Imbalance in tournament designs

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Abstract

We introduce two measures of imbalance, the team imbalance, and the field imbalance, in a tournament design. In addition to an exhaustive study of imbalances in tournament designs with up to eight teams, we present some bounds on the imbalances, as well as recursive constructions for homogeneous tournaments.

1. Introduction

In a (round-robin) *tournament* of $2n$ teams, each team plays each other team exactly once. The $n(2n-1)$ games are played in $2n-1$ rounds, with n games in each round; each team sees action in each round.

A schedule for a tournament is equivalent to a 1-factorization of the complete graph K_{2n}, i.e. to a partition of the edges of K_{2n} into 1-factors (i.e. perfect matchings).

A *tournament design* is a tournament, together with an assignment of games to n given *fields*; in each round, exactly one game is assigned to a field.

It is usually, but not always, desirable to strive for some sort of balance in assigning teams to play games at particular fields. It is the associated notion of *imbalance* (both *team imbalance* and *field imbalance*) that we attempt to take a closer look at in this article.

In Section 2, we provide the necessary definitions and briefly survey the known results on balanced tournament designs. In Section 3, we provide an exhaustive study and report on computer enumeration of imbalances in tournament designs with up to 8 teams. In Section 4, we provide some bounds for the imbalances as well as recursive constructions for homogeneous tournaments.

2. Tournament designs

A *tournament design* (TD) is a quadruple $(V, \mathcal{F}, P, \alpha)$ where V is a $2n$-element set whose elements are *teams*, $\mathcal{F} = \{F_1, \ldots, F_{2n-1}\}$ is a set of 1-factors such that $(V, \mathcal{F})$ is a 1-factorization of K_{2n}, and $\alpha = (\alpha_1, \ldots, \alpha_{2n-1})$ is the *field assignment*, i.e. a set whose elements are mappings; the mapping $\alpha_i : F_i \to P$ maps the n 2-subsets of F_i onto the set of fields P.

The elements of $\mathcal{F}$ are called *rounds*; the elements of P (fields) are usually denoted by $P_1, P_2, \ldots, P_n$ (or by $A, B, C, \ldots$) and the elements of V (teams) are usually denoted by $T_1, T_2, \ldots, T_{2n}$ (or sometimes just by integers $1, 2, \ldots, 2n$).

The *appearance matrix* of a tournament design TD(n) is an $n \times 2n$ matrix $A = (a_{ij})$ where the entry a_{ij} is the number of times the team T_j plays on the field P_i. The row sums of the appearance matrix of a TD(n) all equal $4n - 2$ while the column sums all equal $2n - 1$.

We will represent a tournament design $\mathcal{T}$ by an $n \times (2n-1)$ array whose rows are indexed by the fields and whose columns are indexed by the rounds, and whose entry in the row i and the column j is the pair of teams playing in round F_j on the field P_i.

From the appearance matrix A, we can define $I_T(j)$, the *team imbalance* of the team T_j:

$$I_T(j) = \max_{i,k}\{|a_{ij} - a_{kj}| : i, k \in \{1, \ldots, n\}\}$$

and $I_F(i)$, the *field imbalance* of the field P_i:

$$I_F(i) = \max_{j,l}\{|a_{ij} - a_{il}| : j, l \in \{1, \ldots, n\}\}.$$

The (total) team imbalance $IT(\mathcal{T})$ of the tournament design $\mathcal{T}$, and the (total) field imbalance $IF(\mathcal{T})$ of $\mathcal{T}$ are defined respectively by

$$IT(\mathcal{T}) = \sum_{j=1}^{2n} I_T(j), \ IF(\mathcal{T}) = \sum_{i=1}^{n} I_F(j).$$

A TD*(n) is a TD(n) whose appearance matrix contains no zeros.

