide us with a supply of self–dual transversal designs which are available to be used in applying the construction methods which appear in Sections 4 and 5.

REMARK. For a survey of constructions for generalized Hadamard matrices see [6].

4. A CONSTRUCTION METHOD

In this section we give a construction method for semiregular group divisible designs which can be used to construct many classes of semiregular group divisible designs whose duals are semiregular group divisible designs.

Let $\mathscr{T}$ be a tactical configuration whose dual $\mathscr{T}^d$ is a semiregular GDD or a BIBD with parameters

$$m_1\bar{r},\ m_1\bar{k},\ \bar{k},\ \bar{r};\ \bar{\bar{m}}_1,\bar{\bar{m}}_2;\ \bar{\rho}_1,\bar{\rho}_2$$

and with a parallelism with m_1 blocks in each parallel class. (If $\mathscr{T}^d$ is a BIBD we take $\bar{\bar{m}}_2 = 1$ and $\bar{\rho}_2$ has no value.) Let $\mathscr{C}_i$, $i = 1,\cdots,\bar{k}$, be tactical configurations with mutually disjoint point sets $\mathscr{P}_i$. Suppose further that each $\mathscr{C}_i^d$ is a semiregular GDD or a BIBD with parameters

$$v,b,r,k;\ m_1,m_2;\ \lambda_1,\lambda_2\ .$$

(If $\mathscr{C}_i^d$ is a BIBD, then we take $m_2 = 1$ and λ_2 has no value.)

Let

(i) $\mathscr{B}_\gamma = \{\bar{B}_{\gamma\beta}: \beta = 1,\cdots,\bar{\bar{m}}_1\}$, $\gamma = 1,\cdots,\bar{\bar{m}}_2$ be the groups of $\mathscr{T}^d$,

(ii) $\mathscr{P}_i = \{\bar{P}_{i\alpha} : \alpha = 1,\cdots,m_1\}$, $i = 1,\cdots,\bar{k}$, be the parallel classes of $\mathscr{T}^d$, and

(iii) $\mathscr{B}_{ji} = \{B_{ji\alpha} : \alpha = 1,\cdots,m_1\}, j = 1,\cdots,m_2$, be the groups of $\mathscr{C}_i^d$.

Define sets $B^*_{\delta\gamma\beta}$ by $B^*_{\delta\gamma\beta} = \cup(B_{\delta i\alpha})$, where the union taken over all $i\alpha$ such that $\bar{P}_{i\alpha}$ is on $\bar{B}_{\gamma\beta}$ in $\mathscr{T}$. Let $\mathscr{P}^* = \bigcup_{i=1}^{\bar{k}} \mathscr{P}_i$, $\mathscr{B}^* = \{B^*_{\delta\gamma\beta} : \delta = 1,\cdots,m_2, \gamma = 1,\cdots,\bar{\bar{m}}_2,$ $\beta = 1,\cdots,\bar{\bar{m}}_1\}$ and $\mathscr{G}^{*d} = (\mathscr{P}^*, \mathscr{B}^*, \mathscr{I}^*)$, where $\mathscr{I}^*$ is defined by set–theoretical inclusion. We then have that $\mathscr{G}^*$ is a semiregular GDD with groups $\mathscr{B}^*_{\delta\gamma} = \{B^*_{\delta\gamma\beta}: \beta = 1,\cdots,\bar{\bar{m}}_1\}, \delta = 1,\cdots,m_2, \gamma = 1,\cdots,\bar{\bar{m}}_2$. The parameters of $\mathscr{G}^*$ are $V = \bar{\bar{m}}_1\, \bar{\bar{m}}_2\, m_2$, $B = \bar{k}b$, $R = \bar{k}r$, $K = k\bar{r}$, $M_1 = \bar{\bar{m}}_1$, $M_2 = \bar{\bar{m}}_2\, m_2$, $\Lambda_1 = \bar{\rho}_1 r + (\bar{k} - \bar{\rho}_1)\lambda_1$ and $\Lambda_2 = \bar{\rho}_2 r + (\bar{k} - \bar{\rho}_2)\lambda_1$, provided $\bar{\bar{m}}_2 \neq 1$. When $\bar{\bar{m}}_2 = 1$ and $m_2 \neq 1$ we get a GDD with all parameters as given except for Λ_2 which is given in this case by $\Lambda_2 = \bar{k}\lambda_2$. When $\bar{\bar{m}}_2 = m_2 = 1$ $\mathscr{G}^*$ is a BIBD. In this case $M_2 = \bar{\bar{m}}_2\, m_2 = 1$, Λ_1, as given, is the index of $\mathscr{G}^*$ and Λ_2 has no value.

