Planar Graphs, Regular Graphs, Bipartite Graphs and Hamiltonicity

by

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Abstract

This paper seeks to review some ideas and results relating to Hamiltonian graphs. We list the well known results which are to be found in most undergraduate graph theory courses and then consider some old theorems which are fundamental to planar graphs. By restricting our attention to 3-connected cubic planar graphs (a class of graphs of interest to Four Colour Theorists), we are able to report on recent results regarding the smallest non-Hamiltonian graphs. We then consider regular graphs generally and what might be said about when the number of Hamiltonian cycles is greater than one. Another interesting class of graphs are the bipartite graphs. In general these are not Hamiltonian but there is a famous conjecture due to Barnette that suggests that 3-connected cubic bipartite planar graphs are Hamiltonian. In our final two sections we consider this along with another open conjecture due to Barnette.

1. Introduction

Although Hamiltonian graphs were named after Hamilton, this is not because of precedence. Kirkman wrote a paper in 1856 [23] in which he considered "a circuit which passes through each vertex once and only once." This paper was actually presented to the Royal Society in 1855 (see Biggs, Lloyd and Wilson [5]), while Hamilton's first communication on the subject was his letter of 1856 on the *icosian calculus*.

Whoever originated the idea, many people have now been worrying for well over a century about graphs which have a cycle which goes through every vertex once and only once. Such graphs we'll call **Hamiltonian graphs** and such cycles are **Hamiltonian cycles**.

We consider two examples, see Figure 1. These are the dodecahedron and the Petersen graph, P. In Figure 1(a) we show a Hamiltonian cycle by labelling the vertices in order from 1 to 20. Vertex 20 is joined to vertex 1 to complete the cycle.

We've used the dodecahedron here because it was on this graph that Hamilton played cycle games. In fact he even marketed that game, though, it seems, without a great deal of financial success.

On the other hand, it is well known that the Petersen graph is not Hamiltonian. A proof of this can be found by realizing that an even number of the edges 16, 2t, 39, 48 and 57 must

be used in a Hamiltonian cycle. This together with the property exemplified in Figure 2 quickly leads to a contradiction.

The dodecahedron
(a)

The Petersen Graph
(b)
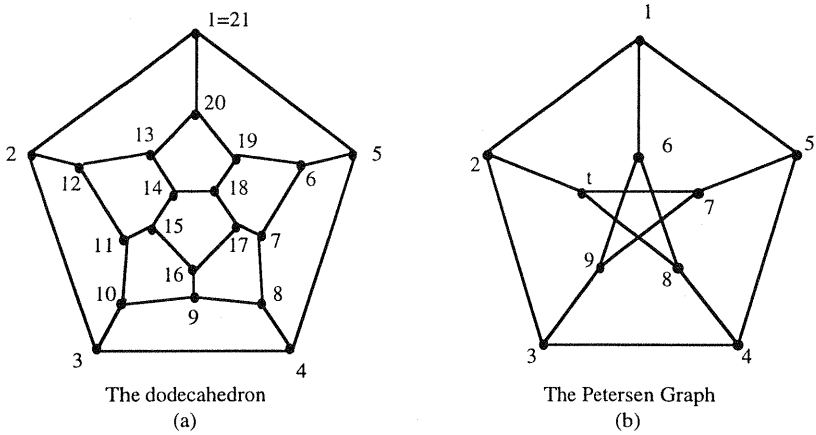
Figure 1

Both of the graphs in Figure 1 are cubic graphs. The advantage of cubic graphs (every vertex of degree 3) when looking for a Hamiltonian cycle is that if two edges incident with a vertex are in a Hamiltonian cycle, then the other incident edge isn't. But what's more, the edge that isn't, forces two more edges to be **in** the Hamiltonian cycle. We show this in Figure 2, where if u_1u, uu_2 are in the Hamiltonian cycle, then uv isn't. Hence v_1v, vv_2 must be on that cycle.
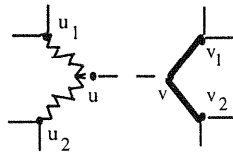
u_1u, $u u_2$ in a Hamiltonian cycle force $v_1 v$, $v v_2$ to be in that cycle.

Figure 2

Now given that Euler had invented Euler tours in 1736, it's perhaps a surprise that we had to wait until 1855 for Hamiltonian cycles. What's more, graphs with Euler tours have a simple characterisation - every vertex has even degree. So the other surprise is that Hamiltonian graphs do **not** have a simple characterisation. Somehow one would expect it to be possible to take a dual of a graph (by interchanging vertices and edges, say) and then apply the Eulerian tours characterisation to the dual hoping it would lead to a Hamiltonian characterisation in the original graph. This does not seem to be a workable idea.

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The importance of Hamiltonian graphs seems to rely partly on the difficulty of characterising them. But it also relies on the fact that Hamiltonian graphs are related to the travelling salesperson problem. So there are potential practical applications.

We know that the Hamiltonian problem is hard because it is an example of an NP-complete problem. If we could find a good algorithm for finding a Hamiltonian cycle in a graph, we'd be able to find good algorithms for other things which are also difficult to solve.

In general we know very little about Hamiltonian graphs. One of the nicest simple results is the following by Bondy and Chvátal [6].

Theorem 1. *Let G be a simple graph with n vertices and let u, v be non-adjacent vertices in G such that*

$$deg(u) + deg(v) \geq n.$$

Let $G + uv$ denote the graph obtained from G by joining u, v by an edge. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

The first part of this result generalizes a theorem of Ore [26], where, when the degree condition applies to **all** pairs of distinct non-adjacent vertices in a graph, the graph is Hamiltonian. Theorem 1 also implies a number of well known results. In particular it implies Dirac's Theorem [12] which follows.

