A linear algebraic approach to directed designs

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Abstract

A t-(v, k, λ) directed design (or simply a t-(v, k, λ)DD) is a pair $(V, \mathcal{B})$, where V is a v-set and $\mathcal{B}$ is a collection of (transitively) ordered k-tuples of distinct elements of V, such that every ordered t-tuple of distinct elements of V belongs to exactly λ elements of $\mathcal{B}$. (We say that a t-tuple belongs to a k-tuple, if its components are contained in that k-tuple as a set, and they appear with the same order). In this paper with a linear algebraic approach, we study the t-tuple inclusion matrices $D^v_{t,k}$, which sheds light to the existence problem for directed designs. Among the results, we find the rank of this matrix in the case of $0 \le t \le 4$. Also in the case of $0 \le t \le 3$, we introduce a semi-triangular basis for the null space of $D^v_{t,t+1}$. We prove that when $0 \le t \le 4$, the obvious necessary conditions for the existence of t-(v, k, λ) signed directed designs, are also sufficient. Finally we find a semi-triangular basis for the null space of $D^{t+1}_{t,t+1}$.

Australasian Journal of Combinatorics 23(2001), pp.119–134

1 Introduction

Let $0 < t \leq k \leq v$ and $\lambda \geq 0$ be integers, and let V be a set of v elements. Throughout the paper we will assume a total order on the elements of V. Let the set of all k-subsets of V be ordered lexicographically from 1 to $\binom{v}{k}$, and the set of its t-subsets from 1 to $\binom{v}{t}$. A *t-inclusion matrix* $W_{t,k}^{v} = [w_{ij}]$ is a $\binom{v}{t} \times \binom{v}{k}$ matrix defined by $w_{ij} = 1$ if the i-th t-subset is included in the j-th k-subset, and $w_{ij} = 0$ otherwise. A $\binom{v}{k} \times 1$ vector $F = [f_i]$ represents a t-(v,k,λ) *design*, if each f_i is a non-negative integer and

$$W_{t,k}^{v} F = \lambda e_t \tag{1}$$

where $e_t = (1, \cdots, 1)^t$.

An integer vector which satisfies (1) but in which the components are not necessarily positive, represents a t-(v,k,λ) *signed design*. A signed design is called a (v,k,t) *trade* if $\lambda = 0$. The sum of the non-negative components in a trade, which is equal to the absolute value of the sum of the negative components, is called the *volume* of a trade, and usually is denoted by s. Also the *foundation* of a trade $T = [t_i]$ may be defined as

$$\text{found}(T) = \{x \in V \mid x \in i\text{–th block, for some } i \text{ with } t_i \neq 0\}.$$

A trade with a minimum volume and with a minimum foundation size is called a *minimal* trade.

For given v, k, t, the set of all t-(v,k,λ) signed designs forms a $\mathbf{Z}$-module. The set of all (v,k,t) trades is a submodule of this module and is denoted by $M_{t,k}^{v}$. Clearly this submodule is a subset of the null space of $W_{t,k}^{v}$. Graver and Jurkat [2] and independently Wilson [14] proved the following theorem about the rank of the matrix $W_{t,k}^{v}$.

Theorem 1.1 ([2] and [14]).

$$\text{rank } W_{t,k}^{v} = \begin{cases} \binom{v}{t}, & \text{if } t \leq k \leq v-t; \\ \binom{v}{k}, & \text{if } v-t \leq k \leq v. \end{cases}$$

Graver and Jurkat, in the same paper, introduced a basis of (v,k,t) trades for the module $M_{t,k}^{v}$. Other papers have appeared since, which introduce bases for this module with easier algorithms; for example see [3], [6] and [7]. In [5] a very simple algorithm for producing a basis is given,

Theorem 1.2 [5]. There exists a semi-triangular basis for trades.

The basis given in [5] is semi-triangular and consists of minimal trades. This basis is also a module basis for $M_{t,k}^{v}$.

The following is also a well known theorem.

Theorem 1.3 ([2] and [14]). Let $t, k, v, \lambda_1, \ldots, \lambda_t = \lambda$ be integers where $v \geq 1$ and $0 \leq t, k \leq v$. There exists a t-(v,k,λ) signed design if and only if $\lambda_{s+1} = \frac{k-s}{v-s}\lambda_s$, for

$0 \leq s < t$.

Wilson in [15] has studied the matrix $W_{t,k}^{v}$ in detail.

There is also a linear algebraic approach to the other combinatorial designs such as orthogonal arrays; for example see [8]. In this paper we look at the directed designs with this approach.

By an n-tuple of V, we mean a transitively ordered n-subset of V. Each k-tuple of distinct elements of V is called a *block*. Note that a t-tuple is said to appear in a block if its components are contained in that block as a set, and if they appear with the same order. For example the 4-tuple *abcd* contains the 3-tuples *abc*, *abd*, *acd* and *bcd*. Let all k-tuples of V be ordered lexicographically from 1 to $k!\binom{v}{k}$, and its t-tuples from 1 to $t!\binom{v}{t}$. A *t-inclusion matrix* $D_{t,k}^{v} = [d_{ij}]$ is a $t!\binom{v}{t} \times k!\binom{v}{k}$ matrix defined by $d_{ij} = 1$ if the i-th t-tuple is included in the j-th k-tuple, and $d_{ij} = 0$ otherwise. A $k!\binom{v}{k} \times 1$ integral vector $F = [f_i]$ is said to represent a t-(v, k, λ) *signed directed design* (or simply t-(v, k, λ)SDD), if

$$D_{t,k}^{v} F = \lambda e_t \tag{2}$$

where $e_t = (1, \cdots, 1)^t$ is a $t!\binom{v}{t} \times 1$ vector.