We establish the values of Λ_1 and Λ_2 and leave the rest to the reader. Consider two blocks $B^*_1 = B^*_{\delta_1\gamma_1\beta_1}$ and $B^*_2 = B^*_{\delta_2\gamma_2\beta_2}$ of $\mathscr{G}^{*d}$. We split the analysis into three cases.

<u>Case 1</u> $\delta_1 = \delta_2, \gamma_1 = \gamma_2, \beta_1 \neq \beta_2$.

B^*_1 and B^*_2 "contain" a unique block from each of the $\mathscr{C}_i$ and, in fact, B^*_1 and B^*_2 contain $\bar{\rho}_1$ such blocks in common. Each of the remaining $\bar{k} - \bar{\rho}_1$ pairs of blocks from the same $\mathscr{C}_i$ in B^*_1 and B^*_2 meet in λ_1 points.

<u>Case 2</u> $\delta_1 = \delta_2, \gamma_1 \neq \gamma_2$.

First, this case does not arise if $\bar{\bar{m}}_2 = 1$. If $\bar{\bar{m}}_2 \neq 1$, then a similar argument to that of Case 1 show that B^*_1 and B^*_2 meet in $\bar{\rho}_2 r + (\bar{k} - \bar{\rho}_2)\lambda_1$ points.

<u>Case 3</u> $\delta_1 \neq \delta_2$.

This case does not arise if $m_2 = 1$. Assuming $m_2 \neq 1$ we have that B_1^* and B_2^* contain no common blocks of the $\mathscr{C}_i$. Each of the pairs of blocks from the same $\mathscr{C}_i$ (one in B_1^* and one in B_2^*) are from different groups of the $\mathscr{C}_i^d$. So B_1^* and B_2^* have $\bar{k}\lambda_2$ common points.

The results for Λ_1 and Λ_2 are immediate unless $\bar{\bar{m}}_2 \neq 1 \neq m_2$. So suppose $\bar{\bar{m}}_2 \neq 1 \neq m_2$. Since $\mathscr{T}^d$ is a semiregular GDD we have $\bar{\rho}_2 = \dfrac{\bar{r}\ \bar{k}}{m_1\bar{r}} = \dfrac{\bar{k}}{m_1}$. So

$$\begin{aligned}\bar{\rho}_2 r + (\bar{k} - \bar{\rho}_2)\lambda_1 &= \bar{\rho}_2(r + (m_1 - 1)\lambda_1) \\ &= \bar{\rho}_2\, m_1\, \lambda_2\end{aligned}$$

using (5) (which applies since $\mathscr{C}_i^d$ is a semiregular GDD, rather than a BIBD). But then $\bar{\rho}_2 r + (\bar{k} - \bar{\rho}_2)\lambda_1 = \bar{k}\,\lambda_2$ since $\bar{k} = \bar{\rho}_2 m_1$. Thus blocks of $\mathscr{G}^{*d}$ in different classes $\mathscr{B}^*_{\gamma\delta}$ meet in $\bar{\rho}_2 r + (\bar{k} - \bar{\rho}_2)\lambda_1$ $(= \bar{k}\,\lambda_2)$ points of $\mathscr{P}^*$.

REMARKS. (a) $\mathscr{G}^*$ is a transversal design if and only if $\mathscr{T}^d$ and the $\mathscr{C}_i^d$ are transversal designs. In this situation we must have $m_1 = \bar{\bar{m}}_1$.

(b) $\mathscr{G}^*$ is symmetric if and only if $\bar{k}r = k\bar{r}$.

(c) If $\mathscr{T}$ is a self–dual transversal design and the $\mathscr{C}_i$ are (v,b,r,k,λ)–designs, then $\mathscr{G}^*$ has the parameters of the GDD we can construct using Construction 3.1 of Rajkundlia [16].

