On the spectral radius of nonnegative matrices*

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Abstract

We give lower bounds for the spectral radius of nonnegative matrices and nonnegative symmetric matrices, and prove necessary and sufficient conditions to achieve these bounds.

1 Introduction and Preliminaries

In this note, we will be concerned with nonnegative matrices. Let A be an $n \times n$ nonnegative matrix. The spectral radius of A is denoted by $\rho(A)$. Due to the Perron-Frobenius theorem, $\rho(A)$ is an eigenvalue, also known as the Perron root of A. For a matrix X, X^t denotes the transpose of X.

A nonnegative matrix is row-regular if all of its row sums are equal. A matrix A is row-semiregular if there is a permutation matrix P such that $P^tAP = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ where both B and C are row-regular. Column-regular and column-semiregular are defined similarly. The matrix A is regular if A is both row-regular and column-regular. Semiregular is defined similarly.

An $n \times n$ nonnegative matrix A all of whose row sums $d_1, \ldots, d_n$ are positive is almost row-regular if $a_{ij} > 0$ implies that d_id_j is a constant. Almost column-regular is defined similarly. A is almost regular if A is both almost row-regular and almost column-regular.

In this paper, we give lower bounds for the spectral radius of nonnegative matrices and nonnegative symmetric matrices, and prove necessary and sufficient conditions to achieve these bounds.

Lemma 1.1 *Let $A = (a_{ij})$ be an $n \times n$ nonnegative irreducible matrix with positive row sums $d_1, d_2, \ldots, d_n$. Then the following are equivalent:*
(1) A is almost row-regular;

*Project supported by Guangdong Provincial Natural Science Foundation of China (990447).

(2) A is row-regular or row-semiregular;

(3) for each $1 \leq i \leq n$ *and any number* $a \neq 0$, $\frac{\sum_{j=1}^{n} a_{ij}(d_i d_j)^a}{d_i}$ *is a constant.*

Proof. Note that (2)⇒(1) and (1)⇒(3) are obvious. We need only to prove (3)⇒(2).

Suppose for each $1 \leq i \leq n$ and any number $a \neq 0$, $\frac{\sum_{j=1}^{n} a_{ij}(d_i d_j)^a}{d_i} = r$. If all the d_i are equal, then A is row-regular. Otherwise set $\delta = \min_{1 \leq i \leq n} d_i$ and $\Delta = \max_{1 \leq i \leq n} d_i$.

Note that A is irreducible. Choose u and v such that $d_u = \delta$ and $d_v = \Delta$. Suppose without loss of generality that $a > 0$. Then we have

$$r = \frac{\sum_{j=1}^{n} a_{uj}(\delta d_j)^a}{\delta} \leq (\delta\Delta)^a$$

and

$$r = \frac{\sum_{j=1}^{n} a_{vj}(\Delta d_j)^a}{\Delta} \geq (\delta\Delta)^a.$$

It follows that $r = (\delta\Delta)^a$, and whenever $a_{ij} > 0$, then $d_i = \delta$ and $d_j = \Delta$ or vice versa. Note that $\delta < \Delta$. Then $d_i = d_j$ implies that $a_{ij} = 0$. Set $\alpha = \{i : d_i = \delta\}$ and $\beta = \{i : d_i = \Delta\}$. Then $\alpha \cap \beta = \emptyset$, $\alpha \cup \beta = \{1, \ldots, n\}$, $A[\alpha, \alpha] = 0$, and $A[\beta, \beta] = 0$. Hence there is a permutation matrix P such that $P^t A P = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ where each row sum of B is δ and each row sum of C is Δ. We conclude that A is row-semiregular.

The proof is now completed. □

For a nonnegative symmetric matrix, the same argument as in the proof of Lemma 1.1 will apply to all irreducible components of A. Hence we have the following.

Lemma 1.2 *Let* $A = (a_{ij})$ *be an* $n \times n$ *nonnegative symmetric matrix with positive row sums* $d_1, d_2, \ldots, d_n$. *Then the following are equivalent:*
(1) A is almost regular;
(2) A is regular or semiregular;
(3) for each $1 \leq i \leq n$ *and any number* $a \neq 0$, $\frac{\sum_{j=1}^{n} a_{ij}(d_i d_j)^a}{d_i}$ *is a constant.*

2 Nonnegative matrices and digraphs

The following Lemma is contained in [1].

Lemma 2.1 *Let A be an* $n \times n$ *nonnegative matrix. Then*

$$\rho(A) \geq \min \frac{(Ax)_i}{x_i}. \tag{2.1}$$

If A is irreducible, then equality holds in (2.1) if and only if x is an eigenvector corresponding to $\rho(A)$.