Here f_i is called the frequency of the i-th block (or k-tuple) in the signed directed design. A t-(v, k, λ) *directed design* (or simply t-(v, k, λ)DD) is a t-(v, k, λ)SDD in which $f_i \geq 0$ for all i. A t-(v, k, λ)SDD with $\lambda = 0$ is said to represent a null directed design, or a (v, k, t) *directed trade* (or simply a (v, k, t)DT).

Directed designs were first introduced by Hung and Mendelsohn in [4]. Some further work has been done on the construction of these designs, for references see [1], [9], [10], [11], [12], and [13].

It should be noted that here we consider directed designs and directed trades as vectors, but they can be defined in a traditional way. For example

Definition. A (v, k, t) *directed trade* (or simply a (v, k, t)DT) of volume s consists of two disjoint collections T_1 and T_2 , each of s blocks, such that the number of blocks containing any t-tuple of V is the same in T_1 and T_2.

A (v, k, t)DT of volume s will be represented by

$$T = T_1 - T_2 = \sum_{i=1}^{s} B_{1i} - \sum_{i=1}^{s} B_{2i},$$

where B_{1i}'s and B_{2i}'s are the blocks contained in T_1 and T_2, respectively.

It is clear that if in a directed trade we consider the blocks without order, then we obtain a trade.

In this paper we use arrays to represent directed trades. For example

T_1	T_2
$xyzw$	$xywz$
$yxwz$	$yxzw$

is a $(4,4,3)$DT of volume 2, where $xyzw$ and $yxwz$ are blocks both with frequency $+1$ and $xywz$ and $yxzw$ are blocks both with frequency -1.

The set $C^v_{t,k}$ of all signed directed designs is a $\mathbf{Z}$-module, and the set $N^v_{t,k}$ of all directed trades is a submodule of this module. In other words $N^v_{t,k}$ is the following set:

$$N^v_{t,k} = \{F \mid F \text{ is an integral vector of size } k!\binom{v}{k} \times 1, \quad D^v_{t,k}F = 0\}.$$

Clearly this submodule is a subset of the null space of $D^v_{t,k}$ which we denote by Ker $D^v_{t,k}$. Here we consider Ker $D^v_{t,k}$ as a vector space over the rational field.

From the vector representation it can be easily seen that:

(i) if F_1 and F_2 are two t-(v,k,λ)DDs, then $F_1 - F_2$ is a (v,k,t)DT;

(ii) let F be a t-(v,k,λ)DD and T be a (v,k,t)DT, then $F+T$ is a t-(v,k,λ)DD if and only if $F+T$ is a positive integral vector;

(iii) if T' and T'' are two (v,k,t)DTs, then each of $T'-T''$ and $T'+T''$ is also a (v,k,t)DT.

Here, first we determine the dimension of Ker $D^v_{t,k}$ for $0 \le t \le 4$. Then for $0 \le t \le 3$ we introduce a semi-triangular basis of directed trades for Ker $D^v_{t,t+1}$, such that it is also a module basis for the $\mathbf{Z}$-module $N^v_{t,t+1}$. Next for any given t, we introduce a semi-triangular basis of directed trades for Ker $D^{t+1}_{t,t+1}$, such that it is also a module basis for the $\mathbf{Z}$-module $N^{t+1}_{t,t+1}$. Finally we show that for $0 \le t \le 4$ the necessary conditions for the existence of a t-(v,k,λ)SDD are also sufficient.

2 Some results about $N^v_{t,k}$

In this section we state some lemmas about $N^v_{t,k}$. First we need the following definition.

Definition. A directed trade is called *strictly directed* if when we consider its blocks without order then we obtain a trade of volume 0.
The following lemma is immediate from the definition.

Lemma 2.1. If $T_1, T_2 \in N^v_{t,k}$ are two strictly directed trades, then $T_1 + T_2$ is also a strictly directed trade.

Corollary 2.2. An integral linear combination of strictly directed trades is also a strictly directed trade.

Lemma 2.3. When $t \le k < v-t$, there does not exist a basis for $N^v_{t,k}$ which consists only of strictly directed trades.

Proof. We know that in this case there exists a non-void (v, k, t) trade (Theorem 1.1), and from this trade we may construct a directed trade, for example by writing elements of each block in increasing order. By Corollary 2.2 this directed trade can not be obtained from a linear combination of strictly directed trades. ■

Notation. A basis of $N^v_{t,k}$, will be denoted by $\beta_{t,k}$, which may be partitioned as $\beta_{t,k} = \beta'_{t,k} \cup \beta''_{t,k}$, where $\beta''_{t,k}$ consists of all strictly directed trades in this basis.

We know that $N^v_{t,k}$ may be identified with the integral vectors in the null space of $D^v_{t,k}$. So the following lemma is as an easy exercise in linear algebra.

Lemma 2.4. The module dimension of $N^v_{t,k}$ is equal to dim Ker $D^v_{t,k}$.