Next, we suppose that $\mathscr{T}$ is a transversal design with parameters $m_1\bar{k}$, $m_1\bar{r}$, $\bar{r}$, $\bar{k}$; m_1, $\bar{k}$; 0, $\bar{\lambda}_2$ and also that each $\mathscr{C}_i$ is a semiregular GDD or a BIBD with parameters b,v,k,r; $\overline{m}_1$, $\overline{m}_2$; ρ_1, ρ_2. (If the $\mathscr{C}_i$ are BIBDs, then we take $\overline{m}_2 = 1$ and ρ_2 has no value.) In this situation $\mathcal{G}^{*d}$ is a semiregular GDD with parameters B, V, K, R; $\overline{M}_1 = \overline{m}_1$, $\overline{M}_2 = \overline{k}\,\overline{m}_2$; $\underline{P}_1 = \bar{r}\,\rho_1$, $\underline{P}_2 = \dfrac{k^2\overline{\lambda}_2}{m_2}$. The groups of $\mathcal{G}^{*d}$ are the groups of the $\mathcal{C}_i$.

To establish the values for $\underline{P}_1$ and $\underline{P}_2$ we consider two points Q and T of $\mathcal{G}^{*d}$ and split the analysis into three cases.

<u>Case 1</u> Q and T belong to the same group of some $\mathcal{C}_i$.

In $\mathcal{C}_i$ Q and T are on ρ_1 common blocks. These blocks are each "substituted" into $\bar{r}$ blocks of $\mathcal{T}$. So Q and T are on $\bar{r}\,\rho_1$ common blocks of $\mathcal{G}^{*d}$.

<u>Case 2</u> Q and T belong to different groups of some $\mathcal{C}_i$.

This case does not arise if $\overline{m}_2 = 1$. If $\overline{m}_2 \neq 1$, then an argument similar to that of Case 1 shows that Q and T are on $\bar{r}\,\rho_2$ common blocks of $\mathcal{G}^{*d}$.

<u>Case 3</u> Q and T are points of $\mathcal{C}_i$ and $\mathcal{C}_h$, respectively, where $i \neq h$.

First, for each $j = 1,\cdots,m_2$, Q is on $\dfrac{k}{m_2}$ blocks of $\mathcal{B}_{ji}$ and T is on $\dfrac{k}{m_2}$ blocks of $\mathcal{B}_{jh}$. Each pair of blocks, one from $\mathcal{B}_{ji}$ on Q and one from $\mathcal{B}_{jh}$ on T, are substituted together into $\overline{\lambda}_2$ blocks of $\mathcal{T}$. So Q and T are on $m_2 \times \left[\dfrac{k}{m_2}\right]^2 \times \overline{\lambda}_2$ common blocks of $\mathcal{G}^{*d}$.

The results for $\underline{P}_1$ and $\underline{P}_2$ are immediate if $\overline{m}_2 = 1$. So suppose $\overline{m}_2 \neq 1$. Now, since $\mathcal{T}$ is a transversal design we have $\bar{r} = m_1\,\overline{\lambda}_2$ and, since the $\mathcal{C}_i$ are semiregular GDDs (rather than BIBDs) we also have that $\rho_2 = \dfrac{rk}{b} = \dfrac{k^2}{v} = \dfrac{k^2}{m_1 m_2}$. We thus obtain

$$\bar{r}\,\rho_2 = \frac{k^2\overline{\lambda}_2}{m_2} \text{ when } \overline{m}_2 \neq 1.$$

REMARK. $\mathcal{G}$ is a self–dual transversal design if and only if $\mathcal{T}$ and the $\mathcal{C}_i$ are self–dual transversal designs. In this situation $\mathcal{G}^*$ has the parameters of the transversal design arising from the direct product construction ([16], p.66) applied to $\mathcal{T}$ and a $\mathcal{C}_i$.

If $\mathcal{T}^d$ is an affine 1–resolvable BIBD and the $\mathcal{C}_i^d$ are symmetric BIBDs (resp. affine $\frac{r}{m_2}$ – resolvable BIBDs), then $\mathcal{G}^*$ is an affine r–resolvable BIBD (resp. an affine $\frac{r}{m_2}$ – resolvable BIBD).

REMARK. The construction for affine r–resolvable BIBDs with $\mathcal{C}_i^d$ each a symmetric BIBD is a dualized version of the construction for affine r–resolvable BIBDs due to Shrikhande and Raghavarao [18].

