P_{2k}-factorization of complete bipartite multigraphs

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Abstract

We show that a necessary and sufficient condition for the existence of a P_{2k}-factorization of the complete bipartite multigraph $\lambda K_{m,n}$ is $m = n \equiv 0 \pmod{k(2k-1)/d}$, where $d = \gcd(\lambda, 2k-1)$.

1. Introduction

Let $K_{m,n}$ be the complete bipartite graph with two partite sets having m and n vertices respectively. The graph $\lambda K_{m,n}$ is the disjoint union of λ graphs each isomorphic to $K_{m,n}$. A subgraph F of $\lambda K_{m,n}$ is called a spanning subgraph of $\lambda K_{m,n}$ if F contains all the vertices of $\lambda K_{m,n}$. It is clear that a graph with no isolated vertices is uniquely determined by the set of its edges. So in this paper, we consider a graph with no isolated vertices to be a set of 2-element subsets of its vertices. For positive integer k, a path on k vertices is denoted by P_k. A P_k-factor of $\lambda K_{m,n}$ is a spanning subgraph F of $\lambda K_{m,n}$ such that every component of F is a P_k and every pair of P_k's have no vertex in common. A P_k-factorization of $\lambda K_{m,n}$ is a set of edge-disjoint P_k-factors of $\lambda K_{m,n}$ which is a partition of the set of edges of $\lambda K_{m,n}$. (In paper [4] a P_k-factorization of $\lambda K_{m,n}$ is defined to be a resolvable (m, n, k, λ) bipartite P_k design.) The multigraph $\lambda K_{m,n}$ is called P_k-factorable whenever it has a P_k-factorization.

P_k-factorizations of $\lambda K_{m,n}$ have been studied by several researchers. When $k = 3$, the spectrum problem for P_3-factorization of the complete bipartite graph $K_{m,n}$ has been completely solved (see [2]). When k is an even number, the spectrum problem for P_k-factorization of the complete bipartite graph $K_{m,n}$ has been completely solved (see [3,4,5]). In this paper, we show that a necessary and sufficient condition for the existence of a P_{2k}-factorization of the complete multigraph $\lambda K_{m,n}$ is that $m = n \equiv 0 \pmod{k(2k-1)/d}$, where $d = \gcd(\lambda, 2k-1)$. (For integers x and y, we use $\gcd(x, y)$ to denote the greatest common divisor of x and y.)

2. Main result

The following theorem gives a necessary condition for the complete bipartite multigraph $\lambda K_{m,n}$ to be P_{2k}-factorable.

Theorem 2.1. *Let k, m and n be positive integers. If $\lambda K_{m,n}$ is P_{2k}-factorizable then $m = n \equiv 0 \pmod{k(2k-1)/d}$, where $d = \gcd(\lambda, 2k-1)$.*

Proof: Let X and Y be the two partite sets of $\lambda K_{m,n}$ with $|X| = m$ and $|Y| = n$. Let $\{F_1, F_2, \cdots, F_r\}$ be a P_{2k}-factorization of $\lambda K_{m,n}$. Let F_i have t components. Since F_i is a spanning subgraph of $\lambda K_{m,n}$, we have $m = kt = n$ and $|F_i| = m(2k-1)/k$ is an integer that does not depend on the individual P_{2k}-factors. Hence

$$m = n \equiv 0 \pmod{k}. \tag{1}$$

Let b be the total number of components; then $b = \lambda m^2/(2k-1)$ and $r = b/t = \lambda km/(2k-1)$, that is, $\lambda km/(2k-1)$ is an integer. Since $\gcd(k, 2k-1) = 1$, the number $\lambda m/(2k-1)$ must be an integer, and therefore $\lambda m \equiv 0 \pmod{(2k-1)}$. Let $d = \gcd(\lambda, 2k-1)$. Then we have

$$m \equiv 0 \pmod{(2k-1)/d}. \tag{2}$$

By combining equalities (1) and (2), we have $m = n \equiv 0 \pmod{k(2k-1)/d}$.

To show that the condition is also sufficient, we need the following results.

Lemma 2.2. *If $\lambda K_{m,n}$ has a P_{2k}-factorization, then $s\lambda K_{n,n}$ has a P_{2k}-factorization for every positive integer s.*

Proof: Construct a P_{2k}-factorization of $\lambda K_{n,n}$ repeatedly s times; this gives a P_{2k}-factorization of $s\lambda K_{n,n}$.