Definition. The smallest block (in the lexicographical ordering) of a directed trade is called a *starting block.*

It is clear that a set of directed trades with distinct starting blocks are linearly independent. If a set of directed trades with distinct starting blocks forms a basis for $N^v_{t,k}$, then this basis is called a semi-triangular basis. It means that if we consider each element of this basis as a column vector, by a suitable permutation a semi-triangular matrix may be produced.

A semi-triangular basis construction. For constructing a semi-triangular basis $\beta_{t,k}$ it is sufficient that:

(*i*) the sets $\beta'_{t,k}$ and $\beta''_{t,k}$ are semi-triangular;

(*ii*) the starting blocks of directed trades in $\beta'_{t,k}$ are distinct from the starting blocks of directed trades in $\beta''_{t,k}$.

A semi-triangular set $\beta'_{t,k}$ may be constructed as follows:
Khosrovshahi and Ajoodani in [5] constructed a semi-triangular basis of minimal trades for the **Z**-module $M^v_{t,k}$. Let T be an element of this basis with starting block $\{x_1, \ldots, x_k\}$, $x_1 < \ldots < x_k$,. By arranging elements of each block of this trade in decreasing order, we obtain a (v, k, t)DT with the starting block, $x_k \ldots x_1$. Let $\beta'_{t,k}$ be the set of all directed trades obtained in that manner. Now for any semi-triangular basis $\beta''_{t,k}$, of strictly directed trades, always the condition (*ii*) holds. For a block $\{y_1 \ldots y_k\}$, where $y_1 > \ldots > y_k$, can not be a starting block in any strictly directed trades.

3 Results about $D^v_{t,k}$

The structure of $D^v_{t,k}$ is obvious for some values of t and k:

$$D^v_{t,k} = 0 \quad \text{if } k < t$$
$$D^v_{t,t} = I$$
$$D^v_{0,k} = J.$$

where $J = (1, \ldots, 1)$. And we have the following matrix equation:

$$\binom{k-s}{t-s} D^v_{s,k} = D^v_{s,t} D^v_{t,k} \quad \text{where} \quad s \le t \le k. \tag{3}$$

To prove (3), let S be an s-tuple and K be a k-tuple, such that S is contained in K. Then the number of t-tuples T such that T is contained in K and contains S is $\binom{k-s}{t-s}$.

Theorem 3.1. The map $D^v_{t+1,k}$: Ker $D^v_{t,k} \to$ Ker $D^v_{t,t+1}$ is a linear transformation and we have:

(i) dim Ker $D^v_{t+1,k}$ = dim Ker $D^v_{t,k}$− dim Im $D^v_{t+1,k}$;

(ii) if T is a strictly directed trade, then $T' = D^v_{t+1,k}T$ is also a strictly directed trade.

Proof. (i) It is obvious by Equation (3) that one may interpret $D^v_{t+1,k}$ as operating on Ker $D^v_{t,k}$ and mapping each element of Ker $D^v_{t,k}$ to Ker $D^v_{t,t+1}$. Assume that

$$U = \{\beta \in \text{Ker } D^v_{t,k} \mid D^v_{t+1,k}\beta = 0\}.$$

By a familiar theorem from linear algebra we have:

$$\dim U = \dim \text{Ker } D^v_{t,k} - \dim \text{Im } D^v_{t+1,k}.$$

It is sufficient to show that U=Ker $D^v_{t+1,k}$. It is obvious that $U \subseteq$ Ker $D^v_{t+1,k}$. Suppose $\beta' \in$ Ker $D^v_{t+1,k}$, thus $D^v_{t+1,k}\beta' = 0$ and $D^v_{t,t+1}D^v_{t+1,k}\beta' = 0$. Therefore by (3) we have $D^v_{t,k}\beta' = 0$. This means that $\beta' \in$ Ker $D^v_{t,k}$, so $\beta' \in U$ and finally Ker $D^v_{t+1,k} \subseteq U$. This completes the proof of (i).

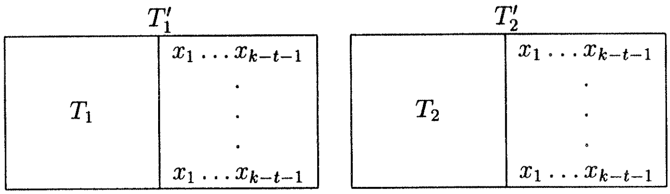
(ii) Let $T \in$ Ker $D^v_{t,k}$ be a strictly directed trade. Then $T = T_1 - T_2 = \sum_{i=1}^s B_i - \sum_{i=1}^s B_i\alpha_i$, where α_i's are permutations. Each $(t+1)$-tuple, which is contained in a given block B_i, is a block in T_1'. And each $(t+1)$-tuple contained in a $B_i\alpha_i$ is a block in T_2'. Thus for each block in T_1' a permutation of it is in T_2' and viceversa. This completes the proof. ■

Theorem 3.2. Suppose that there exists a basis $\beta_{t,t+1}$ for Ker $D^v_{t,t+1}$ such that $\beta_{t,t+1} = \beta'_{t,t+1} \cup \beta''_{t,t+1}$, where for each $T \in \beta'_{t,t+1}$, $|\text{found}(T)| = 2t+2$ and for each $T \in \beta''_{t,t+1}$, $|\text{found}(T)| \le 2t$. Then for the linear mapping $D^v_{t+1,k}$: Ker $D^v_{t,k} \to$ Ker $D^v_{t,t+1}$ we have:

$$\text{Im } D^v_{t+1,k} = \begin{cases} \text{Ker } D^v_{t,t+1} & \text{if } k < v-t, \quad \text{i.e. the mapping is onto;} \\ \langle B''_{t,t+1} \rangle & \text{if } v-t \le k \le v-t+1, \end{cases}$$

where $\langle B''_{t,t+1} \rangle$ is a subspace generated by the elements of $\beta''_{t,t+1}$.