Theorem 2.2 *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with positive row sums $d_1, d_2, \ldots, d_n$. Then*

$$\rho(A) \geq \min_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} a_{ij} d_j}. \tag{2.2}$$

If A is reducible, then equality holds in (2.2) if and only if A is row-regular or row-semiregular.

Proof. Let $A^2 = B = (b_{ij})$. On setting $x = (1, \ldots, 1)^t$, by Lemma 2.1 we obtain

$$\begin{aligned} \rho(B) &\geq \min_{1 \leq i \leq n} \tfrac{(Bx)_i}{x_i} = \min_{1 \leq i \leq n} \textstyle\sum_{s=1}^{n} b_{is} \\ &= \min_{1 \leq i \leq n} \textstyle\sum_{s=1}^{n} \sum_{j=1}^{n} a_{ij} a_{js} = \min_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij} d_j. \end{aligned} \tag{2.3}$$

Observe that $\rho(A) = \sqrt{\rho(B)}$. From (2.3) we have (2.2), as desired.

Suppose that A is irreducible. Now we are going to prove the second part of Theorem 2.2.

First suppose that A is row-regular or row-semiregular. By Lemma 1.1, for each $1 \leq i \leq n$, $\sum_{j=1}^{n} a_{ij} d_j$ is a constant. Then for some constant r and each i,

$$\sum_{s=1}^{n} b_{is} = \sum_{j=1}^{n} \sum_{s=1}^{n} a_{ij} a_{js} = \sum_{j=1}^{n} a_{ij} d_j = r. \tag{2.4}$$

It follows from (2.4) that r is an eigenvalue of B corresponding to $x = (1, \ldots, 1)^t$, which implies that $\rho(B) = r$. It follows that

$$\rho(A) = \sqrt{\rho(B)} = \sqrt{\sum_{j=1}^{n} a_{ij} d_j}. \tag{2.5}$$

Conversely, suppose equality in (2.2) holds. Then by Lemma 2.1,

$$B(1, \ldots, 1)^t = \rho(B)(1, \ldots, 1)^t.$$

Hence

$$\sum_{j=1}^{n} a_{ij} d_i = \sum_{s=1}^{n} b_{is} = \rho(B) = \rho(A)^2$$

is a constant for each i. By Lemma 1.1 (let a=1), A is row-regular or row-semiregular. This completes the proof of Theorem 2.2. $\square$

Let $D = (V, E)$ be a directed graph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set E. The adjacency matrix of D is the $n \times n$ (0-1) matrix $A = (a_{ij})$ in which $a_{ij} = 1$ if and only if vertex i is adjacent to vertex j (that is, there is an arc from vertex i

to vertex j). Then the i-th row sum d_i of A is just the out-degree of vertex i in D. The directed graph D is out-regular (out-semiregular) if the adjacency matrix is row-regular (row-semiregular). Clearly G is out-semiregular if and only D is bipartite, and each vertex in the same part of the bipartition has the same out-degree. The spectral radius of D, denoted by $\rho(G)$, is defined to be the spectral radius of its adjacency matrix A. An immediate corollary of Theorem 2.2 is given as follows.

Corollary 2.3 *Let G be a directed graph of order n with positive out-degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\rho(G) \geq \min_{1 \leq i \leq n} \sqrt{\sum_{(i,j) \in E} d_j}. \tag{2.6}$$

If G is strongly connected, then equality holds in (2.6) if and only if D is out-regular or out-semiregular.

3 Symmetric matrices and graphs

We need the following lemma.

Lemma 3.1 *Let A be an $n \times n$ nonnegative symmetric matrix. Then*

$$\rho(A) \geq \frac{x^t(Ax)}{x^t x} \tag{3.1}$$

with equality if and only if x is an eigenvector corresponding to $\rho(A)$.

The proof of Lemma 3.1 is a routine exercise in linear algebra.

We are now ready to prove the main result of this section.

Theorem 3.2 *Let $A = (a_{ij})$ be an $n \times n$ nonnegative symmetric matrix with positive row sums $d_1, d_2, \ldots, d_n$. Then*

$$\rho(A) \geq \sqrt{\frac{\sum_{i=1}^{n} d_i^2}{n}} \tag{3.2}$$

with equality if and only if A is regular or semiregular.

Proof. Let $A^2 = B = (b_{ij})$. On setting $x = (1, \ldots, 1)^t$, by Lemma 3.1 we obtain

$$\rho(B) \geq \frac{x^t B x}{x^t x} = \frac{\sum_{i=1}^{n} \sum_{s=1}^{n} \sum_{j=1}^{n} a_{ji} a_{is}}{n} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \sum_{s=1}^{n} a_{is}}{n} = \frac{\sum_{i=1}^{n} d_i^2}{n}.$$

Note that $\rho(A) = \sqrt{\rho(B)}$. We have (3.2).

Now we are going to prove the second part of Theorem 3.2.