Theorem 2. *If G is a simple graph with n vertices, $n \geq 3$, and $deg(v) \geq \frac{1}{2}n$ for all $v \in VG$, then G is Hamiltonian.*

One of the concepts to arise out of Bondy and Chvátal's work is that of the **closure** of a graph. Suppose $u, v \in VG$, $uv \notin EG$, and $\deg u + \deg v \geq n$. Then join u to v. Continue adding edges in this way until no such pair u, v exists. The graph obtained is the **closure** of G, $c(G)$. (It should be noted that the closure of a given graph is unique.) The following result follows from Theorem 1.

Theorem 3. *G is Hamiltonian if and only if $c(G)$ is Hamiltonian.*

Corollary. *If $c(G) = K_n$, the complete graph on n vertices, then G is Hamiltonian.*

Clearly $c(G)$ is often going to be Hamiltonian but not complete. It would seem to be of interest to try to determine which graphs have incomplete Hamiltonian closures and how such graphs are related. This, like most Hamiltonian questions appears to be difficult. Some work in a related area where the closure concept is different, but related to that of Bondy and Chvátal, can be found in Broersma, [9], and Broersma and Schiermeyer, [10].

A closure concept for claw-free graphs (graphs which contain no induced copy of $K_{1,3}$) has been introduced by Ryjácek [29]. Here edges are added to connected induced graphs on the neighbourhood of vertices of the graph G to give a graph $c\ell(G)$. Ryjácek proves the following result.

Theorem 4. *Let G be a claw-free graph. Then*

(1)   *c ℓ (G) is unique;*

(2)   *there exists a triangle-free graph H such that c ℓ (G) is the line graph of H;*

(3)   *the size of the largest cycle in G is the same as that of the largest cycle in c ℓ (G).*

As a result of this theorem G is Hamiltonian if and only if $c\ell$ (G) is Hamiltonian. Theorem 4 also has the following corollary.

Corollary.   *Every 7-connected claw-free graph is Hamiltonian.*

2. Planarity In General

Why would you want to consider the Hamiltonicity of planar graphs? First of all, there is no doubt that planar graphs are an interesting class of graphs in their own right. That is in itself enough to justify an investigation into which of them are Hamiltonian.

But the second good reason is that P. G. Tait [31] conjectured that all cubic planar graphs were Hamiltonian. If the conjecture had been true, it would have provided an elegant proof of the Four Colour Theorem. The argument runs this way. Every cubic graph has an even number of vertices. Hence a Hamiltonian cycle in a cubic graph has an even number of edges. Colour those edges alternately in red and white. Every other edge can be coloured blue to give a 3-edge colouring of the graph. Then there is a theorem that says that the Four Colour Theorem and the 3-edge colourability of cubic planar graphs are equivalent. Hence the 3-edge colouring in red, white and blue proves the Four Colour Theorem.

Unfortunately Tait's Conjecture is false, as we shall see in the next section. Before we get to that though, we present two interesting results about Hamiltonian planar graphs. The first of these is Grinberg's Theorem (see [16]). This shows that there is a useful connection between the "inside" and "outside" faces of a Hamiltonian planar graph.

Theorem 5.   *Let G be a graph drawn in the plane with no crossings. Suppose G has a Hamiltonian cycle H. Let α_i, β_i, be the number of faces bounded by i-cycles, respectively, inside and outside of H. Then*

$$\sum_i (i - 2)\left(\alpha_i - \beta_i\right) = 0,$$

where the summation is over all possible values of i.

This theorem has been used to show that certain planar graphs are non-Hamiltonian. The approach here is to set up the equation of the theorem using the values of i known from the graph. Then number theoretic considerations show that the equation cannot hold.

For instance, no planar graph with faces where i = 5, 8 and 9 are Hamiltonian, if there is only one 9-face. Assume that such a graph is Hamiltonian.

Then
$$3(\alpha_5 - \beta_5) + 6(\alpha_8 - \beta_8) + 7(\alpha_9 - \beta_9) = 0.$$

Hence
$$3[(\alpha_5 - \beta_5) + 2(\alpha_8 - \beta_8)] = 7(\beta_9 - \alpha_9).$$

But if there is only one 9-face, then $\beta_9 - \alpha_9 = \pm 1$.

Hence
$$3[(\alpha_5 - \beta_5) + 2(\alpha_8 - \beta_8)] = \pm 7.$$

Since the left hand side is non-zero and divisible by 3, so is the right hand side. This is clearly a contradiction. Hence no such planar graph (and such graphs do exist) is Hamiltonian.

The other interesting result of this section, that every 4-connected planar graph is Hamiltonian, is by Tutte [37]. Thomassen [33] has given a nice proof of this theorem. At the same time he extended Tutte's result to settle a conjecture by Plummer [28] which suggested that every edge might be in a Hamiltonian cycle.

Theorem 6. *Let G be a 4-connected planar graph. Then G is Hamiltonian. Further, there is a Hamiltonian cycle through every edge of G.*
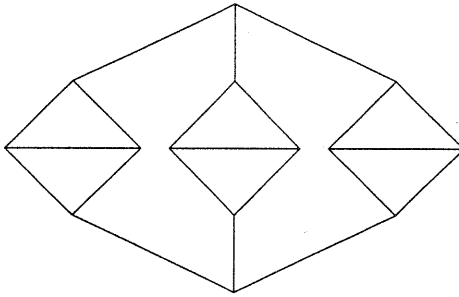
(In fact, there is an error in [33]. The proof is completed successfully in Chiba and Nishezeki [11].)