Proof. (i) Let $T \in \beta_{t,t+1}$ and $k < v - t$. By assumption $|\text{found}(T)| \leq 2t + 2$. Then $v \geq k + t + 1$, and we may choose $k - t - 1$ elements $x_1, \ldots, x_{k-t-1}$ of $V-$ found(T). Then the directed trade $T' = T'_1 - T'_2$, with blocks given as follows, is the required trade and we have $D^v_{t+1,k} T' = T$.



(ii) If $v - t \leq k \leq v - t + 1$, then $v \leq k + t$. Thus if β is a basis with integral vectors (directed trades) for Ker $D^v_{t,k}$, then it consists only of strictly directed trades. Thus by Theorem 3.1 (ii), Im $D^v_{t+1,k} \subseteq \langle B''_{t,t+1} \rangle \subseteq$ Ker $D^v_{t,t+1}$. In this case for each $T \in \beta''_{t,t+1}$ we have $|\text{found}(T)| \leq 2t$, by assumption and we may choose $k - t - 1$ elements $x_1, \ldots, x_{k-t-1}$ of $V-$ found(T). As in (i) the directed trade T' is the required trade and $D^v_{t+1,k} T' = T$. The proof is complete. ■

Now by applying the above two theorems we see that to determine the dimension of Ker $D^v_{t,k}$ (or the rank of $D^v_{t,k}$) for given positive integers t, k, v ($t \leq k \leq v - t + 1$), it is sufficient that:

(*i*) we know the dimension of Ker $D^v_{t,k}$ and,

(*ii*) in the special case of $k = t + 1$ we be able to construct a basis for Ker $D^v_{t,t+1}$ such that it satisfies the assumptions of Theorem 3.2.

In that way the rank of $D^v_{t,k}$ may be obtained inductively.

4 Dimension of Ker $D^v_{t,k}$ or dimension of $N^v_{t,k}$

In this section we determine the dimension of Ker $D^v_{t,k}$ where $0 \leq t \leq 4$. We also introduce a semi-triangular basis of directed trades for Ker $D^v_{t,t+1}$ where $0 \leq t \leq 3$, which is a module basis for **Z**-module $N^v_{t,t+1}$.

Theorem 4.0. For $t = 0$

(*i*) dim Ker $D^v_{0,k} = k!\binom{v}{k} - 1$ for each $0 \leq k \leq v$;

(*ii*) there exists a semi-triangular basis $\beta^v_{0,1}$ of directed trades which satisfies the assumption of Theorem 3.2.

Proof. (i) By definition $D^v_{0,k} = [1 \cdots 1]$, thus (i) is obvious.
(ii) Take $(v, 1, 0)$ directed trades T with blocks as follows

$$\begin{array}{cc} T_1 & T_2 \\ \hline x & x+1 \end{array}$$

for each x such that $1 \leq x \leq v-1$. Note that $|\text{found}(T)| = 2$.
These $v-1$ directed trades form the desired $\beta^v_{0,1}$. ∎

Theorem 4.1. For $t = 1$,

(i) $\dim \operatorname{Ker} D^v_{1,k} = k!\binom{v}{k} - v$ if $1 \leq k \leq v-1$;
$\dim \operatorname{Ker} D^v_{1,v} = v! - 1$;

(ii) there exists a semi-triangular basis $\beta^v_{1,2}$ of directed trades for $\operatorname{Ker} D^v_{1,2}$, which satisfies the assumptions of Theorem 3.2. .

Proof. (i) The first equation follows from Theorem 3.2 and Theorem 4.0. Also one may see it by applying a suitable permutation on the columns of $D^v_{1,k}$, which may be represented as follows:

$$D^v_{1,k} = \left[\begin{array}{c|c} W^v_{1,k} & C \end{array}\right].$$

Since for $k \leq v-1$, $W^v_{1,k}$ is full rank, therefore $D^v_{1,k}$ is full rank. For the second equation, we note that $D^v_{1,v} = J$ is of size $v \times v!$. Thus $\dim \operatorname{Ker} D^v_{1,v} = v! - 1$.
(ii) We let $\beta^v_{1,2} = \beta'_{1,2} \cup \beta''_{1,2}$, where $\beta''_{1,2}$ contains $(v, 2, 1)$ strictly directed trades as follows

$$\begin{array}{cc} T_1 & T_2 \\ \hline xy & yx \end{array}$$

for $1 \leq x < y \leq v$. So we have $|\text{found}(T)| = 2$, and $|\beta''_{1,2}| = \binom{v}{2}$.

If $v \geq 4$, $\beta'_{1,2}$ contains the directed trades which were introduced at the end of Section 2. And $|\beta'_{1,2}| = \binom{v}{2} - \binom{v}{1}$. Thus $|\beta_{1,2}| = 2!\binom{v}{2} - v$, and proof is complete. ∎

Example 1. A basis for $\operatorname{Ker} D^4_{1,2}$.