Example 1. Consider the following tournament design $\mathcal{T}$ with 6 teams 1,2,3,4, 5,6, and 3 fields A, B, C:

	F_1	F_2	F_3	F_4	F_5
A	16	26	36	46	14
B	25	13	24	35	23
C	34	45	15	12	56

The appearance matrix of this tournament is

	1	2	3	4	5	6	
A	2	1	1	2	0	4	4
B	1	3	3	1	2	0	3
C	2	1	1	2	3	1	2
	1	2	2	1	3	4	

The last (appended) row in the above appearance matrix is the vector ($I_T(1)$, $I_T(2), I_T(3), I_T(4), I_T(5), I_T(6)$) of team imbalances, while the rightmost (appended) column is the vector $(I_F(A), I_F(B), I_F(C))^T$ of field imbalances. The (total) team imbalance $IT(\mathcal{T}) = 13$, while the (total) field imbalance $IF(\mathcal{T}) = 9$.

It is easily seen that in any tournament design $\mathcal{T}$, $I_T(j) \geq 1$ for any j, and $I_F(i) \geq 1$ for any i, and thus $IT(\mathcal{T}) \geq 2n$ and $IF(\mathcal{T}) \geq n$. It is also easily seen that $IT(\mathcal{T}) = 2n$ implies $IF(\mathcal{T}) = n$, and vice versa. A tournament design $\mathcal{T}$ with $IT(\mathcal{T}) = 2n$ (and thus $IF(\mathcal{T}) = n$) is said to be *balanced.* Balanced tournament designs (BTDs) have been introduced in [GO], and their existence settled in [SVV]. Since then, many articles have been devoted to BTDs, often satisfying additional properties (see, e.g. [LV], [C], [H], [F], [L]).

However, we want to take a look at tournament designs with larger than minimum imbalance, including those with the largest possible imbalance. The motivation for considering such TDs comes from practical considerations (just as is the case for BTDs): for instance, in a tournament, the home team may want to play each of its games at the field accomodating the largest number of spectators; another team may require avoiding playing on a specified field altogether, etc.

In a tournament design, we may associate with each field P_i a graph G_i (the *field graph*) with $2n$ vertices and $2n-1$ edges; the vertices are the teams $T_1, \ldots, T_{2n}$, and T_jT_k is an edge of G_i if T_j and T_k play their match on the field P_i (in some one of the $2n-1$ rounds). Clearly, the row of the appearance matrix A corresponding to P_i is the degree sequence of G_i.

A tournament design is *field-homogeneous* if for any two rows R_j, R_k of the appearance matrix A there exists a permutation matrix Q such that $R_jQ = R_k$ (i.e. the field graphs corresponding to P_j and P_k have the same degree sequences). Similarly, a tournament design is *team-homogeneous* if for any two columns C_m, C_l of the appearance matrix A there exists a permutation matrix Q' such that $Q'C_m = C_l$.

A (field-homogeneous) tournament is said to be *field-uniform* if for any two fields P_i, P_j, the corresponding field graphs G_i, G_j are isomorphic. Clearly, every balanced tournament design is necessarily both, field-homogeneous and team-homogeneous. But not every field-homogeneous tournament design is field-uniform. One important class of field-uniform BTDs that has been studied by several authors are Hamiltonian BTDs (HBTDs), i.e. those in which every field graph is the Hamiltonian path. There exists no HBTD for $n = 2$ or $n = 3$. Horton [H] proved that there exists a Hamiltonian BTD(n) for all $n \geq 1$, n not divisible by 2,3, or 5.

Another field-uniform and team-homogeneous TD is given in Example 2.

Example 2.

	F_1	F_2	F_3	F_4	F_5
A	16	26	24	46	14
B	25	13	15	12	23
C	34	45	36	35	56

The appearance matrix of this tournament design is

	1	2	3	4	5	6	
A	2	2	0	3	0	3	3
B	3	3	2	0	2	0	3
C	0	0	3	2	3	2	3
	3	3	3	3	3	3	

Here each field graph is $K_4 - e$. We have $IT(\mathcal{T}) = 18, IF(\mathcal{T}) = 9$.

On the other hand, not every field-homogeneous (or even field-uniform) TD is team-homogeneous, as witnessed by Example 3.