3 . 3-Connected Cubic Planar Graphs

Tait's Conjecture that every cubic planar graph is Hamiltonian was clearly false. We show two counterexamples in Figure 3.
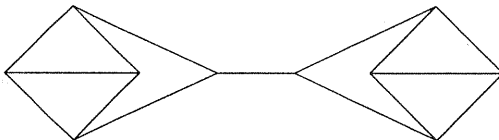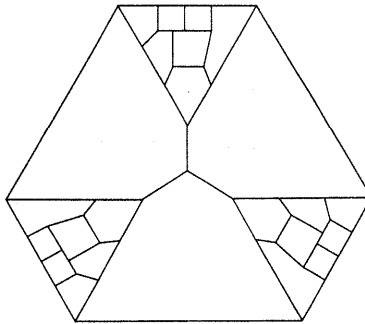

a 1-connected cubic planar non-Hamiltonian graph

a 2-connected cubic planar non-Hamiltonian graph

Figure 3

On the basis of the graphs in Figure 3, a Hamiltonian cubic planar graph would need to be 3-connected. Despite that hiccup (and Tait's original conjecture is so strangely worded that he may have required 3-connectivity), it is perhaps a little surprising that a counterexample took so long to appear. It is hard to say exactly why this was. Possibly the emphasis was on the "map" side of the Four Colour Theorem rather than on its "cubic graph" side. Whatever the reason, it was not until 1946 that Tutte [36] produced his famous non-Hamiltonian 3-connected cubic planar graph (see Figure 4).

Tutte's non-Hamiltonian 3-connected cubic planar graph

Figure 4

It must be said, of course, that Tutte's graph has a relatively large number of vertices (46) and so would not have been easy to find. Having established that there was a non-Hamiltonian cubic 3-connected planar graph, though, it then remained to find the smallest such graph.

Lederberg, Barnette and Bosák, [8], separately contributed a total of six non-Hamiltonian 3-connected cubic planar graphs. They are shown in Figure 5.
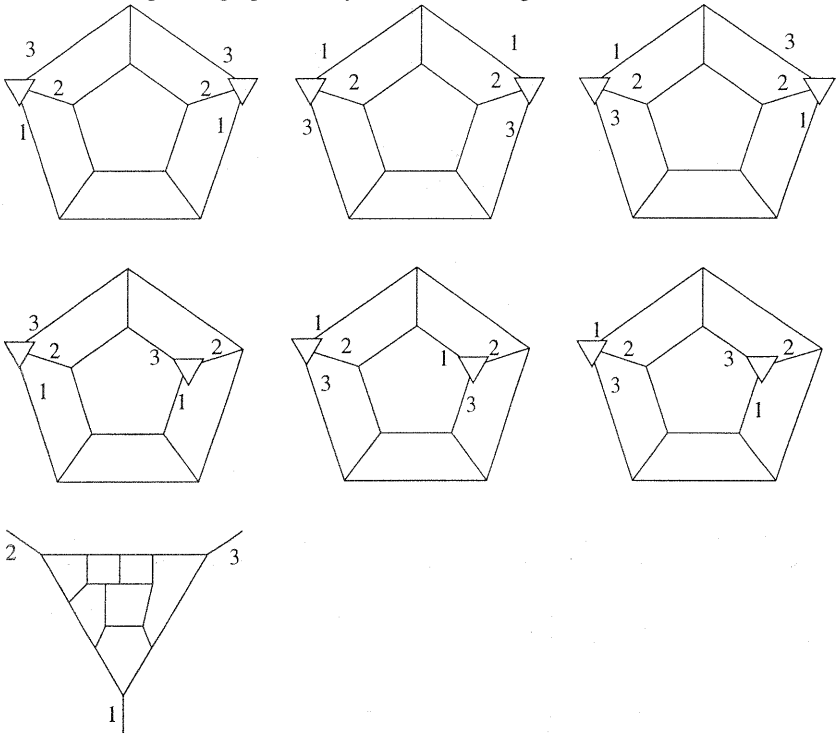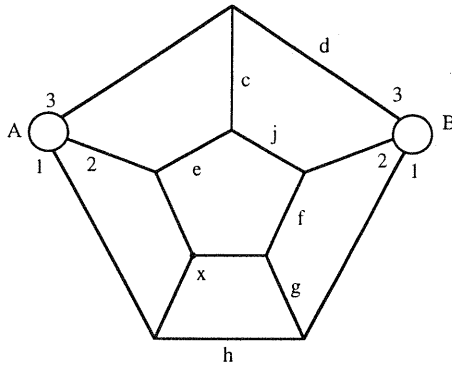

Figure 5

The configuration at the bottom of Figure 5 is to be inserted in the pentagonal prisms at the positions marked with a triangle, so that the edges 1, 2 and 3 of the configuration correspond with the edges 1, 2 and 3 of the prisms. Now the configuration is such that there is no path from 1 to 3 which passes through all the vertices of the configuration. (There are, however, paths from 1 to 2 and from 2 to 3 which do pass through all the vertices of the configuration.) This is the key to showing that none of the graphs is Hamiltonian.