We call an affine 1–resolvable BIBD with t blocks in each parallel class, and each pair of non–parallel blocks meeting in μ points, an ARD(μ,t). An ARD (μ,t) is a $\left[\mu t^2, \frac{t(\mu t^2-1)}{t-1}, \frac{\mu t^2-1}{t-1}, \mu t, \frac{\mu t-1}{t-1}\right]$–design ([7] p.73). The known ARD(μ,t)s have t a prime power. If $\mathcal{T}^d$ is an ARD(μ,t) and the $\mathcal{C}_i$ are $\left[t, r, \frac{r(r-1)}{t-1}\right]$–designs, then $\mathcal{G}^*$ is an affine r–resolvable $\left[\mu t^2, \frac{t(\mu t^2-1)}{t-1}, \frac{r(\mu t^2-1)}{t-1}, \mu rt, \frac{r(\mu rt-1)}{t-1}\right]$–design with t blocks in each affine r–resolution class, each pair of blocks in each affine r–resolution class meeting in $\frac{\mu rt(r-1)}{t-1}$ points, and each pair of blocks in different affine r–resolution classes meeting in μr^2 points. If we then use an ARD($\bar{\mu}$,t) as $\mathcal{T}^d$ and replicas of $\mathcal{G}^*$ as the $\mathcal{C}_i$ we obtain $\mathcal{G}^{**}$ with parameters

$$B = \mu t^2 \left\{ \frac{\bar{\mu}t^2-1}{t-1} \right\}, \qquad V = \bar{\mu}t^2 \left\{ \frac{\mu t^2-1}{t-1} \right\},$$

$$R = \mu rt \left\{ \frac{\bar{\mu}t^2-1}{t-1} \right\}, \qquad K = \bar{\mu}rt \left\{ \frac{\mu t^2-1}{t-1} \right\},$$

$$\Lambda_1 = \frac{\mu rt(\bar{\mu}rt-1)}{t-1}, \qquad \underline{P}_1 = \frac{\bar{\mu}rt(\mu rt-1)}{t-1}, \tag{13}$$

$$\Lambda_2 = \mu r^2 \left\{ \frac{\bar{\mu}t^2-1}{t-1} \right\}, \qquad \underline{P}_2 = \bar{\mu} r^2 \left\{ \frac{\mu t^2-1}{t-1} \right\},$$

$$M_1 = \bar{\mu}\, t^2, \qquad M_2 = \frac{\mu t^2-1}{t-1},$$

$$\overline{M}_1 = \mu t^2, \qquad \overline{M}_2 = \frac{\bar{\mu} t^2-1}{t-1}.$$

REMARK. GDDs with parameters (13) are symmetric if and only if $\mu = \bar{\mu}$.

Note that semiregular GDDs with semiregular group divisible duals having parameters (13) with $r = 1$ are constructible by using an ARD($\bar{\mu}$,t) as $\mathcal{S}^d$ and replicas of an ARD(μ,t) as the $\mathcal{C}_i$ in our construction. Some semiregular GDDs with semiregular group divisible duals having parameters of the form (13) with $r = 1$ have appeared in the literature prior to this. Values of μ, $\bar{\mu}$ and t for such GDDs and a reference are given in the following list.

1. $\mu = q^{h-2} = \bar{\mu}$, $t = q$, where q is a prime power and $h \geq 2$, Jungnickel [11].
2. $\mu = q^{h-2}$, $\bar{\mu} = q^{\bar{h}-2}$, $t = q$, where q is a prime power, $h \geq 2$ and $\bar{h} \geq 2$, Rahilly [14].
3. $\mu = n_1$, $\bar{\mu} = n_2$, $t = 2$, where n_1 and n_2 are orders of Hadamard designs, Rahilly [14].

Since a variety of symmetric BIBDs with a prime power number of points are known we can obtain many classes of semregular GDDs whose duals are semiregular with parameters (13) and $r > 1$. To the best of the author's knowledge these GDDs are new. Note that from each $(t, r, \frac{r(r-1)}{t-1})$ – design with t a prime power we can obtain infinitely many such designs by varying μ and $\bar{\mu}$.

Consider an application of the construction method for GDDs of this section out of which it arises that $\mathcal{G}^*$ and $\mathcal{G}^{*d}$ are each a semiregular GDD. It is easy to show that

$\mathscr{G}^*$ is a transversal design such that $\mathscr{G}^{*d}$ is not a transversal design if and only if $\mathscr{T}$ is a self–dual transversal design, $\mathscr{C}_i$ is a BIBD or a semiregular GDD which is not a transversal design and $\mathscr{C}_i^d$ is a transversal design. Similarly, it is easy to show that $\mathscr{G}^{*d}$ is a transversal design such that $\mathscr{G}^*$ is not a transversal design if and only if

(a) $\mathscr{T}^d$ is an affine 1–resolvable BIBD, $\mathscr{C}_i$ is a transversal design and $\mathscr{C}_i^d$ is a semiregular GDD, or

(b) $\mathscr{T}$ is a transversal design, $\mathscr{T}^d$ is a semiregular GDD and $\mathscr{C}_i^d$ is an affine 1–resolvable BIBD, or

(c) $\mathscr{T}$ and $\mathscr{C}_i$ are transversal designs and $\mathscr{T}^d$ and $\mathscr{C}_i^d$ are semiregular GDDs with at least one of $\mathscr{T}^d$ and $\mathscr{C}_i^d$ not a transversal design.

