An Upper Approximation to the BIBD (15, 21, 7, 5, 2)

R.G. Stanton and R.W. Buskens
Department of Computer Science
University of Manitoba
Winnipeg, Canada
R3T 2N2

Abstract The design (15,21,7,5,2) is the only BIBD of block size 5 that does not exist. If it did exist, it would provide an exact bicovering, in 21 blocks of size 5, of the pairs from 15 points. However, we show that 22 quintuples are sufficient to provide a bicover of the pairs from 15 points; thus there are only 10 repetitions required in the bicovering.

1. Introduction.

If one could form the Balanced Incomplete Block Design (15,21,7,5,2), then it would provide an exact bicovering of all pairs from 15 points. However, it is well known that this is the only BIBD with k = 5 that does not exist. Currently (cf. **[2]** for an extensive set of references), there is considerable interest in coverings by quintuples; in this paper, we determine the classical bicovering numer $N_2(2,5,15)$, that is, the minimal number of quintuples needed to cover all pairs from 15 elements at least twice.

An upper bound of 23 blocks can be obtained as follows. Cycle on the block (1 2 6 7 9) to give 15 blocks; cycle on the block (1 4 7 10 13) to give three further blocks. The differences 2 and 4 do not appear in these eighteen blocks; however, all pairs of elements that are distance 2 or 4 apart occur twice in the five blocks :

(1 3 5 7 9), (7 9 11 13 15), (13 15 2 4 6), (4 6 8 10 12), (10 12 14 1 3).

This construction establishes that

$$22 \leq N_2(2,5,15) \leq 23.$$

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2. Requirements for a Bicovering in 22 Blocks.

We first suppose that $N_2(2,5,15) = 22$. Since any element must occur at least seven times, the number of excess occurrences of elements is given by $22(5) - 15(7) = 5$. The number of excess pairs is $22(10) - 2(105) = 10$. As in [3], we represent the excess pairs by edges of the excess graph. Since an element normally appears 7 times in the bicovering, we refer to the elements that occur more than 7 times as exceptional elements. Each additional occurrence of an element contributes four additional excess pairs to the excess graph; thus the excess graph is made up of points whose degrees are 4, 8, 12, 16, or 20.

If there is only one exceptional point, then the excess graph contains a single point with 10 loops. But repeated pairs of the form aa do not occur in blocks.

If two elements A and B appear in the excess graph, then all ten excess pairs must be of the form AB; but 10 is not a possible degree for a point in the excess graph.

If there are three points in the excess graph, the exceptional elements may have frequencies 10, 8, 8; or they may have frequencies 9, 9, 8.The first possibility leads to a graph with points of degrees 12, 4, 4; such a graph can not be constructed without loops.

If element P has frequency 8 and elements Q and R each have frequency 9, then P has degree 4 and Q and R each have degree 8 in the excess graph. Then the excess graph must contain the edge PQ twice, PR twice, and QR six times. Now, the pair QR appears a total of 8 times (6 times in excess, twice normally) and both Q and R appear 9 times each (see Figure 1).

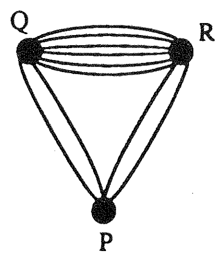

Figure 1

Since Q and R occur twice with the non-exceptional elements, we see that there must be two blocks of the form Q1234 and R1234. Now we introduce the weight function of a block B, as in [3], namely,

$$w(B) = x_0 + x_3 + 3x_4 + 6x_5,$$

where x_i is the number of blocks meeting B in i points. From [3], we also quote the following result.

Lemma 1. $w(B) = (b-1) - \Sigma (r_i - 1) + \Sigma (\lambda_{ij} - 1)$, where b is the number of blocks, r_i is the frequency of element i in the bicovering, and λ_{ij} is the frequency of the pair (i,j) in the bicovering.

From Lemma 1, w(Q1234) = 21 - 32 + 10 = -1; this is a contradiction, since the definition of w(B) ensures that w(B) can not be negative. We thus have

Lemma 2. If a bicovering of the pairs from 15 points can be achieved in 22 blocks, then the excess graph must contain either 4 or 5 points.