Consider the example below (see Figure 6). If the configuration is placed at A and B in the first graph of Figure 5, then at A we have to use the sequence of edges 1, 2 or 2, 3, because the sequence 1, 3 will not pick up all of the vertices of A. If the edges 1, 2 are used at A, then 3 is not used. Hence edges c and d have to be used. But d is a 3 edge at B. To use all the vertices of B we have to exit on the 2 edge. This then forces edges e and f to be in the assumed Hamiltonian cycle (otherwise we have the cycle c,d,2,j,c inside the Hamiltonian cycle). Since the 1 edge at B is not in this Hamiltonian cycle, the edges g and h are. But then the cycle has closed up and the vertex x has not been included.

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Figure 6

A similar argument can be used if we assume that the sequence of edges 2, 3 is used at A instead of 1, 2. Further, analogous arguments apply to the other graphs in Figure 5.

The interesting fact about the configuration at the bottom of Figure 5 is that it is precisely the configuration that Tutte exploited, and in the same way, in his 1946 counterexample to Tait's conjecture.

In [20], Holton and McKay were able to prove that the six graphs of Figure 5 were the only non-Hamiltonian cubic 3-connected planar graphs on 38 vertices with cyclic connectivity 3. What's more the six graphs had the fewest vertices of all non-Hamiltonian cubic 3-connected graphs. There was the possibility though, that there were other non-Hamiltonian graphs on 38 vertices.

Theorem 7. *Let G be a 3-connected cubic planar graph on 36 or fewer vertices. Then G is Hamiltonian.*

If $|VG| = 38$, then G is Hamiltonian unless G is one of the graphs of Figure 5, or G is cyclically 4-connected. (A cyclically 4-connected graph requires the removal of at least four edges before it breaks into two components each of which contains a cycle.)

The proof technique followed a method developed by Okamura [25]. A variety of decomposition techniques, as well as some computation, show that the graph G of the Theorem has no 3-cuts or essential 4-cuts. Then any remaining possibilities for a non-Hamiltonian G with $|VG| \leq 36$, could be reduced to a smaller non-Hamiltonian 3-connected cubic planar graph by applying one of Okamura's 15 reductions. (A 4-cut is essential if each of the two components left on removal of the 4-cut contains more than 4 vertices.)

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A number of years were spent unsuccessfully trying to improve this result by hand. In the process a very long argument was developed which could probably have been tightened. The length of the argument and the lack of any ancillary benefits finally led to that approach being abandoned. It was eventually decided to bring the strength of the computer to bear again. As a result, the open case (of cyclically 4-connected graphs on 38 vertices) of Theorem 6 was settled and the theorem extended. The authors involved in the effort were Aldred, Sheng Bau, Holton and McKay [1].

Theorem 8.   *Let G be a cyclically 4-connected cubic planar graph.*

If |VG| ≤ 40, then G is Hamiltonian.

If |VG| = 42, then G is one of the non-Hamiltonian graphs of Figure 7.
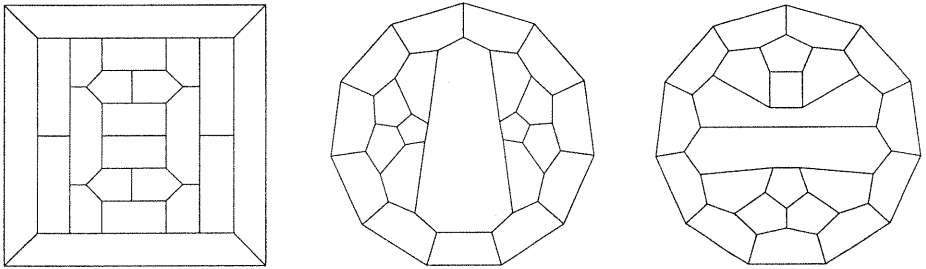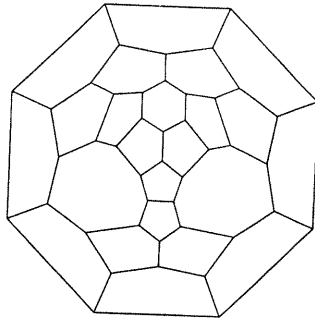

The three non-Hamiltonian cyclically 4-connected cubic planar graphs on 42 vertices
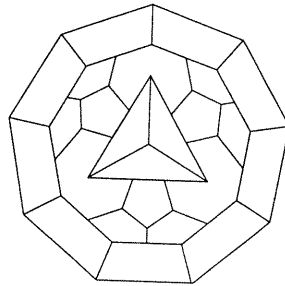
Figure 7