Example 3.

	F_1	F_2	F_3	F_4	F_5
A	16	26	36	46	14
B	25	13	24	12	23
C	34	45	15	35	56

The appearance matrix of this tournament design is

	1	2	3	4	5	6	
A	2	1	1	2	0	4	4
B	2	4	2	1	1	0	4
C	1	0	2	2	4	1	4
	1	4	1	1	4	4	

Each field graph is isomorphic to the "dragon" (i.e. a triangle, with further two pendant edges attached to one of its vertices), so this TD is field-uniform – but it is not team-homogeneous. We have $IT(\mathcal{T}) = 15$, $IF(\mathcal{T}) = 12$.

There exist further examples of tournament designs TD(3) which are field-uniform but not team homogeneous, with field graphs isomorphic to G_1 and to G_2, respectively, where G_1 is the "bull", and G_2 is the graph "E", i.e. the tree obtained by appending a pendant edge to the central vertex of a path with four edges.

The ultimate aim of this study would be to determine, for each n, the spectrum for pairs $(IT(\mathcal{T}), IF(\mathcal{T}))$ where $\mathcal{T}$ runs through all tournament designs TD(n). In the next section, we report on the results of a computer determination of this spectrum when $n = 6$ and $n = 8$. However, it appears that to determine this spectrum for arbitrary n is too ambitious an undertaking at present.

3. Tournament designs with up to 8 teams

For any 1-factorization $(V, \mathcal{F})$ of K_{2n}, we define the *team imbalance spectrum* $S_T(V, \mathcal{F})$ and the *field imbalance spectrum* $S_F(V, \mathcal{F})$ as follows:

$S_T(V, \mathcal{F}) = \{IT(\mathcal{T}) : \mathcal{T} = (V, \mathcal{F}, P, \alpha) \text{ is a TD}\}$

$S_F(V, \mathcal{F}) = \{IF(\mathcal{T}) : \mathcal{T} = (V, \mathcal{F}, P, \alpha) \text{ is a TD}\}$.

The team imbalance spectrum $S_T(n)$ and the field imbalance spectrum $S_F(n)$ are defined as

$S_T(n) = \bigcup S_T(V, \mathcal{F})$, $S_F(n) = \bigcup S_F(V, \mathcal{F})$

where the union is taken over all 1-factorizations $(V, \mathcal{F})$ of K_{2n}.

Similarly, the (combined) imbalance spectrum $S(V, \mathcal{F})$ is the set of ordered pairs:

$S(V, \mathcal{F}) = \{(IT(\mathcal{T}), IF(\mathcal{T})) : \mathcal{T} = (V, \mathcal{F}, P, \alpha) \text{ is a TD}(n)\}$.

The *imbalance spectrum* $S(n)$ of n (n a positive integer) is then defined as

$$S(n) = \bigcup S_n(V, \mathcal{F})$$

where the union is taken over all 1-factorizations $(V, \mathcal{F})$ of K_{2n}. The imbalance spectra $S^*(V, \mathcal{F})$ and $S^*(n)$ for TD$^*(n)$ are defined similarly.

It is well known that there exists a unique 1-factorization of K_{2n} when $n = 2$ or $n = 3$. It is easily seen that there exists, up to an isomorphism, a unique tournament design TD(2) whose appearance matrix is

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix}.$$

Moreover, this tournament design has $IT(\mathcal{T}) = 6$, $IF(\mathcal{T}) = 4$; it is neither a TD*, nor Hamiltonian, nor is it field- or team- homogeneous.

But already the next case $n = 3$ is more complicated, in spite of the fact that the 1-factorization of K_6 is unique. The imbalance spectrum for $n = 3$ is given in Table 1. In Table 1 (and in Table 2 below, the symbol + in row i and column j indicates $(i, j) \in S(n)$, while the symbol * indicates $(i, j) \in S^*(n)$; blank entry indicates that $(i, j) \notin S(n)$.