3. Excess Graphs on Four and Five Points

Figure 2 shows the excess graphs on 4 points; Figure 3 shows the excess graphs on 5 points. We shall discuss excess graph 5(a) in the next section.

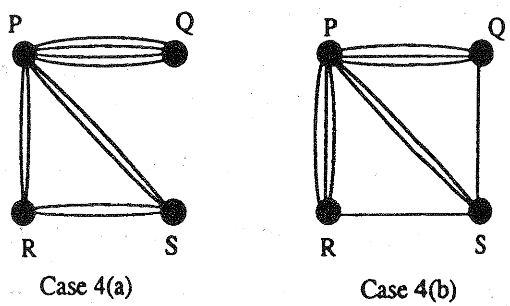

Figure 2: Excess graphs on 4 points

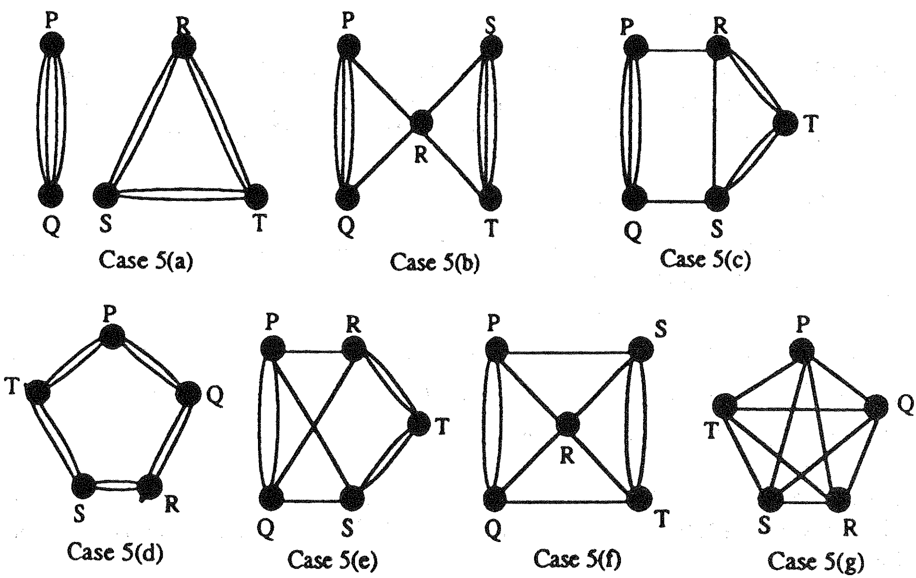

Figure 5: Excess graphs on 5 points

4. Excess Graph 5(a).

All elements have frequency 8, and thus degree 4, in all of the excess graphs on 5 points. In Case 5(a), the pair PQ occurs 4 excess times, where RS, RT, and ST each appear 2 excess times.Consequently, the bicovering must contain six blocks of the form PQxxx, two blocks of the form Pxxxx, and two blocks of the form Qxxxx.

To determine how elements R, S, and T appear in the bicovering, we define element types, for X is in {R,S,T}, as follows.

Type a. X occurs twice in the PQ blocks; six more times.

Type b. X occurs once with each of PQ, P, and Q; five more times.

Type c. X occurs twice with each of P and Q; 4 more times.

Most of the mathematical possibilities for (a,b,c) can be ruled out. We restrict our discussion to the case (a,b,c) = (1,2,0). For this case, let T be the type a element, and R and S be the type b elements.

Now, PR and QR must appear. Since w(PRnnn) = -1, where n represents any non-exceptional element, PRS must appear. Also, the multiset of elements in the P-blocks is the same as the multiset of elements in the Q-blocks.

The weight of block PRSnn is zero; so choice of PRS12 as a block forces Q12 to appear. Similarly, selecting QRS34 as a block forces P34 to appear. We can thus write the P and Q blocks as PRS12, QRS34, P3456, Q1256.