As an example, let $\mathscr{T}$ be a self–dual transversal design with parameters q^{n+2}, q^{n+2}, q^{n+1}, q^{n+1}; q, q^{n+1}; 0, q^n, where $n \geq 0$ and q is a prime power. Also, let $\mathscr{C}_i$ be an ARD(q^{h-2}, q), where $h \geq 2$. Then $\mathscr{G}^*$ has parameters

$$
\begin{aligned}
B &= q^{h+n+1}, & V &= q^{n+2}(q^{h-1} + \cdots + 1), \\
R &= q^{h+n}, & K &= q^{n+1}(q^{h-1} + \cdots + 1), \\
\Lambda_1 &= 0, & \underline{P}_1 &= q^{n+1}(q^{h-2} + \cdots + 1), \qquad (14) \\
\Lambda_2 &= q^{h+n-1}, & \underline{P}_2 &= q^n(q^{h-1} + \cdots + 1), \\
M_1 &= q, & M_2 &= q^{n+1}(q^{h-1} + \cdots + 1), \\
\overline{M}_1 &= q^h, & \overline{M}_2 &= q^{n+1}
\end{aligned}
$$

REMARKS. (a) Let $q = p^\sigma$, $\sigma \geq 2$. $\mathscr{G}^*$ with parameters (14) could be used in our construction method in the role of $\mathscr{T}$ with (for example) the $\mathscr{C}_i$ being semiregular

GDDs with semiregular group divisible duals, with parameters given by (13) and with $q = \bar{\mu}t^2$.

(b) $\mathscr{G}^*$ with parameters given by (14) could also be used in our construction method in the role of $\mathscr{C}_i$ or $\mathscr{C}_i^d$ with (for example) $\mathscr{T}$ an appropriately chosen self–dual transversal design. Constructions of this sort yield further transversal designs whose duals are semiregular GDDs.

(c) With one exception GDDs with parameters (14) or arising as in (a) and (b) appear to be new. With $h = q = 2$ and $n = 0$ in (14) we obtain a GDD with the parameters of SR66 in Clatworthy [5].

5. SELF–DUAL REGULAR GROUP DIVISIBLE DESIGNS

Consider two self–dual semiregular GDDs $\mathscr{G}_i = (\mathscr{P}_i, \mathscr{B}_i, \mathscr{I}_i)$, $i = 1,2$, with parameters $v,v,k,k; m_1,m_2; \lambda_1,\lambda_2$. We suppose that $\mathscr{P}_1$ and $\mathscr{P}_2$ are disjoint. Let the groups of $\mathscr{G}_i$ be $\mathscr{P}_{ij} = \{P_{ij\ell} : \ell = 1,\cdots,m_1\}$, $j = 1,\cdots m_2$, and the groups of $\mathscr{G}_i^d$ be $\mathscr{B}_{ij} = \{B_{ij\ell} : \ell = 1,\cdots,m_1\}$, $j = 1,\cdots,m_2$. Define sets $B'_{ij\ell}$, where $i = 1,2$, $j = 1,\cdots,m_2$ and $\ell = 1,\cdots,m_1$, by

$$B'_{1j\ell} = (B_{1j\ell}) \cup \mathscr{P}_{2j}$$

and

$$B'_{2j\ell} = (B_{2j\ell}) \cup \mathscr{P}_{1j}\,.$$

Let $\mathscr{B}' = \{B'_{ij\ell} : i = 1,2,\ j = 1,\cdots,m_2,\ \ell = 1,\cdots,m_1\}$ and define an incidence structure $\mathscr{G}' = (\mathscr{P}_1 \cup \mathscr{P}_2, \mathscr{B}', \mathscr{I}')$, $\mathscr{I}'$ being given by set–theoretical inclusion.