Lemma 2.3. *If $\lambda K_{n,n}$ has a P_{2k}-factorization, then $\lambda K_{sn,sn}$ has a P_{2k}-factorization for every positive integer s.*

Proof: Since $K_{s,s}$ is 1-factorable (see [1, pp72–75]), we can let $\{F_1, F_2, \cdots, F_s\}$ be a 1-factorization of it. For each i with $1 \le i \le s$, replace every edge of F_i with a $\lambda K_{n,n}$ to get a spanning subgraph G_i of $\lambda K_{sn,sn}$ such that the G_i $(1 \le i \le s)$ are pairwise edge disjoint and their sum is $\lambda K_{sn,sn}$. Since $\lambda K_{n,n}$ is P_{2k}-factorable, each G_i is also P_{2k}-factorable. And consequently, $\lambda K_{sn,sn}$ is P_{2k}-factorable.

We are now in a position to prove that the condition is also sufficient.

Lemma 2.4. *When $2k-1 = t\lambda$ and $n = t(t\lambda+1)/2$ then $\lambda K_{n,n}$ has a P_{2k}-factorization.*

Proof: Let $u = (t\lambda+1)/2$ and $v = t$. Let X and Y be the two partite sets of $\lambda K_{n,n}$ and set

$$X = \{x_{i,j} \mid 1 \le i \le u,\ 1 \le j \le v\},$$
$$Y = \{y_{i,j} \mid 1 \le i \le u,\ 1 \le j \le v\}.$$

We will construct a P_{2k}-factorization of $\lambda K_{n,n}$. We remark in advance that the additions in the first subscript of the $x_{i,j}$'s and $y_{i,j}$'s are taken modulo u in $\{1, 2, \cdots, u\}$ and the additions in the second subscript are taken modulo v in $\{1, 2, \cdots, v\}$.

For each i with $1 \le i \le u$, let

$$E_{2i-1} = \{x_{i,j}\, y_{i,j+2i-2} \mid 1 \le j \le v\}.$$

For each i with $2 \le i \le u$, let

$$E_{2i-2} = \{x_{i,j}\, y_{i-1,j+2i-3} \mid 1 \le j \le v\}.$$

Let $F = \bigcup_{1 \le i \le 2u-1} E_i$, then the graph F is a P_{2k}-factor of $\lambda K_{n,n}$. Define the bijection σ from $X \cup Y$ onto $X \cup Y$ by $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$ for all i, j with $1 \le i \le u$ and $1 \le j \le v$. For each i and j with $1 \le i, j \le u$, let

$$F_{i,j} = \{\sigma^i(x)\sigma^j(y) \mid x \in X,\ y \in Y,\ xy \in F\}.$$

It is easy to show that the $F_{i,j}$ $(1 \le i, j \le u)$ are pairwise disjoint P_{2k}-factors of $\lambda K_{n,n}$ and their sum is $\lambda K_{n,n}$. Thus $\{F_{i,j} \mid 1 \le i \le u,\ 1 \le j \le u\}$ is a P_{2n}-factorization of $\lambda K_{n,n}$. This proves the lemma.

Theorem 2.5. *Let $d = \gcd(\lambda, 2k-1)$. When $n \equiv 0 \pmod{k(2k-1)/d}$, then $\lambda K_{n,n}$ has a P_{2k}-factorization.*

Proof: Put $\lambda = hd$, $2k - 1 = td$, $\gcd(h, t) = 1$, $n = sk(2k-1)/d$, and $N = k(2k-1)/d$. Then we have $n = st(td+1)/2$ and $N = t(td+1)/2$. By lemma 2.4, $dK_{N,N}$ has a P_{2k}-factorization. By applying Lemmas 2.2 and 2.3, we see that $\lambda K_{n,n}$ has a P_{2k}-factorization.

Combining Theorems 2.1 and 2.5 gives the main result:

Theorem 2.6. *Let $d = \gcd(\lambda, 2k-1)$. The graph $\lambda K_{m,n}$ has a P_{2k}-factorization if and only if $m = n \equiv 0 \pmod{k(2k-1)/d}$.*

References

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