$\beta''_{1,2}$ consists of the following strictly directed trades:

$$\begin{array}{cc|cc|cc|cc|cc|cc} T_1 & T_2 & T_1 & T_2 & T_1 & T_2 & T_1 & T_2 & T_1 & T_2 & T_1 & T_2 \\ \hline 12 & 21 & 13 & 31 & 14 & 41 & 23 & 32 & 24 & 42 & 34 & 43 \end{array}$$

and $\beta'_{1,2}$ consists of the following directed trades,

$$\begin{array}{cc|cc} T_1 & T_2 & T_1 & T_2 \\ \hline 21 & 31 & 31 & 32 \\ 43 & 42 & 42 & 41 \end{array}$$

A representation of the elements of $\beta_{1,2}$ as vectors is given below.

$$
\begin{array}{c}
12\\ 13\\ 14\\ 21\\ 23\\ 24\\ 31\\ 32\\ 34\\ 41\\ 42\\ 43
\end{array}
\begin{pmatrix}
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
-1 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & -1 & & -1 & & & 1 & \\
 & & & & -1 & & -1 & \\
 & & & & & & & 1 \\
 & & -1 & & & & -1 & \\
 & & & -1 & & -1 & 1 & \\
 & & & 1 & & & & -1
\end{pmatrix}
$$

Theorem 4.2. For $t = 2$

$$
(i)\ \dim \operatorname{Ker} D^v_{2,k} = \begin{cases} k!\binom{v}{k} - 2!\binom{v}{2} & \text{if } 2 \le k \le v-2, \\ v! - \binom{v+1}{2} & \text{if } k = v-1, \\ v! - \binom{v}{2} - 1 & \text{if } k = v; \end{cases}
$$

(*ii*) there exists a semi-triangular basis $\beta^v_{2,3}$ of directed trades which satisfies the assumptions of Theorem 3.2.

Proof. (i) The equations in (i) may be obtained from Theorem 3.2 and Theorem 4.1. Also we may obtain the first equation by a suitable permutation on the columns of $D^v_{2,k}$, which may result as follows:

$$
D^v_{2,k} = \left[\begin{array}{c|c} W^v_{2,k} & \mathrm{C} \\ \hline 0 & W^v_{2,k} \end{array}\right].
$$

Since for $k \le v-2$, $W^v_{2,k}$ is full rank, thus $D^v_{2,k}$ is full rank.

(ii) We let $\beta^v_{2,3} = \beta'_{2,3} \cup \beta''_{2,3}$, where $\beta''_{2,3}$ contains strictly directed trades as follows.

For each $x, y, z \in V$ (such that $x < y < z$), we have the following $(v,3,2)$DTs of volume 2 and with foundation size 3 or 4, in each of which the first block is a starting block. And these are the maximum possible numbers of such trades.

T_1	T_2
xyz	yxz
zyx	zxy

T_1	T_2
xzy	yxz
yzx	zxy

T_1	T_2
yxz	yzx
wzx	wxz
$(y < w \le v,$	$w \ne z)$

T_1	T_2
yzx	zyx
zyw	yzw
$(x < w \le v,$	$w \ne y, z)$

T_1	T_2
zxy	zyx
wyx	wxy
$(z < w \le v)$	

And we have $|\beta''_{2,3}| = 5\binom{v-1}{3} + 4\binom{v-2}{2} + 3(v-3) + 2$.

If $v \geq 6$. $\beta'_{2,3}$ contains the directed trades which were introduced at the end of Section 2. Each directed trade in $\beta'_{2,3}$ has foundation 6, and $|\beta'_{2,3}| = \binom{v}{3} - \binom{v}{2}$. By a simple computation, the total number of directed trades obtained is equal to dim Ker $D^v_{2,3}$. ■

Theorem 4.3. For $t = 3$,

(*i*) $$\dim \operatorname{Ker} D^v_{3,k} = \begin{cases} k!\binom{v}{k} - 3!\binom{v}{3} & \text{if } 3 \leq k \leq v-3, \\ \dim \operatorname{Ker} D^v_{2,k} - |\beta''_{2,3}| & \text{if } v-2 \leq k \leq v-1; \end{cases}$$

(*ii*) there exists a semi-triangular basis $\beta^v_{3,4}$ of directed trades which satisfies the assumption of Theorem 3.2.

Proof. (i) The equations in (i) may be obtained by Theorem 3.2 and Theorem 4.2.

For (ii), let $\beta_{3,4} = \beta'_{3,4} \cup \beta''_{3,4}$, where $\beta''_{3,4}$ contains strictly directed trades as follows.

For each $x, y, z, w \in V$ (such that $x < y < z < w$), we have the following $(v, 4, 3)$DTs of volume 2 or 4, and with foundation size 4, 5 or 6, in each of which the first block is a starting block. And these are the maximum possible numbers of such trades.