First suppose that A is regular or semiregular. Then By Lemma 1.2 (let a=1), for some constant r and each j,

$$\sum_{s=1}^{n} b_{js} = \sum_{i=1}^{n} a_{ji} \sum_{s=1}^{n} a_{is} = \sum_{i=1}^{n} a_{ji} d_i = r. \tag{3.3}$$

Hence $\rho(B) = r$. On the other hand, by (3.3) we also have $\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} d_i = nr$, i.e., $\frac{\sum_{i=1}^{n} d_i}{n} = r$. Hence

$$\rho(A) = \sqrt{\rho(B)} = \sqrt{r} = \sqrt{\frac{\sum_{i=1}^{n} d_i}{n}}.$$

Conversely suppose equality in (3.2) holds. Then by Lemma 3.1, $(1, \ldots, 1)^t$ is an eigenvector of B corresponding to $\rho(B)$. Hence for each i $\sum_{j=1}^{n} a_{ij} d_j = \sum_{s=1}^{n} b_{is} = \rho(B) = \rho(A)^2$. By Lemma 1.2 (let a=1), A is regular or semiregular. This completes the proof of Theorem 3.2. $\square$

Let $G = (V, E)$ be an undirected simple graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. The adjacency matrix of G is the $n \times n$ (0-1) symmetric matrix $A = (a_{ij})$ in which $a_{ij} = 1$ if and only if vertex i is adjacent to vertex j (that is, there is an edge between vertices i and j). Then the i-th row sum d_i of A is just the degree of vertex i in G. The graph G is regular (semiregular) if the adjacency matrix is regular (semiregular). Clearly G is semiregular if and only G is bipartite, and each vertex in the same part of bipartition has the same degree. The spectral radius of G, denoted by $\rho(G)$, is defined to be the spectral radius of its adjacency matrix A. An immediate corollary of Theorem 3.2 is given as follows.

Corollary 3.3 *Let G be an undirected simple graph of order n with positive degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\rho(G) \geq \sqrt{\frac{\sum_{i=1}^{n} d_i}{n}} \tag{3.4}$$

with equality if and only if G is regular or semiregular.

Let A be an $n \times n$ nonnegative symmetric matrix with positive row sums $d_1, \ldots, d_n$. Hoffman, Wolfe and Hofmeister [3] have proved that

$$\rho(A) \geq \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\sqrt{d_i d_j}}{\sum_{i=1}^{n} d_i}. \tag{3.5}$$

By Lemma 1.2 and the result in [3], equality holds in (3.5) if and only if A is regular or semiregular.

It is natural to wonder how (3.2) compares with (3.5); the examples below shows that (3.2) is sharper than (3.5) in many cases.

First, for any $n \geq 4$, let A be the $n \times n$ matrix with $a_{1,i} = 1$ for $2 \leq i \leq n$, $a_{i,i+1} = a_{i+1,i} = a_{2,n} = a_{n,2} = 1$ for $2 \leq i \leq n-1$ and all other entries 0. Note that A is the adjacency matrix of a wheel of order n, and $d_1 = n-1$, $d_2 = \ldots = d_{n-1} = 3$. The right hand side of (3.2) is $\sqrt{\frac{(n-1)(n+8)}{n}}$. The right hand side of (3.5) is $\frac{\sqrt{3(n-1)}+3}{2}$. Since

$$\lim_{n\to\infty} \frac{\sqrt{\frac{(n-1)(n+8)}{n}}}{\frac{\sqrt{3(n-1)}+3}{2}} = \frac{2}{\sqrt{3}} > 1,$$

for the matrix A, (3.2) is a sharper bound than (3.5) if n is sufficiently large.

Next, we give another example. For $n \geq 5$, Let A be the $n \times n$ matrix with $a_{1,i} = a_{2,i} = 1$ for $3 \leq i \leq n$, $a_{i,i+1} = a_{i+1,i} = 1$ for $3 \leq i \leq n$ and all other entries 0. Then $d_1 = d_2 = n-2$, $d_3 = d_n = 3$, $d_4 = \ldots = d_n = 4$. The right hand side of (3.2) is $\sqrt{\frac{2(n-2)^2+16(n-4)+9}{n}}$. However, the right hand side of (3.5) is $\frac{2(n-4)\sqrt{4(n-2)}+4\sqrt{3(n-2)}+4(n-5)+4\sqrt{3}}{3n-7}$. Note that

$$\lim_{n\to\infty} \frac{\sqrt{\frac{2(n-2)^2+16(n-4)+9}{n}}}{\frac{2(n-4)\sqrt{4(n-2)}+4\sqrt{3(n-2)}+4(n-5)+4\sqrt{3}}{3n-7}} = \frac{3\sqrt{2}}{4} > 1.$$

We see that for A, (3.2) is sharper than (3.5) again if n is sufficiently large.

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(Received 14/1/2000)