In [15] Faulkner and Younger proved that the smallest cyclically 5-connected cubic planar graph which is non-Hamiltonian has 44 vertices. The work in [1] established that the graph of Figure 8, announced by Faulkner and Younger in [14], is unique. Aldred et al continued their search up to 52 vertices.
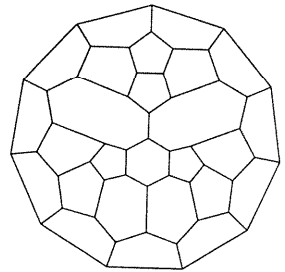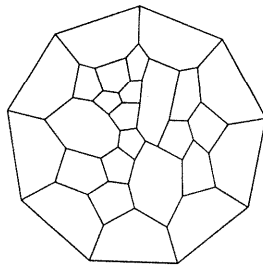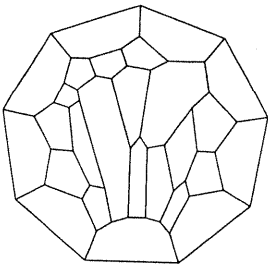
Theorem 9.   *Let G be a cyclically 4-connected cubic planar graph. If |VG| ≤ 52, then G is one of the graphs in Figure 8.*
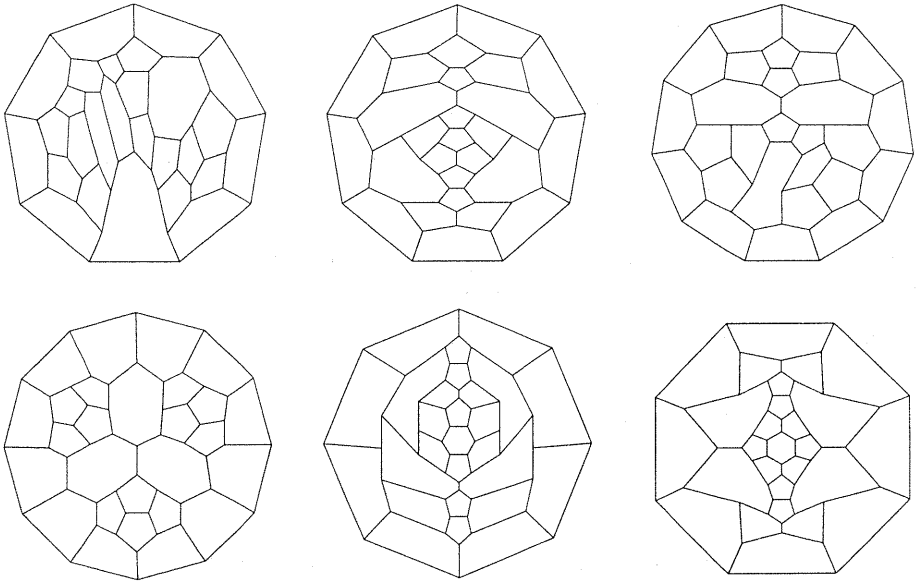
The Faulkner-Younger graph - the 44-vertex vertex graph

The 46-vertex non-Hamiltonian graph

The 50-vertex non-Hamiltonian graphs

The 52-vertex non-Hamiltonian graphs

Smallest non-Hamiltonian cyclically 5-connected cubic planar graphs

Figure 8

As a consequence of the results of Aldred et al plus further computation, it has been possible to prove the following theorem (see [2]). This extends results in [19] and [20].

Theorem 10. *Let G be a 3-connected cubic planar graph. Let $A \subseteq VG$. Then for $|A| \leq 23$, there exists a cycle C in G such that $VA \subseteq VC$.*

So any 23 vertices lie on a cycle in a 3-connected cubic planar graph. This result is best possible as there exist 24 vertices in the graphs of Figure 5, which do not lie in a cycle.

The method of proof here is inductive and relies on reductions. These, however, are fairly straightforward (see [1] and [21]). One of the interesting things about this result (and others in this area), is that the most difficult part of the induction is the initial step. Once the property has been assumed for k vertices, it is relatively easy to show that it holds for k + 2 vertices. (Recall that cubic graphs must contain an even number of vertices.) The difficulty is showing that the result is true for the base cases. This is where the computer was vital in generating and checking the thousands of graphs involved.

(It should be noted here that this base case work involved more than simply generating all the graphs and checking them for the desired property. A great deal of mathematical effort

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was first required to reduce the number of graphs which had to be tested and so bring the computation down to a manageable size.)

4. How Many Hamiltonian Cycles?

In this section we shall assume that all our graphs are Hamiltonian (though not necessarily planar, 3-connected or cubic).

The first question that comes to mind in this context is whether there exist graphs other than cycles, which have a **unique** Hamiltonian cycle. It's very readily seen that adding just a few edges to a cycle can produce graphs with only one Hamiltonian cycle.

In [30], Sheehan looked at the maximum edge set that a graph on n vertices can have and still be uniquely Hamiltonian.

Theorem 11. *Let G be a graph on n vertices with a single Hamiltonian cycle. Then*
$$|EG| \leq \left[\frac{n^2}{4}\right] + 1 .$$

Sheehan also claimed there that there is a unique extremal graph (we show such a graph on 12 vertices in Figure 9). However, it is possible to find other extremal graphs by rearranging the adjacencies slightly. For instance, let v_3 be adjacent to v_1, v_2, v_4, v_5, v_6, v_8, v_9, v_{10}, v_{11} and let v_5 be adjacent to v_3, v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}.
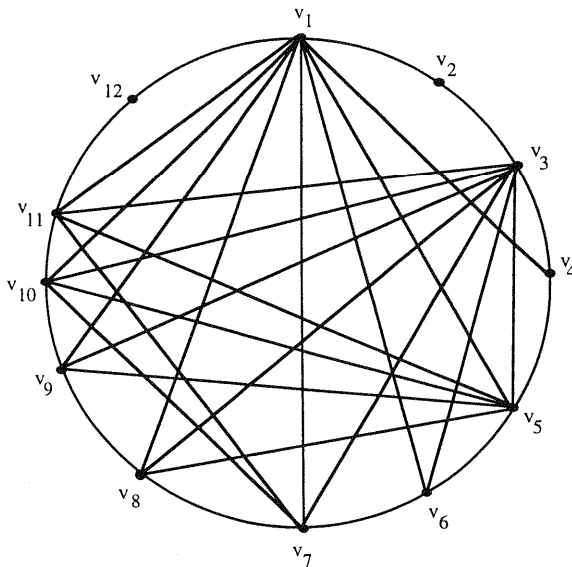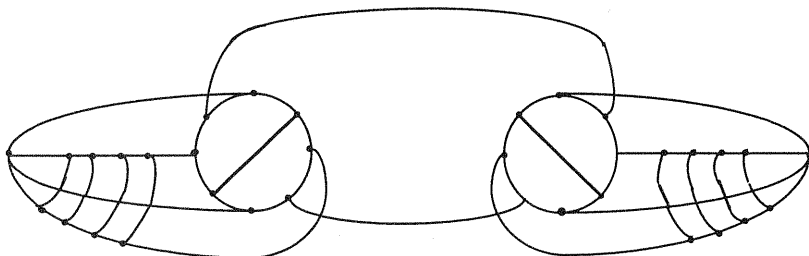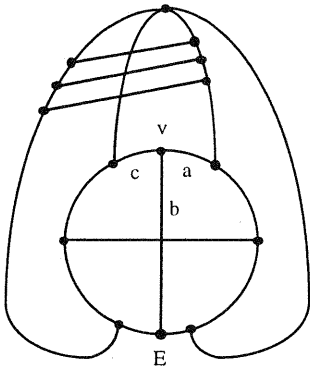