If the parameters of $\mathscr{G}_1$ and $\mathscr{G}_2$ satisfy $\lambda_2 \neq \frac{2k}{m_2}$ and

$$\lambda_1 + m_1 = \lambda_2\,, \tag{15}$$

then $\mathscr{G}'$ is a self–dual GDD with parameters $2v, 2v, k+m_1, k+m_1; v, 2; \lambda_2, \frac{2k}{m_2}$. The groups of $\mathscr{G}'$ are $\mathscr{P}_1$ and $\mathscr{P}_2$ and the groups of $\mathscr{G}'^d$ are $\mathscr{B}'_i = \{B'_{ij\ell} : j = 1,\cdots,m_2,$

$\ell = 1,\cdots,m_1\}, i = 1,2$.

We show that $\mathscr{B}'_1$ and $\mathscr{B}'_2$ form a group division of $\mathscr{G}'^d$ and leave the rest to the reader. We consider two blocks $B'_{ij\ell}$ and $B'_{\alpha\beta\gamma}$ of $\mathscr{G}'$ and split the analysis into three cases.

<u>Case 1</u> $i = \alpha$, $j = \beta$, $\ell \neq \gamma$.

In this case $(B'_{ij\ell}) \cap (B'_{\alpha\beta\gamma}) = ((B_{ij\ell}) \cap (B_{ij\gamma})) \cup \mathscr{P}_{\nu j}$, where $\nu = 2$ or 1 as $i = 1$ or 2. So $B'_{ij\ell}$ and $B'_{\alpha\beta\gamma}$ have $\lambda_1 + m_1 = \lambda_2$ points in common.

<u>Case 2</u> $i = \alpha$, $j \neq \beta$.

Here $(B'_{ij\ell}) \cap (B'_{\alpha\beta\gamma}) = (B_{ij\ell}) \cap (B_{i\beta\gamma})$. So $B'_{ij\ell}$ and $B'_{\alpha\beta\gamma}$ have λ_2 common points.

<u>Case 3</u> $i \neq \alpha$.

Here $(B'_{ij\ell}) \cap (B'_{\alpha\beta\gamma}) = ((B_{ij\ell}) \cap \mathscr{P}_{i\beta}) \cup ((B_{\alpha\beta\gamma}) \cap \mathscr{P}_{\alpha j})$. But each block of $\mathscr{G}_i$ meets each group of $\mathscr{G}_i$ in $\frac{k}{m_2}$ points, and similarly for $\mathscr{G}_\alpha$. So $B'_{ij\ell}$ and $B'_{\alpha\beta\gamma}$ have $\frac{2k}{m_2}$ common points.

We now show that $\mathscr{G}'$ is a regular GDD.

$\mathscr{G}'$ cannot be singular since $k + m_1 > \lambda_1 + m_1 = \lambda_2$. If $\mathscr{G}'$ is semiregular, then $\frac{4vk}{m_2} = (k + m_1)^2$ and so $4m_1k = (k + m_1)^2$, whence $k = m_1$. But then we have $\lambda_2 - \lambda_1 = m_1$ and, from (5), $m_1\lambda_2 - (m_1 - 1)\lambda_1 = m_1$. Solving these equations for λ_1 yields $\lambda_1 = m_1 - m_1^2$. But $\lambda_1 \geq 0$ and so we must have $m_1 = 1$, a contradiction.

Suppose that $\mathscr{G}$ is a self–dual semiregular GDD with parameters v,v,k,k; m_1,m_2; λ_1,λ_2 satisfying (15). By solving $\lambda_2 - \lambda_1 = m_1$ and $m_1\lambda_2 - (m_1-1)\lambda_1 = k$ for λ_2 we obtain $\lambda_2 = k - m_1^2 + m_1$. But $v\lambda_2 = k^2$ and so we have $v = \frac{k^2}{k-m_1^2+m_1}$. It is straightforward to show that, in the special case where $k - m_1^2 + m_1$ divides k, the parameters of $\mathcal{G}$ have the form

$$q^2\delta\beta,\ q^2\delta\beta,\ q\delta\beta,\ q\delta\beta;\ \delta,\ q^2\beta;\ \delta(\beta-1),\ \delta\beta, \tag{16}$$

where $\delta = \beta(q-1) + 1$ and $\beta = \gamma q + 1$ for some $\gamma \geq 0$. The case $\gamma = 0$ yields parameters $q^3,q^3,q^2,q^2;\ q,q^2;\ 0,q$. From Section 3 self–dual GDDs with these parameters exist for all prime powers q. The self–dual regular GDDs we can obtain from these designs have parameters

$$2q^3,\ 2q^3,\ q^2+q,\ q^2+q;\ q^3,\ 2;\ q,\ 2, \tag{17}$$

where q is a prime power greater than two.