T_1	T_2
$xzwy$	$zxwy$
$wzxy$	$wxzy$
$yzxw$	$yxzw$
$ywxz$	$ywzx$

T_1	T_2
$xzyw$	$xzwy$
$zxwy$	$zxyw$

T_1	T_2
$xywz$	$yxwz$
$wyxz$	$wxyz$
$zyxw$	$zxyw$
$zwxy$	$zwyx$

T_1	T_2
$xyzw$	$xywz$
$yxwz$	$yxzw$

T_1	T_2
$xwyz$	$xwzy$
$wxzy$	$wxyz$

T_1	T_2
$ywxz$	$ywzx$
$wyzx$	$wyxz$

T_1	T_2
$xwyz$	$xwzy$
$wxzy$	$wxyz$

T_1	T_2
$zwxy$	$zwyx$
$wzyx$	$wzxy$

T_1	T_2
$yxwz$	$ywxz$
αwxz	αxwz
$zywx$	$zyxw$
$z\alpha xw$	$z\alpha wx$
$(z < \alpha < w$	or $y < \alpha < z$ or $w < \alpha \leq v)$

T_1	T_2
$yxzw$	$yzxw$
αzxw	αxzw
$wyzx$	$wyxz$
$w\alpha xz$	$w\alpha zx$
$(z < \alpha < w$	or $y < \alpha < z$ or $w < \alpha \leq v)$

T_1	T_2
$xwzy$	$zxwy$
$zwxy$	$wxzy$
$yzxw$	$yxwz$
$ywxz$	$yzwx$

T_1	T_2
$ywzx$	$wyzx$
$zwyx$	$zywx$
$wyz\alpha$	$ywz\alpha$
$zyw\alpha$	$zwy\alpha$
$(y < \alpha < z$ or	$x < \alpha < y$
$w < \alpha \leq v$ or	$z < \alpha < w)$

T_1	T_2
$yzwx$	$ywzx$
$ywz\alpha$	$yzw\alpha$
$wzxy$	$zwxy$
$zw\alpha y$	$wz\alpha y$
$(z < \alpha < w$ or	$x < \alpha < y$
$y < \alpha < z$ or	$z < \alpha \leq v)$

T_1	T_2
$zywx$	$zwyx$
$zwy\alpha$	$zyw\alpha$
$\alpha' wyx$	$\alpha' ywx$
$\alpha' yw\alpha$	$\alpha' wy\alpha$
$(z < \alpha < w$ or	$x < \alpha < y$
$y < \alpha < z$ or	$w < \alpha \leq v$
$w < \alpha' \leq v$ or	$z < \alpha' < w)$

T_1	T_2
$zyxw$	$zywx$
$z\alpha wx$	$z\alpha xw$
$\alpha' ywx$	$\alpha' yxw$
$\alpha' \alpha xw$	$\alpha' \alpha wx$
$(z < \alpha < w$	or $y < \alpha < z$
	or $w < \alpha \leq v$
$w < \alpha' \leq v$	or $z < \alpha' < w)$

T_1	T_2
$wyxz$	$wyzx$
$w\alpha zx$	$w\alpha xz$
$\alpha' yzx$	$\alpha' yxz$
$\alpha' \alpha xz$	$\alpha' \alpha zx$
$(z < \alpha < w$	or $y < \alpha < z$
	or $w < \alpha \leq v$
	$w < \alpha' \leq v)$

T_1	T_2
$zwyx$	$wzyx$
$wz\alpha x$	$zw\alpha x$
$wzy\alpha'$	$zwy\alpha'$
$zw\alpha\alpha'$	$wz\alpha\alpha'$
$(z < \alpha < w$ or	$y < \alpha < z$
	$w < \alpha \leq v$
$y < \alpha' < z$ or	$x < \alpha' < y$
$w < \alpha' \leq v$ or	$z < \alpha' < w)$

T_1	T_2
$zxwy$	$zyxw$
$zywx$	$zwxy$
αyxw	αxwy
αwxy	αywx
$(w < \alpha \leq v$ or	$z < \alpha < w)$

T_1	T_2
$zxyw$	$zyxw$
$zwyx$	$zwxy$
αyxw	αxyw
αwxy	αwyx
$(w < \alpha \leq v$ or	$z < \alpha < v)$

T_1	T_2
$wxzy$	$wyxz$
$wyzx$	$wzxy$
αyxz	αxzy
αzxy	αyzx
$(w < \alpha \leq v)$	

T_1	T_2
$wxyz$	$wyxz$
$wzyx$	$wzxy$
αyxz	αxyz
αzxy	αzyx
$(w < \alpha \leq v\ )$	

T_1	T_2
$wzxy$	$wzyx$
$w\alpha yx$	$w\alpha xy$
$\alpha' zyx$	$\alpha' zxy$
$\alpha' \alpha xy$	$\alpha' \alpha yx$
$(w < \alpha \leq v$ or	$z < \alpha < w$
$w < \alpha' \leq v)$	

T_1	T_2
$wyzx$	$wzyx$
$wzy\alpha$	$wyz\alpha$
$\alpha' zyx$	$\alpha' yzx$
$\alpha' yx\alpha$	$\alpha' zy\alpha$
$(y < \alpha < z$ or	$x < \alpha < y$
$w < \alpha \leq v$ or	$z < \alpha < w$
$w < \alpha' \leq v)$	

We have $\mid \beta''_{3,4} \mid = 23\binom{v-2}{4} + 41\binom{v-3}{3} + 54\binom{v-4}{2} + 11(v-4) + 51(v-5) + 54$, for $v > 4$ and $|\beta''_{3,4}| = 9$, for $v = 4$.