Figure 9

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All of these graphs have vertices of degree 2 though, so it seemed reasonable to conjecture that all Hamiltonian graphs with a single cycle have vertices of degree 2. That such a conjecture is false can be seen by an example of Entringer and Swart [13]. They produced the graph of Figure 10 which has minimum degree 3. In a sense it is almost regular as all but two of its vertices have degree 3.

A graph with a unique Hamiltonian cycle, all of whose vertices except
two (of degree 4) are of degree 3

Figure 10

It is worth noting that the graphs of Sheehan, and of Entringer and Swart are 2-connected but not 3-connected. This raises the question as to how connected a graph with a unique Hamiltonian cycle can be. Aldred and Thomassen [3] were able to produce a 3-connected graph which has only one Hamiltonian cycle (see Figure 11). Their construction relies on the fact that the graph on which that of Figure 10 is based (see E of Figure 11(a)) has a unique Hamiltonian cycle through the edges a,c and a unique Hamiltonian cycle through the edges b,c but no Hamiltonian cycle through ab. They therefore take two copies of E and label the edges a,b,c on one graph and label the corresponding edges a',b',c', on the other graph. Let v and v' be the vertices incident with a,b,c and a',b',c', respectively. Remove v and v' and join the "hanging" edges a,b,c to c',a',b', respectively. To cover all the vertices in the first copy of E we have to use the edges a,c or b,c. Since the latter edges are joined to a',b' in the second copy of E, the vertices of the second copy cannot be completely covered. However, the route through a,c in one copy of E to b',c' in the second copy provides the unique Hamiltonian cycle.

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(b)

(a)

Figure 11

This, of course, raises the question as to whether or not there is a 4-connected uniquely Hamiltonian graph.

So if G contains a Hamiltonian cycle, under what circumstances does it contain another? How many Hamiltonian cycles can it contain?

A classical result of Cedric Smith (announced in [36]) shows that every Hamiltonian cubic graph contains at least three Hamiltonian cycles.

Theorem 12. *Every edge in a cubic graph lies on an even number of Hamiltonian cycles.*

From doodling on a diagram similar to Figure 12, it is easy to prove the Corollary below. The "worst" case of the two cycles through e still allows another cycle through f.
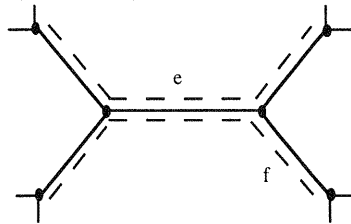
Figure 12

Corollary. *Every Hamiltonian cubic graph contains at least three Hamiltonian cycles.*

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In 1975, Sheehan [30] conjectured that every Hamiltonian 4-regular graph has at least one more Hamiltonian cycle. This conjecture was followed three years later by the following result of Thomason [32] which generalised that of Smith [37].

Theorem 13. *Let G be a graph, all of whose vertices have odd degree. Then every edge is contained in an even number of Hamiltonian cycles.*

Thomason's proof is rather elegant. We illustrate the method first on cubic graphs. Suppose that G is a Hamiltonian cubic graph. Let $H = (x_1, x_2, ..., x_n)$ be a Hamiltonian cycle in G. We now form a new graph $\mathcal{G}$ whose vertices are Hamiltonian paths starting with the edge $x_1 x_2$. Before describing adjacencies in $\mathcal{G}$, consider the Hamiltonian path $H_1 = x_1, x_2, ..., x_n$ shown in Figure 13(a).

(a) H_1:

$x_1 \quad x_2 \quad x_3 \qquad x_r \qquad x_{n-1} \quad x_n$

(b) H_2:

$x_1 \quad x_2 \quad x_3 \qquad x_s \qquad x_{r+2} \quad x_{r+1}$

(c) H_3:
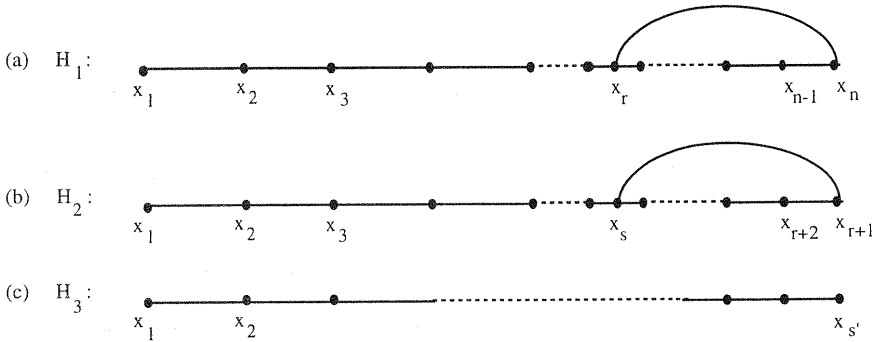
$x_1 \quad x_2 \qquad\qquad x_{s'}$

Figure 13

Now in G we know that x_n is adjacent to x_1. Since G is cubic, x_n is adjacent to another vertex, x_r, say. Consider H_2, the Hamiltonian path $x_1, x_2, ..., x_r, x_n, x_{n-1}, ..., x_{r+1}$. We show this in Figure 13(b).