REMARK. If $q = 2$, then $\lambda_2 = \frac{2k}{m_2} = 2$ and we obtain a (16,6,2)–design.

We note that, if the parameters of a semiregular GDD satisfy (15), then its complement also has parameters which satisfy (15). From the complements of GDDs with parameters $q^3,q^3,q^2,q^2;\ q,q^2,0,q$, where $q > 2$, we obtain self–dual regular GDDs with parameters

$$2q^3,2q^3,q^3-q^2+q,\ q^3-q^2+q;\ q^3,\ 2;\ q(q-1)^2,\ 2(q-1). \tag{18}$$

For fixed $q > 2$ we have that the parameter lists (17) and (18) are different. They are also not parameter lists for complementary GDDs.

Next, we show that self–dual GDDs with parameters (16) exist when $\gamma = 1$ and q is a prime power.

First, affine planes of order q (that is, ARD(1,q)s) exist for each prime power q. Using an affine plane of order q as $\mathcal{T}^d$ and replicas of such a plane as the $\mathcal{C}_i$ in the construction of Section 4 we obtain a self–dual GDD with parameters

$$q^2(q+1),\ q^2(q+1),\ q(q+1),\ q(q+1);\ q^2,\ q+1;\ q,\ q+1 \tag{19}$$

(These parameters are those of (13) with $\mu = \bar{\mu} = r = 1$ and $t = q$.) Then apply the construction of Section 4 again in the following way: Let $\mathcal{T}$ be a self–dual transversal design with parameters $q^4,q^4,q^2,q^2; q^2,q^2; 0,1$, where q is a prime power, and, for each $i = 1,\cdots,q^2$, let $\mathcal{C}_i$ be a self–dual GDD with parameters (19). The construction method of Section 4 then yields a self–dual semiregular GDD $\mathcal{G}^*$ with parameters

$$q^4(q+1),q^4(q+1),q^3(q+1),q^3(q+1);q^2,q^2(q+1);q^3,q^2(q+1). \tag{20}$$

Clearly $\mathcal{G}^*$ has parameters which satisfy (15). Using the construction method of this section we obtain a self–dual regular GDD with parameters

$$2q^4(q+1),\ 2q^4(q+1),\ q^2(q^2+q+1),\ q^2(q^2+q+1);\ q^4(q+1),\ 2;\ q^2(q+1),\ 2q,$$

where q is any prime power.

From the complements of the designs with parameters (20) we can obtain self–dual regular GDDs with parameters

$$2q^4(q+1),\ 2q^4(q+1),\ q^2(q^3-q+1),\ q^2(q^3-q+1);\ q^4(q+1),2;q^2(q-1)(q^2-1),2q(q-1),$$

where q is any prime power.

Self–dual regular GDDs with two point and block groups can also be constructed in the following way: Let $\mathcal{S}_i = (\mathcal{P}_i, \mathcal{B}_i, \mathcal{I}_i)$, $i = 1,2$, be (v,k,λ)–designs such that $\mathcal{P}_1 \cap \mathcal{P}_2 = \phi = \mathcal{B}_1 \cap \mathcal{B}_2$. Complementing $(\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{B}_1 \cup \mathcal{B}_2, \mathcal{I}_1 \cup \mathcal{I}_2)$ yields a self–dual regular GDD $\mathcal{G}'$ with parameters 2v, 2v, 2v–k, 2v–k; v, 2; 2(v–k)+λ, 2(v–k). The groups of $\mathcal{G}'$ are $\mathcal{P}_1$ and $\mathcal{P}_2$ and the groups of $\mathcal{G}'^d$ are $\mathcal{B}_1$ and $\mathcal{B}_2$. If a self–dual regular GDD $\mathcal{G}'$ with two point and block groups is such that its complement is the union of a pair of disjoint symmetric BIBDs, then we say that the complement $\mathcal{G}'^c$ of $\mathcal{G}'$ is *trivial*. The self–dual regular GDDs with two point and block groups listed in [5] (namely R94, 133, 166, 173, 187, 195, 204 and 206) all have trivial complements.