In order to appear twice each with P and Q, elements 1 through 6 must each appear once in the PQ blocks. Since w(Pnnnn) = 0 = w(Qnnnn), elements from {1,2,5,6} must appear in different blocks, as must elements from {3,4,5,6}. Without loss of generality, take partial blocks PQ1, PQ2, PQ5, and PQ6. Now elements 3 and 4 may appear 2, 1, or 0 times with PQ1 and PQ2. In this section, we shall only look at the third possibility, namely, elements 3 and 4 do not appear with either PQ1 or PQ2. Consequently, the six PQ blocks can be written in the form PQ1, PQ2, PQ3, PQ4, PQ5, PQ6.

The other 12 blocks must contain all remaining pairs among the elements of A = {1,2,3,4}. So these blocks have the form 13, 24, 13, 24, 14, 23, 14, 23, -, -, -, -, where we have written only occurrences of elements from A. We refer to the last 4 blocks, that do not contain elements from A, as free blocks.

We now consider placement of the elements 5 and 6. The pair 56 appears twice already, and both 5 and 6 appear twice with P and twice with Q. Elements 5 and 6 appear four more times and must appear with each element from A. This can be done in two blocks; hence 5 and 6 appear separately (twice each) in the free blocks. There are two choices for the placement of 5 and 6 at this point: either they appear with the same or different one-factors of A. We do not yet specify with which one-factors 5 and 6 appear.

We now consider placement of R and S. Let a_i denote the number of blocks in which R appears with i elements from N = {1,2,3,4,5,6}; then $1 \leq i \leq 3$. We count occurrences of R and of pairs Ry (y in N) to give

$$a_1 + a_2 + a_3 = 6,$$
$$a_1 + 2a_2 + 3a_3 = 8.$$

Only two solutions are possible for (a_1,a_2,a_3): (5,0,1) or (4,2,0). If element R has pattern (5,0,1), 5R and 6R must each appear twice, and PQRn must appear. Since R must appear in a block containing three elements from N, either 5R or 6R will appear three times. This is not permitted.

Consequently, both R and S must have the pattern (4,2,0). Since both R and S pick up elements of {1,2,3,4} from the one-factors in the blocks containing two elements from N, and they must appear with PQ once each, blocks PQ5R and PQ6S may be taken without loss of generality. This forces 5RS, 5S, 6RS, and 6R. Now R and S must be placed with the remaining one-factors from {1,2,3,4}. Again, we do not say with which one-factors they appear.

We use the same notation as for R and S to determine the placement of the element T. We have

$$a_1 + a_2 + a_3 = 8,$$
$$a_1 + 2a_2 + 3a_3 = 12.$$

The solutions for (a_1,a_2,a_3) are (6,0,2), (5,2,1), and (4,4,0). If T has the (6,0,2) pattern, too many 5T or 6T pairs will appear. If T has the pattern (5,2,1), it can not appear with all four R and S elements. Thus, T has the pattern (4,4,0), that is, T must appear with R and S in the a_2 blocks. So that T may occur with the right number of 5, 6, R, and S elements, we must have 5ST, 6RT, PQ5RT, and PQ6ST as blocks.

Finally, we must place elements from M = {7,8,9,0}. There are 12 pairs from M and exactly 12 blocks in which these pairs may be placed. Thus, the elements from M must be split across the a_2 blocks. We arbitrarily take RT7, RT8, ST9, ST0. At this point, we split the problem into two cases.

Case1. Elements 5 and 6 appear with the same one-factors.

Case 2. Elements 5 and 6 appear with different one-factors.

In Case 1, our partial construction has now taken the following form.

PQ1, PQ2, PQ5RT, PQ6ST, PQ3, PQ4;
PRS12, P3456, QRS34, Q1256;
135, 245, 136, 246, 14RT7, 23RT8, 14ST9, 23ST0;
5RS, 5ST, 6RS, 6RT.

Both elements 7 and 9 must appear with 13 and 24 in the a_3 blocks; this requires the blocks PQ279 and PQ379. A similar argument on elements 8 and 0 results in the blocks PQ180 and PQ480.