	3	4	5	6	7	8	9	10	11	12
6	*									
7										
8			*							
9				*						
10						+				
11					+	+				
12					+		+			
13							+			
14					+	+	+	+		
15							+		+	+
16								+		
17										
18							+			

Table 1.

There are 21 essentially different tournament designs TD(3) of which three are TD*(3) (one of the latter is the balanced TD). Of the five TD(3) that are field-uniform, two are also team-homogeneous (besides the unique balanced TD, it is the TD in Example 2). There is no Hamiltonian BTD(3), as is well known.

For $n = 4$, we used a simple backtracking algorithm to determine the imbalance spectrum for each of the 6 nonisomorphic 1-factorizations of K_8. The individual spectra $S(V, \mathcal{F}), S^*(V, F)$ are all distinct, although those for the 1-factorizations $\mathcal{F}_3, \mathcal{F}_5$ and $\mathcal{F}_6$ (numbering as in [W], p.88) are similar, and that for $\mathcal{F}_2$ somewhat similar. On the other hand, Corriveau [C] has already established that 1-factorizations $\mathcal{F}_1$ (the Steiner 1-factorization) and $\mathcal{F}_4$ do not underlie a TD(4) [and thus $(8,4) \notin S(\mathcal{F}_1) \cup S(\mathcal{F}_4)$], while the other four 1-factorizations do. Interestingly enough, the Steiner 1-factorization which has automorphism group of order 1344 – by far the largest order from among the 6 nonisomorphic 1-factorizations of K_8 – has the smallest size imbalance spectra: $|S(\mathcal{F}_1)| = 153$, $|S^*(\mathcal{F}_1)| = 11$. By contrast, $|S(\mathcal{F}_5)| = 226, |S^*(\mathcal{F}_5)| = 42$. Not only does the Steiner 1-factorization $\mathcal{F}_1$ not underlie a TD(4); it turns out that the smallest possible team imbalance in a TD(4) that $\mathcal{F}_1$ admits is 11, while the smallest field imbalance is 8 (these can be attained simultaneously, i.e. $(11,8) \in S(\mathcal{F}_1)$). Moreover, the "smallest" element of $S^*(\mathcal{F}_1)$ is (12,8). Compare this with $(IT(\mathcal{T}), IF(\mathcal{T})) = (8,4)$ whenever $\mathcal{T}$ is a BTD(4).

Table 2 depicts the imbalance spectrum for $n = 4$. We omit listing the imbalance spectra for the individual 1-factorizations of K_8. However, we note, that, for example, the spectra of all six 1-factorizations contain the pairs (32,18) and (28,22), so that the largest team imbalance (32), as well as the largest field imbalance (22) may be attained in a TD(4) with any underlying 1-factorization.

In addition to the 47 nonisomorphic BTD(4) including the 18 Hamiltonian BTDs determined by Corriveau [C], we found many interesting field- and team-homogeneous TD(4).

Example 4 below shows a field- and team-homogeneous tournament design for 8 teams 0,1,2,...,7 and 4 fields A, B, C, D which not only is not field-uniform but, in fact, has the property that no two field graphs are isomorphic.

Example 4. Tournament design array

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
A	01	02	03	17	46	23	25
B	27	13	15	04	05	57	16
C	36	47	24	35	12	06	34
D	45	56	67	26	37	14	07

