On the normality of Cayley digraphs of valency 2 on nonabelian groups of odd square free order*

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Abstract

In this paper, we prove that all Cayley digraphs of valency 2 on non-abelian groups of odd square-free order are normal.

For a given subset S of a finite group G without the identity element 1, the Cayley *digraph* on G with respect to S is denoted by $\Gamma = \mathrm{Cay}(G, S)$ where $V(\Gamma) = G$, $E(\Gamma) = \{(g, sg) \mid g \in G, s \in S\}$. It is clear that Aut (Γ), the automorphism group of Γ, contains the right regular representation G_R of G as a subgroup. Moreover Γ is connected if and only if $G = \langle S \rangle$, and Γ is undirected if and only if $S^{-1} = S$.

Γ is called normal if G_R is a normal subgroup of Aut (Γ). The concept of normality for Cayley digraphs is known to be important in the study of arc-transitive digraphs and half-tranisitive graphs. A natural problem is, for a given finite group G, to determine all normal or nonnormal Cayley digraphs of G. However this is a very difficult problem. The groups for which complete information about the normality of Cayley digraphs is available are cyclic groups of prime order (see [1]) and groups of order $2p$ (see [3]). Wang, Wang and Xu [9] determined all disconnected normal Cayley digraphs. Therefore we always suppose, in this paper, that the Cayley digraph $\mathrm{Cay}(G, S)$ is connected, that is, S is a generating subset of G. Xu [11, Problem 6] asked the following question: when S is a minimal generating set of G, are the corresponding Cayley digraph and graph normal? For abelian groups, Feng and Gao [5] proved that if the Sylow 2-subgroups of G are cyclic then the answers to the question are positive, and otherwise negative in general.

About nonabelian groups, Feng and Xu [6] proved that there are only two non-normal connected Cayley digraphs of valency 2 on nonabelian groups of order p^3 and p^4. This also implies that there are few nonnormal connected Cayley digraphs. Feng [4] determined all nonnormal Cayley digraphs of valency 2 on nonabelian groups of order $2p^2$. Wang and Li [10] also proved that the Cayley graphs of nonabelian groups

*Supported by the National Natural Science Foundation of China (Grant no. 19671077) and Doctoral Program Foundation of the National Education Department of China

of order $2pq$ and of degree 2 are normal. In this paper we discuss the normality of connected Cayley digraphs of valency 2 on nonabelian groups of odd square-free order. Our result is the following:

Main Theorem *Let G be a nonabelian group of odd square-free order and let $|S| = 2$. Then $\Gamma =$Cay(G, S) is normal.*

To prove our result, we need the following lemmas:

Lemma 1 ([11, Prop. 1.5]) *Let $A = \mathrm{Aut}(\Gamma)$ be the automorphism group of the Cayley digraph Γ of a group G with respect to its generating subset S and let A_1 be the stabilizer subgroup of A fixing the identity element 1 of G. Then Γ is normal if and only if A_1 is contained in the automorphism group $\mathrm{Aut}(G)$ of G.*

Lemma 2 ([4]) *Let $S = \{e, f\}$ be a two-generating subset of G without the identity 1 and let A_1^* be the subgroup of A which fixes the elements 1, e and f of G. Then Γ is normal if and only if $A_1^* = 1$.*

In this paper, we mainly discuss a normal subgroup A of the automorphism group of the Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ of valency 2 to determine whether Γ is normal. It is clear that $|A : G|$ is a power of 2. To prove our theorem, we can assume that Cay(G, S) is not normal, where G is the smallest counterexample of odd square-free order. Let N be a smallest normal subgroup of A. Then $N = T_1 \times T_2 \times \cdots \times T_k$ where T_i is isomorphic to Z_p or a simple group. Since G is of odd square-free order, $k = 1$. When N is simple, since G is a Hall odd-subgroup of A, $N \cap G$ is also a Hall odd-subgroup of N. Hence, by Corollary 5.6 of [2], $N \cong PSL(2,p)$ where p is a Mersenne prime. Moreover, by Theorem II.8.27 of [7], G is the semidirect product of Z_p by $Z_{(p-1)/2}$.

Now, we deal with the case when N is transitive on the set $V(\Gamma)$ of the digraph Γ.