If $v \geq 8$, $\beta'_{3,4}$ contains the directed trades which were introduced at the end of Section 2. Each directed trade in $\beta'_{3,4}$ has foundation of size 8, and $\mid \beta'_{3,4} \mid = \binom{v}{4} - \binom{v}{3}$. By a simple computation the total number of directed trades obtained in (ii) equals dim Ker $D^v_{3,4}$. ■

Theorem 4.4. For $t = 4$,

$$\text{dim Ker } D^v_{4,k} = \begin{cases} k!\binom{v}{k} - 4!\binom{v}{4} & \text{if } 4 \leq k \leq v-4, \\ \text{dim Ker } D^v_{3,k} - \mid \beta''_{3,4} \mid & \text{if } v-3 \leq k \leq v-2. \end{cases}$$

Proof. These results follow immediately from Theorem 3.2 and Theorem 4.3. ■

5 A semi-triangular basis for $N^{t+1}_{t,t+1}$

In this section we introduce a semi-triangular basis of directed trades for Ker $D^{t+1}_{t,t+1}$. This will be done by the following lemmas.

Lemma 5.1 Let $k = v = t+1$. Each $(t+1)$-tuple such as $x_1 \ldots x_m y_1 y_2 y_3 \cdots y_{t-m+1}$, where $0 \leq m \leq t$ and $y_2 < y_1 < x_i$ $(i = 1, \ldots, m)$, can not be a starting block in any strictly directed trade.

Proof. Since $v = k = t+1$, every directed trade is strict and thus, basis $\beta^{t+1}_{t,t+1}$ of integral vectors (directed trades) also consists only of strictly directed trades. Now if a block such as $b = x_1 \cdots x_m y_1 y_2 \cdots y_{t-m+1}$ where $0 \leq m \leq t$ and $y_2 < y_1 < x_i$ $(i = 1, \cdots, m)$ is a starting block in a strictly directed trade T, $T = T_1 - T_2$, then the t-tuple $x_1 \cdots x_m y_2 \cdots y_{t-m+1}$ must appear in a block of T_2, and this block is necessarily a permutation of the block b. But we see that every permutation of block b which contains the t-tuple $x_1 \cdots x_m y_2 \cdots y_{t-m+1}$ is smaller than b (in the lexicographical ordering), and this is a contradiction. ∎

We denote by Q_{t+1} the set of all $(t+1)$-tuples which satisfy the conditions of Lemma 5.1.

Lemma 5.2. We have,

$$|Q_{t+1}| = \frac{(t+1)!}{2} + \frac{t!}{2} + \sum_{m=2}^{t-2} \frac{m!}{2}(t-m-2)! \left[\binom{t+1}{m+2}(t-m-1) - \binom{t}{m+2} \right].$$

Proof. Let $Q_{t+1} = A \cup B \cup C$ where A, B, and C are defined as follows:
A consists of all of $(t+1)$-tuples, $x_1 \dots x_t y$, such that $y = 1$ and $x_1 < x_2$;
B consists of all of $(t+1)$-tuples, $y_1 \dots y_{t+1}$, such that $y_2 < y_1$;
C consists of all of $(t+1)$-tuples, $x_1 \dots x_m y_1 \dots y_{t-m+1}$, such that $2 \leq m \leq t-2$, $x_1 < x_2$ and $y_2 < y_1 < x_i$ $(i = 1, \dots, m)$.
By an easy counting argument we have,

$$|A| = \tfrac{t!}{2}, \qquad |B| = \tfrac{(t+1)!}{2}, \qquad |C| = \sum_{m=2}^{t-2} \tfrac{m!}{2}\tbinom{t+1}{m+2}(t-m-1)!,$$
$$|A \cap C| = \sum_{m=2}^{t-2} \tfrac{m!}{2}\tbinom{t}{m+2}(t-m-2)!, \qquad \text{and} \qquad |A \cap B| = |B \cap C| = 0.$$

Now by the principle of inclusion and exclusion the assertion follows. ∎

In the following we show that every element in Q'_{t+1}, the complement of the set Q_{t+1}, is a starting block in a strictly directed trade. Therefore a semi-triangular set of strictly directed trades will be produced which is maximal. Thus this set will be a basis for Ker $D^{t+1}_{t,t+1}$. First we state two lemmas from [10].

Lemma 5.3 [10]. If there exists a (v,k,t)DT of volume s, then there exists a $(v+1, k+1, t+1)$DT of volume $2s$.

Lemma 5.4 [10]. If there exists a (v,k,t)DT of volume s, then there exists a $(v+2, k+2, t+2)$DT of volume $2s$.

Lemma 5.5. Each $(t+1)$-tuple in Q'_{t+1} is a starting block in a strictly directed trade.

Proof. We proceed by induction on t. For $t = 1$ we have $Q_2 = \{21\}$, then $Q'_2 = \{12\}$. The directed trade $T = T_1 - T_2$ where

T_1	T_2
12	21

is a strictly directed trade which contains 12 as its starting block.
For $t = 2$ we have $Q_3 = \{213, 231, 312, 321\}$ and $Q'_3 = \{123, 132\}$, where 123 and 132 are starting blocks in the following directed trades.

T_1	T_2
123	213
321	312

T_1	T_2
132	213
231	312

Now suppose the theorem holds for all values less than t; we show that it holds for t also. Suppose $x_1 \cdots x_{t+1} \in Q'_{t+1}$.There are two cases: $x_1 < x_{t+1}$ or $x_{t+1} < x_1$.