Now consider a self–dual semiregular GDD $\mathscr{G}$ with parameters v,v,k,k; m_1,m_2; λ_1, $\lambda_1 + m_1$. Using the construction at the start of this section we can construct a self–dual regular GDD $\mathscr{G}'$ with two point and block classes. Consider two points P_1 and P_2 of $\mathscr{G}'$ one from each group of $\mathscr{G}'$. In $\mathscr{G}'$ P_1 and P_2 are on precisely $2v - 2k - 2m_1 + \frac{2k}{m_\lambda}$($= \lambda_2'$ say) blocks. Suppose that $\mathscr{G}'^c$ is trivial. Then we must have $\lambda_2' = 0$ and so $v = k + m_1 - \frac{k}{m_2}$. But, as earlier shown, $v = \frac{k^2}{k-m_1^2+m_1}$. After some algebra we see that

$$k^2 + m_1(m_1m_2 + 1 - m_1 - 2m_2)k + m_1^2\, m_2(m_1-1) = 0. \tag{21}$$

From (21) we must have $m_1 + 2m_2 > m_1m_2 + 1$ and so we have

$$1 + \frac{2m_2}{m_1} > m_2 + \frac{1}{m_1} \tag{22}$$

and

$$\frac{m_1}{m_2} + 2 > m_1 + \frac{1}{m_2}\,. \tag{23}$$

If $m_1 \geq 2m_2$, then (using (22)) $2 \geq 1 + \frac{2m_2}{m_1} > m_2 + \frac{1}{m_1} > 2$, which is impossible. If $m_1 < 2m_2$, then (using (23)) $4 > \frac{m_1}{m_2} + 2 > m_1 + \frac{1}{m_2} > 2$. Thus we have $m_1 = 2$ or 3. If $m_1 = 3$, then $3 + 2m_2 > 3m_2 + 1$, whence $m_2 < 2$, again impossible. If $m_1 = 2$, then (using (21)) $(k - 1)^2 + 4m_2 - 1 = 0$, which is also impossible. We infer that the complement of $\mathscr{G}'$ is not trivial.

We can obtain self–dual regular GDDs with more than two groups from the construction method of this section. For, if $\lambda_2 = \frac{2k}{m_2}$, then this method yields a self–dual GDD $\mathscr{G}'$ with parameters 2v, 2v, $k+m_1$, $k+m_1$; m_1, $2m_2$; λ_1+m_1, λ_2, provided $\lambda_1+m_1 \neq \lambda_2$.

REMARK. If $\lambda_1 + m_1 = \lambda_2 = \frac{2k}{m_2}$, then we obtain a symmetric BIBD. We will shortly show that a symmetric BIBD obtained in this way must be a (16,6,2)–design.

Suppose $\lambda_2 = \frac{2k}{m_2}$. From (2) and (4) we obtain $k = 2m_1$. Immediately we have $m_1 = \frac{\lambda_2 m_2}{4}$ and $k = \frac{\lambda_2 m_2}{2}$. From (6) we obtain $\lambda_1 = \frac{\lambda_2 m_2(\lambda_2-2)}{\lambda_2 m_2-4}$ from which we infer that $\lambda_2 m_2 - 4$ divides $4(\lambda_2 - 2)$. So $\lambda_2 = 2$ or $m_2 = 2$ or 3. If $m_2 = 2$, then $\lambda_1 = \lambda_2$, a contradiction and, if $m_2 = 3$, then we must have $3\lambda_2 - 4$ divides $4\lambda_2 - 8$ which yields $\lambda_2 = 4$. We thus have that the parameters of $\mathcal{G}_1$ and $\mathcal{G}_2$ are either

(i) of the form $2q^2$, $2q^2$, 2q, 2q; q, 2q; 0, 2 or

(ii) 9, 9, 6, 6; 3, 3; 3, 4 (= parameters complementary to 3^2, 3^2, 3, 3; 3, 3; 0, 1).

The only parameters satisfying $\lambda_1 + m_1 = \lambda_2 = \frac{2k}{m_2}$ are now easily verified to be 8, 8, 4, 4; 2, 4; 0, 2. It follows that, if $\mathcal{G}'$ is a symmetric BIBD, then $\mathcal{G}'$ must be a (16,6,2)–design.

From Section 3 self–dual GDDs with parameters as in (i) exist for q any prime power. The self–dual GDDs we can obtain from self–dual GDDs with parameters as in (i) and (ii) have parameters $4q^2$, $4q^2$, 3q, 3q; q, 4q; q, 2 and 18, 18, 9, 9; 3, 6; 6, 4. These GDDs are clearly regular.

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