Let (u, v) be a directed arc of Γ (the direction is from u to v). Then u and v are the tail and head of (u, v) respectively. If Γ has a circuit such that for every vertex u on this circuit, u is the tail of two incident arcs of the circuit or the head of two incident arcs, then the circuit is called an alternating circuit of Γ. Furthermore, if u is the tail of two incident arcs, then there exists at most one alternating circuit containing these two incident arcs; in which case we denote the circuit by $O(u)$. Similarly if u is the head of two incident arcs of an alternating circuit we denote the circuit by $I(u)$.

Claim 3 In Γ, an alternating circuit must be an alternating cycle.

Proof. When an alternating circuit A' of Γ is not an alternating cycle, there exist vertices which appear at least two times in A'. Since Γ is vertex-transitive and of valency 2, each vertex of A' must appear two times in A'. Hence, vertices not in A' are not adjacent to the vertices of A'. However, Γ is connected. Thus, all vertices appear in A'. Hence, the subgroup A_1^*, fixing A' pointwise, must fix all vertices of Γ. In other words, $A_1^* = 1$. By Lemma 2, Γ is normal. This is impossible.

Now, we consider the alternating cycle construction of Γ. Since A is transitive, the length of the alternating cycles is a constant $2m$ where m is the number of vertices of valency 2 in an alternating cycle. Since A_1^* fixes the alternating cycle $O(1)$ pointwise, it must fix the set $I((e^{-1}f)^i)$ for $0 \leq i < m$ (see Figure 1 for m

odd). If $|O(1) \cap I((e^{-1}f)^i)| > 2$ for some i, A_1^* fixes all vertices in $I((e^{-1}f)^i)$. Since Γ is transitive and connected, A_1^* fixes all alternating cycles and all vertices. Hence, $A_1^* = 1$, which is impossible. Similarly, $|O(1) \cap O(f(e^{-1}f)^i)| \leq 2$. Since Γ is transitive, $|O(g) \cap O(h)| \leq 2$, where $O(g)$ and $O(h)$ are distinct alternating cycles. Let k be the number of alternating cycles. Then, $km = |G|$ by calculating the number of vertices of valency 2 in the alternating circuits. If $m \geq k$, then there are i, j with $i \neq j$ such that $I((e^{-1}f)^i) = I((e^{-1}f)^j)$. Moreover, there is a vertex $f((e^{-1}f)^l)$ or $(e^{-1}f)^l (l \neq i, j)$ contained in $O(1) \cap I((e^{-1}f)^i)$, which is impossible. Hence, $m < k$.

We define a new digraph $A(\Gamma)$ as follows (see Figure 1 for m odd): $V(A(\Gamma))$ is the set of different alternating cycles; for $O(g), O(h) \in V(A(\Gamma)), (O(g), O(h)) \in E(A(\Gamma))$ if and only if $O(g) \cap O(h)$ contains vertices which are of valency 2 in $O(h)$.

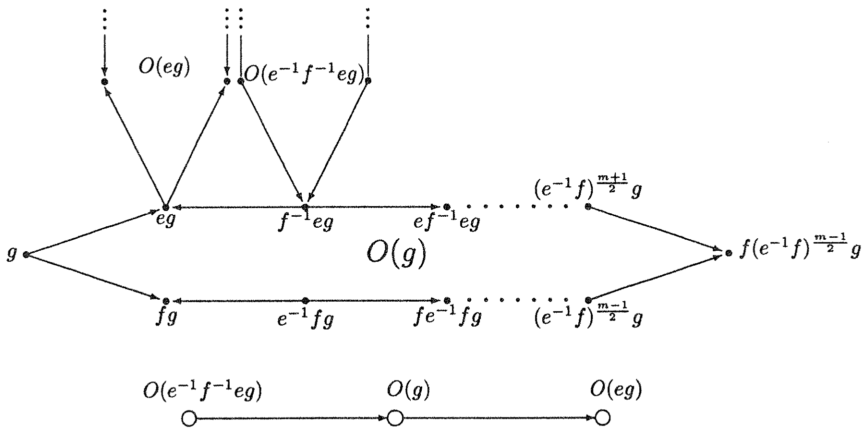

Figure 1

It is clear that there are no loops in $A(\Gamma)$, and that $A(\Gamma)$ is of order k and of out-degree m or $m/2$. Further, we have the following:

Lemma 4 *Two alternating cycles $O(g)$ and $O(h)$ of Γ have at most two common vertices. If $O(g)$ and $O(h)$ have a common vertex, or have two common vertices that have different valencies in the same alternating cycle, then $A \leq Aut(A(\Gamma))$ and $Aut(A(\Gamma))$ has a regular arc-transitive subgroup isomorphic to G.*

Proof. The first conclusion comes from the previous discussion. Since $A = \text{Aut}(\Gamma)$ preserves the alternating cycle construction of Γ, there is a homomorphism from A to $\text{Aut}(A(\Gamma))$ such that the image of A permutes the vertices of $A(\Gamma)$ (that are the alternating cycles of Γ). Let K be the kernel of this homomorphism. When two alternating cycles have only one common vertex or have two common vertices that have different valencies in the same alternating cycle, since K fixes all alternating cycles, K must fix all vertices in Γ. Hence, $K = 1$. So, $A \leq \text{Aut}(A(\Gamma))$. Moreover, as a subgroup of A, G permutes transitively the arcs of the digraph $A(\Gamma)$. It is clear that the action of G on $A(\Gamma)$ is regular arc-transitive.

By the above lemma, we know that N is isomorphic to a subgroup of $\text{Aut}(A(\Gamma))$. When N is transitive on $V(\Gamma)$, it is also transitive on $V(A(\Gamma))$. Hence, the order of its stabilizer subgroup is $(p+1)p(p-1)/(2k)$. However, by Theorem II.8.27 of [7], $PSL(2,p)$ has no subgroup of order $(p+1)p(p-1)/(2k)$. Hence N is not transitive on $V(\Gamma)$. We consider the graph Γ_N, where $V(\Gamma_N)$ is the set of all $N-$orbits on $V(\Gamma)$, and two vertices $U, V \in V\Gamma_N$ are adjacent in Γ_N if and only if there exit $\beta \in U$ and $\alpha \in V$ which are adjacent in Γ. In our case, Γ_N is also a Cayley digraph Cay$(\overline{G}, \overline{S})$ where $\overline{G} = GN/N$, $\overline{S} = SN/N$. Hence, by Lemma 2.5 of [8], Γ_N is a dicycle. We denote the orbits of N by $\{V_0, V_1, \cdots, V_l\}$ where the out-neighbors of vertices in V_i are in V_{i+1} and $l = |G|/|G \cap N|$. Assume that N is isomorphic to $PSL(2,p)$. Let N_α be the stabilizer subgroup of N fixing the vertex $\alpha \in V\Gamma$. Then, by Theorem II.8.27 of [7], N_α is the dihedral group of order $p+1$. Let M be its cyclic subgroup of order $(p+1)/2$. Since M is cyclic, it has an orbit C of order $(p+1)/2$ in some set V_i. Hence, the out-neighbors and in-neighbors of C are of order $p+1$, $(p+1)/2$ or $(p+1)/4$. If its out-neighbors or in-neighbors are of order $(p+1)/4$ or $(p+1)/2$, the length $2|H|$ of the alternating cycle is a divisor of $(p+1)$, where $H = \langle e^{-1}f \rangle$ is a subgroup of G. This is impossible. Hence, the out-neighbors and in-neighbors of C are of order $p+1$ and consist of two orbits of M of order $(p+1)/2$. Thus, Cay(G,S) is not connected. Hence, $G > N = N \cap G \cong Z_p$. Then, by Lemma 2.5 of [8], Γ_N is a dicycle or G/N-arc transitive of valency 2. If Γ_N is a dicycle, there are $2p$ arcs between V_i and V_{i+1} and the number of arcs also is $2|H|$. Since Γ is connected, A_1^* fixes all vertices. Thus Cay(G,S) is normal. So, we assume that Γ_N is G/N-arc transitive of valency 2. Let K be the kernel of A acting on the set $\{V_1, V_2, \cdots, V_l\}$ and h an element in K fixing a vertex $\alpha \in V_1$. Then, since K fixes all V_i, h fixes the in-neighbors and out-neighbors of α and so fixes all vertices. Hence, K must be regular and be N. Then, since G is the smallest counterexample, $G/N \lhd A/N$. Hence, $G \lhd A$ and Γ is normal. The proof of our main Theorem is completed.

Acknowledgement

We would like to thank the referee for useful comments on our paper.

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(Received 21/1/99; revised 23/6/99)