Appearance matrix

	0	1	2	3	4	5	6	7	
A	3	2	3	2	1	1	1	1	2
B	2	3	1	1	1	3	1	2	2
C	1	1	2	3	3	1	2	1	2
D	1	1	1	1	2	2	3	3	2
	2	2	2	2	2	2	2	2	

$IT(\mathcal{T}) = 16, \ IF(\mathcal{T}) = 8.$

	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
8	*																		
9		*																	
10		*	*	+															
11		*	*	*	+	+													
12			*	*	*	+	+												
13				*	*	*	+	+	+										
14				*	*	*	*	+	+										
15				*	*	*	*	*	+	+									
16				*	*	*	*	*	*	+	+								
17					*	*	*	*	*	+	+	+							
18				+	+	*	*	*	*	+	+	+	+						
19					+	+	*	*	*	+	+	+	+	+					
20					+	+	*	*	*	+	+	+	+	+	+				
21						+	+	+	*	+	+	+	+	+	+				
22						+	+	+	+	+	+	+	+	+	+	+			
23							+	+	+	+	+	+	+	+	+	+	+		
24							+	+	+	+	+	+	+	+	+	+	+	+	
25								+	+	+	+	+	+	+	+	+	+	+	
26									+	+	+	+	+	+	+	+	+	+	+
27									+	+	+	+	+	+	+	+	+	+	+
28										+	+	+	+	+	+	+	+	+	+
29											+	+	+	+	+	+	+	+	
30												+	+	+	+	+	+	+	
31													+	+	+	+	+		
32															+				

Table 2. The imbalance spectrum for $n = 4$

While the underlying 1-factorization of the TD(4) in Example 4 is $\mathcal{F}_6$ (i.e. GK_8), field- and team-homogeneous tournament designs $\mathcal{T}$ with $(IT(\mathcal{T}), IF(\mathcal{T})) = (16, 8)$ exist with the underlying 1-factorization being any of the 6 nonisomorphic 1-factorizations of K_8.

On the other hand, field- and team-homogeneous TD(4) with $(IT(\mathcal{T}), IF(\mathcal{T})) = (24, 12)$, and $I_F(i) = I_T(j) = 3$ $(i = 1, 2, 3, 4; j = 0, 1, \ldots, 7)$ exist only if the underlying 1-factorization is $\mathcal{F}_3$ or $\mathcal{F}_4$. The corresponding TDs are given in Example 5 (neither is field-uniform).

Example 5. TD(4) (underlying 1-factorization: $\mathcal{F}_3$).

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
A	01	02	03	16	17	27	36
B	23	13	12	04	05	35	24
C	45	46	47	25	26	06	07
D	67	57	56	37	34	14	15

Appearance matrix

	0	1	2	3	4	5	6	7	
A	3	3	2	2	0	0	2	2	3
B	2	2	3	3	2	2	0	0	3
C	2	0	2	0	3	2	3	2	3
D	0	2	0	2	2	3	2	3	3
	3	3	3	3	3	3	3	3	

(b) TD(4) (underlying 1-factorization: $\mathcal{F}_4$):

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
A	01	02	03	16	17	37	36
B	23	13	12	04	05	14	24
C	45	46	47	27	26	25	15
D	67	57	56	35	34	06	07

Appearance matrix

	0	1	2	34	4	5	6	7	
A	3	3	1	3	0	0	2	2	3
B	2	3	3	2	3	1	0	0	3
C	0	1	3	0	3	3	2	2	3
D	2	0	0	2	1	3	3	3	3
	3	3	3	3	3	3	3	3	

In some contrast to Example 5 is the field-uniform (but not team-homogeneous) TD*(4) given in Example 6.

Example 6. TD*(4) (underlying 1-factorization: $\mathcal{F}_4$).

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
A	01	02	47	04	34	06	15
B	23	13	56	27	05	25	24
C	45	46	03	16	26	37	36
D	67	57	12	35	17	14	07

Appearance matrix

	0	1	2	3	4	5	6	7	
A	4	2	1	1	3	1	1	1	3
B	1	1	4	2	1	3	1	1	3
C	1	1	1	3	2	1	4	1	3
D	1	3	1	1	1	2	1	4	3
	3	2	3	2	2	2	3	3	

$(IT(\mathcal{T}), IF(\mathcal{T})) = (20, 12)$.

Our final example in this section displays a TD(4) with the largest possible team imbalance. While this particular TD has as its underlying 1-factorization the Steiner 1-factorization $\mathcal{F}_1$, each of the 6 nonisomorphic 1-factorizations underlies a TD with exactly the same appearance matrix, and the same field graphs.