Now in G we know that x_{r+1} is adjacent to x_r and x_{r+2}. Since G is cubic, x_{r+1} is also adjacent to another vertex x_s. Consider H_3, the Hamiltonian path $x_1, x_2, ..., x_s, x_{r+1}, x_{r+2}, ..., x_{s'}$, where we do not define s' precisely. We show H_3 in Figure 13(c).

The process described above will continue until we have no new edges of the form $x_n x_r$, $x_{r+1} x_s$ to add to the Hamiltonian paths. To define adjacency in $\mathcal{G}$, we say that $H_i H_j$ is an edge of $\mathcal{G}$ if we can go from one path H_i to the other H_j by removing one edge (such as $x_r x_{r+1}$, $x_s x_t$) and adding another (such as $x_n x_r$, $x_{r+1} x_s$).

The important thing to notice now is that $\deg H_1 = 1$ and $\deg H_i = 2$, unless H_i is at the end of the chain started in Figure 13, when $\deg H_i = 1$. Since $\mathcal{G}$ is a graph it must have an even number of vertices of degree 1. Suppose two of these are H_1 and H_m. What stops H_m from having a neighbour in $\mathcal{G}$ other than H_{m-1}? It must be that there is an edge from the end vertex (not x_1) to x_1. Hence H_m plus this edge is a Hamiltonian cycle. So, every component of $\mathcal{G}$ is either a cycle, corresponding to no Hamiltonian cycles in G,

or a path, corresponding to two Hamiltonian cycles through x_1, x_2 in G. Smith's result therefore follows for cubic graphs.

If G is now regular of degree r where r is odd, then deg H_1 = r - 2. This is because x_n is adjacent to x_1, x_{n-1} and r - 2 other vertices. It then follows that the vertices of $\mathcal{G}$ all have degree r - 2 or r - 1. But there have to be an even number of vertices of degree r - 2. All of these correspond to Hamiltonian cycles through $x_1 x_2$ and the general result follows.

Note that the odd parity of r is essential for Thomassen's argument to succeed. There is some feeling though that the result should be true for even r too. Thomassen [33] has been able to prove that a second Hamiltonian graph exists if the degree of regularity is sufficiently large. This is, of course, not as strong as Thomason's result which tells us that two Hamiltonian cycles exist through every **edge**.

Thomassen's [34] result follows.

Theorem 14. *Let G be a regular Hamiltonian graph of degree r. If r $\geq$ 300, then G contains a second Hamiltonian cycle.*

It is almost certain that 300 can be reduced considerably (at least to 100) using arguments similar to those used in Thomassen's proof.

If Sheehan's conjecture (that every Hamiltonian 4-regular graph has at least one more Hamiltonian cycle) is true, then every Hamiltonian regular graph (except the cycle) has at least two Hamiltonian cycles. This follows by noting that we only have to prove the result for graphs of even degree because Theorem 12 gives us the odd degree case. So suppose that G is regular of even degree at least six. Then the graph obtained from G by removing the edges of a Hamiltonian cycle H is regular of even degree. A longstanding result of Petersen [27], says that even regular graphs have a spanning graph, S, which is the union of cycles. Now S together with H form a 4-regular (connected) graph. By Sheehan's conjecture this graph has at least two Hamiltonian cycles. Hence so does G.

The above discussion suggests that it is is not possible to find regular graphs which have only one Hamiltonian cycle. Certainly if they exist they will have even degrees less than 300.

This then raises the question as to how many Hamiltonian cycles a Hamiltonian regular graph can have. Are there cubic graphs which have precisely three Hamiltonian cycles, for instance? This question was asked by Cantoni (see Tutte [40]).

The graph in Figure 14 has precisely three Hamiltonian cycles. Of course, this graph is only small. However, adding triangles at each vertex doesn't increase the number of Hamiltonian cycles. Hence there exist cubic graphs with an arbitrary number of vertices which have only three Hamiltonian cycles. But do such graphs exist which are triangle free?
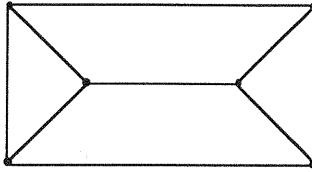
A cubic graph with precisely three Hamiltonian cycles

Figure 14

Another approach to uniquely Hamiltonian graphs might be to consider the size of the vertex of minimum degree in the graph. A recent result of Bondy and Jackson [7] shows that if the minimum degree of a Hamiltonian graph is sufficiently large, then it cannot be uniquely Hamiltonian.

Theorem 15. *Let* G *be a Hamiltonian graph of order* n *with minimum degree at least* $c \log_2 n,$ *where* $c > (2-\log_2 3)^{-1}.$ *Then* G *has at least two Hamiltonian cycles.*

(Note that $(2-\log_2 3)^{-1} \cong 2.41$.)