Case 1: $x_1 < x_{t+1}$

If $x_1 \ldots x_t \in Q'_t$, then by the induction hypothesis there exists a strictly directed trade $(t, t, t-1)$DT which contains $x_1 \ldots x_t$ as a starting block. Then and by Lemma 5.3 there exists a $(t+1, t+1, t)$DT which contains $x_1 \cdots x_t x_{t+1}$ as a starting block.

If $x_1 \ldots x_t \notin Q'_t$, since $x_1 \ldots x_{t+1} \in Q'_{t+1}$, the only possible situation in which $x_1 \ldots x_t \notin Q'_t$, is that where $x_t = 1$. Then necessarily $x_1 \ldots x_{t-1} \in Q'_{t-1}$, and by the induction hypothesis there exists a $(t-1, t-1, t-2)$DT in which $x_1 \ldots x_{t-1}$ is a starting block. By Lemma 5.4 there exists a $(t+1, t+1, t)$DT which contains $x_1 \ldots x_{t-1} x_t x_{t+1}$ as a starting block.

Case 2: $x_{t+1} < x_1$

If $x_2 \ldots x_{t+1} \in Q'_t$, then we proceed as in the previous case.

If $x_2 \ldots x_{t+1} \notin Q'_t$, the only case which may cause trouble is that where $x_2 > x_3$. But then $x_3 \ldots x_{t+1} \in Q'_{t-1}$, and we proceed as in the previous case. ∎

6 Existence of t-(v, k, λ)SDDs

In this section we show that the obvious necessary conditions for the existence of t-(v, k, λ) signed directed designs are also sufficient provided that $t \leq 4$.

Theorem 6.1. Let $t \leq 4$ and $t, k, \lambda_t = \lambda$ be integers and $0 \leq t < k < v - t$. There exists a t-(v, k, λ)SDD if and only if

$$\lambda_i = \frac{\binom{k}{i} P^{t-i}_{v-i}}{\binom{k}{t} P^{k-t}_{v-t}} \lambda_t$$

are positive integers for $0 \leq i < t$.

Proof. First we prove of the necessity of the conditions. Let f be a t-(v, k, λ)SDD. Then by definition

$$D^v_{t,k} f = \lambda_t e_t.$$

Then

$$D^v_{i,t} D^v_{t,k} f = D^v_{i,t} \lambda_t e_t.$$

From Equation (3) we have

$$\binom{k-i}{t-i} D^v_{i,k} f = D^v_{i,t} \lambda_t e_t,$$

and therefore,

$$\binom{k-i}{t-i} D^v_{i,k} f = \lambda_t \binom{t}{i} P^{t-i}_{v-i} e_t.$$

Thus

$$\lambda_i = \frac{\binom{t}{i} P^{t-i}_{v-i}}{\binom{k-i}{t-i}} \lambda_t \quad \text{or} \quad \lambda_i = \frac{\binom{k}{i} P^{t-i}_{v-i}}{\binom{k}{t} P^{k-t}_{v-t}} \lambda_t,$$

for $0 \leq i \leq t$.

Next we prove the sufficiency of these conditions by induction on t. If $t = 0$, then λ_0 blocks (k-tuples) form a $0-(v,k,\lambda_0)$SDD. Assume that theorem holds for some $t \geq 0$, and assume that $\lambda_0 \ldots \lambda_{t+1}$ satisfy these conditions. Then by the induction hypothesis there exists a t-(v,k,λ_t)SDD, namely F_t that $D^v_{t,k} F_t = \lambda_t e_t$.

From Equation (3) we have

$$(k-t) D^v_{t,k} = D^v_{t,t+1} D^v_{t+1,k}.$$

From this we easily obtain

$$(t+1)(v-t) e_t = D^v_{t,t+1} e_{t+1}.$$

Now take $T = D^v_{t+1,k} F_t - \lambda_{t+1} e_{t+1}$. Then T is a $(v, t+1, t)$DT, because

$$\begin{aligned} D^v_{t,t+1} T &= D^v_{t,t+1} D^v_{t+1,k} F_t - \lambda_{t+1} D^v_{t,t+1} e_{t+1} \\ &= (k-t) D^v_{t,k} F_t - \lambda_{t+1}(t+1)(v-t) e_t \\ &= (k-t)\lambda_t e_t - \lambda_{t+1}(t+1)(v-t) e_t \\ &= (k-t)\lambda_t e_t - \frac{(k-t)\lambda_t}{(t+1)(v-t)}(t+1)(v-t) e_t = 0. \end{aligned}$$

Then $T \in N^v_{t,t+1}$ or $T \in \text{Ker } D^v_{t,t+1}$.

Since $t \leq s$ and $k < v-t$, then by Theorem 3.2, there exists $T' \in \text{Ker } D^v_{t,k}$, with integer components (i.e. $T' \in N^v_{t,k}$) such that $D^v_{t+1,k} T' = T$.

If $F_{t+1} = F_t - T'$, then F_{t+1} is a $(t+1)$-(v,k,λ_{t+1})SDD. For, we have $D^v_{t+1,k} F_{t+1} = D^v_{t+1,k} F_t - D^v_{t+1,k} T' = T + \lambda_{t+1} e_{t+1} - T = \lambda_{t+1} e_{t+1}$. The proof is complete. ∎

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(Received 29/2/2000)