For elements 7 and 8 to occur twice with S, we must take the block 5ST78. A similar argument for 9 and 0 requires the block 6RT90.

Suppose now that 5RS90 appears; then we must take 6RS78. Without loss of generality, we may take 13570; this forces us to have 24589, 13689, and 24670. We now have a covering in 22 blocks, given explicitly as

PQ180	PRS12	13570	5RS90
PQ279	P3456	24589	5ST78
PQ5RT	QRS34	13689	6RS78
PQ6ST	Q1256	24670	6RT90
PQ379		14RT7	23RT8
PQ480		14ST9	23ST0

The automorphism group of this covering design was found using "Groups & Graphs", cf. **[1]** , as 4.

If 5RS70 appears, we must take 6RS89. Without loss of generality, we may take 13598; then we must have blocks 24678, 24590, and 13670. This construction produces a bicovering that has an automorphism group of order 2.

PQ180	PRS12	13589	5RS70
PQ279	P3456	24590	5ST78
PQ5RT	QRS34	13670	6RT90
PQ6ST	Q1256	24678	6RS89
PQ379		14RT7	23RT8
PQ480		14ST9	23ST0

We now consider Case 2 in which elements 5 and 6 appear with different one-factors. Under the permutation (5 6)(R S), elements 5 and 6 are interchangeable. Thus, without loss of generality, we take blocks as:

PQ1, PQ2, PQ5RT, PQ6ST, PQ3, PQ4;
PRS12, P3456, QRS34, Q1256;
135, 245, 146, 236, 13RT7, 24RT8, 14ST9, 23ST0.

To ensure that each element from M appears twice with each element from {R,S,T}, we must take 5ST78 and 6RT90. If 5RS7 is a block, then 6RS8 must be a block; then we must take 1467 and 2367. So that element 7 may appear twice with P and with Q, we need PQ27 and PQ47. Without loss of generality, we take 5RS79 and 6RS80. Using the same argument on element 0 forces us to take the blocks 1350, 2450, PQ10, and PQ40. Now, the pair 70 has appeared only once instead of twice.

Thus, we must take 6RS78 as a block. This forces us to have 5RS90. If 1357 appears, then 2367 must appear, in order to avoid three occurrences of the pair 17; but then 37 appears three times. So, 2457 and 1358 are blocks. We first suppose that 2367 is a block; then 1468 must be a block. The only choice for the blocks containing 7 and 8 is: PQ17, PQ47, PQ28, and PQ38. We can not have blocks 13580 and 24579 since this leads to three occurrences of the pair 79. Thus, 13589 and 24570 are blocks. To avoid more than two 19 pairs, we must take blocks 23679 and 14680. The only possible completion is to take blocks PQ170, PQ289, PQ380, and PQ479. Explicitly, the covering now is:

PQ170	PRS12	13589	5RS90
PQ289	P3456	24570	5ST78
PQ5RT	QRS34	13RT7	6RS78
PQ6ST	Q1256	24RT8	6RT90
PQ380		14680	23679
PQ479		14ST9	23ST0

This covering has an automorphism group of order 4, but is not isomorphic to the first covering obtained.

If, in the preceding discussion, we take 1467 and 2368 as blocks, we must place elements 7 and 8 in blocks PQ27, PQ37, PQ18, and PQ48. To avoid extra pairs with element 9, we must take 14670 and 23689. Blocks 13580 and 24579 must appear if 89 is to occur exactly twice. Then we must take blocks PQ189, PQ270, PQ379, PQ480. This completes the bicovering as:

PQ189	PRS12	13580	5RS90
PQ270	P3456	24579	5ST78
PQ379	QRS34	13RT7	6RS78
PQ480	Q1256	24RT8	6RT90
PQ5RT			14670
PQ6ST			23689
			14ST9
			23ST0

Obviously, this covering is isomorphic to the previous one under the permutation: (1 3)(2 4)(9 0)(P Q).

We have thus shown that Excess Graph 5(a) leads to at least three distinct bicoverings in 22 blocks. We sum up the constructions by the

Theorem. $N_2(2,5,15) = 22$.

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