5. Bipartite Graphs

Bipartite graphs are of considerable interest in their own right. Hence it is worth considering them in the context of Hamiltonicity. Clearly, the complete bipartite graphs $K_{m,n}$ with $m \neq n$ are not Hamiltonian. This also means that considering bipartite graphs with high connectivity is not going to produce anything in the Hamiltonian area. In order to achieve Hamiltonicity, the parts of the bipartite graph have to be equal, so this suggests that we first consider regular graphs.

Tutte conjectured in [39], that every 3-connected cubic bipartite graph is Hamiltonian. The first counterexample was found by Horton. It contained 96 vertices. The smallest currently known counterexamples appear to be on 50 vertices. Kel'mans appears to have been the first to produce such graphs, at least one of which is a member of an infinite series.

It would seem that all 3-connected cubic bipartite graphs are, however, Hamiltonian. This result is again due to Kel'mans.

A more detailed history of this problem can be found in Gropp [17].

A conjecture which has stood the test of time and which appears to be very difficult to prove was proposed by Barnette (problem 5 in [4]). Barnette suggested that every 3-connected cubic bipartite planar graph was Hamiltonian. Very little progress has been made on this problem. One of the few results towards the conjecture can be found in Holton, Manvel and McKay [22].

Theorem 16. *Let G be a 3-connected cubic bipartite planar graph. If |VG| ≤ 64, then G is Hamiltonian.*

The proof of this result followed the lines of Theorem 8. Once again certain reductions were employed and then computer calculations completed the argument.

Taking up the theme of the last section regarding the number of Hamiltonian cycles in a Hamiltonian graph, Thomassen [33] has provided a condition for a bipartite Hamiltonian graph to have a second Hamiltonian cycle.

Theorem 17. *Let G be a Hamiltonian graph with bipartition A, B and let H be a Hamiltonian cycle in G. If every vertex of B has degree at least 3, then G has a Hamiltonian cycle other than H.*

This raises the question of whether a bipartite graph can be uniquely Hamiltonian. If there is such a graph, the last theorem implies that it has at least two vertices of degree 2.

6. The Future

In this final section we discuss work in progress and some open questions.

Another conjecture of Barnette [24], also appears to be quite important. It is also proving very difficult to settle one way or the other. Let G be a 3-connected cubic planar graph with maximum face size at most 6. The conjecture suggests that G is Hamiltonian. A particularly interesting subset of the class of graphs to which G belongs is the fullerenes. These graphs are of particular interest to chemists as they provide the potential molecular structure of carbon isomorphs. A **fullerene** is a 3-connected cubic graph with 12 faces of size 5 and all the remaining faces of size 6. These graphs were named after Buckminster Fuller, the architect whose geodesic domes look like 3-dimensional representations of these graphs.

It is of particular interest to chemists to be able to construct fullerenes and to prove that they are Hamiltonian. Aldred et al [2] have shown that fullerenes on up to 176 vertices are Hamiltonian. Their result is actually stronger than this.

Theorem 18. *Let G be a 3-connected cubic planar graph with maximum face size at most 6. If |VG| ≤ 176, then G is Hamiltonian.*

Hence if Barnette's conjecture is false a counterexample may be hard to find by a blind search.

The graphs of Figures 5, 7 and 8, show that as soon as there are two or more faces larger than 6, it is possible to have no Hamiltonian cycle. What if we allow one large face (larger than 6)?

For a start we need 3-connectivity. Figure 17 shows a 2-connected cubic graph with one large face which is not Hamiltonian.
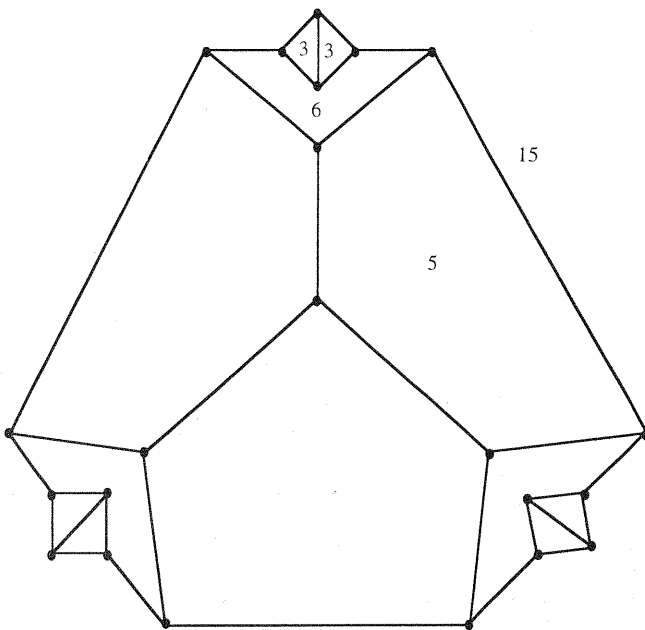
Figure 17

Progress to date on this question is restricted to special classes of graphs. For instance Franzblau [15], has considered what she calls **layered cubic planar graphs**. These are graphs which can be pictured as a sequence of concentric cycles with each consecutive pair of cycles being joined by a matching. The number of edges in the matchings increases along the sequence of cycles. These graphs do not give rise to a counterexample to Barnette's conjecture.

Theorem 19.   *Every layered cubic planar graph is Hamiltonian.*

In [15], Franzblau also considers the non-uniform layered cubic planar graph situation, where the number of edges in the matching between consecutive cycles is unrestricted. This proves difficult and only partial results are obtained.

The two conjectures of Barnette appear to be quite difficult and should provide us with an interesting challenge well into the new century.

Acknowledgement

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