Example 7. TD(4) (underlying 1-factorization: $\mathcal{F}_1$)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
A	01	02	03	04	05	06	07
B	23	13	12	15	14	35	34
C	45	46	47	26	27	24	25
D	67	57	56	37	36	17	16

Appearance matrix

	0	1	2	3	4	5	6	7	
A	7	1	1	1	1	1	1	1	6
B	0	4	2	4	2	2	0	0	4
C	0	0	4	0	4	2	2	2	4
D	0	2	0	2	0	2	4	4	4
	7	4	4	4	4	1	4	4	

$(IT(\mathcal{T}), IF(\mathcal{T})) = (32, 18)$.

Table 3 summarizes our computational results about TD(4). Here Γ is the order of the automorphism group of the underlying 1-factorization.

1-factorization	Γ	$\|S\|$	$\|S^*\|$	BTD
$\mathcal{F}_1$(=Steiner)	1344	153	11	No
$\mathcal{F}_2$	64	210	34	Yes
$\mathcal{F}_3$	16	224	39	Yes
$\mathcal{F}_4$	96	215	37	No
$\mathcal{F}_5$	24	226	42	Yes
$\mathcal{F}_6$(=GK$_8$)	42	225	41	Yes

Table 3.

4. Some general results

The existence of a balanced TD(n) for all $n \geq 3$ implies $(2n, n) \in S(n)$ [in fact, $(2n, n) \in S^*(n)$] for all $n \geq 3$. It is also easy to see that $(4n-2, 4n-4) \in S(n)$ for all $n \geq 2$. This particular imbalance pair arises if in the 1-factorization GK_{2n} all games between pairs of teams with fixed distance d in Z_{2n-1} (as well as those with infinite distance) are assigned to the same field. The resulting appearance matrix is of the form

$$\begin{bmatrix} 2n-1 & 1 & 1 & \ldots & 1 \\ 0 & 2 & 2 & \ldots & 2 \\ 0 & 2 & 2 & \ldots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 2 & 2 & \ldots & 2 \end{bmatrix}$$

Note that many 1-factorizations not isomorphic to GK_{2n} also yield a TD with the same appearance matrix.

How large can the team (field) imbalance be? In the next theorem we construct a TD(n) with large imbalances provided n is even.

Theorem 1. *For every even n there exist a TD(n) $\mathcal{T}$ with $IT(\mathcal{T}) = n(2n-1)$ and $IF(\mathcal{T}) = \frac{n}{2}(3n-1) - 1$.*

Proof. We construct a TD($2m$) for $4m$ teams $T_0, T_1, \ldots, T_{4m-1}$ and $2m$ fields $P_0, P_1, \ldots, P_{2m-1}$ whose appearance matrix $A = [\,A_1 | A_2\,]$, where

$$A_1 = \begin{bmatrix} 4m-1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 4m-2 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 0 & 4m-3 & \ldots & 1 & 1 & 1 \\ \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0 & 0 & 0 & \ldots & 2m+2 & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 2m+1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 2m \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 2 & 2 & 1 & \ldots & 1 & 1 & 1 \\ 2 & 2 & 3 & \ldots & 1 & 1 & 1 \\ \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 2 & 2 & 2 & \ldots & 2m-2 & 1 & 1 \\ 2 & 2 & 2 & \ldots & 2 & 2m-1 & 1 \\ 2 & 2 & 2 & \ldots & 2 & 2 & 2m \end{bmatrix}.$$

To construct such a TD, we proceed in two steps. First we schedule two disjoint subtournaments, one for the teams $T_0, T_1, \ldots, T_{2m-1}$ (the *lower subtournament*), and another one for the teams $T_{2m}, T_{2m+1}, \ldots, T_{4m-1}$ (the *upper subtournament*). Our aim is to schedule the subtournaments in such a way that after the $2m-1$ rounds of both subtournaments, the partial appearance matrix A' will be $A' = [\,A'_1 | A'_2\,]$ where

$$A'_1 = \begin{bmatrix} 2m-1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 2m-2 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 0 & 2m-3 & \ldots & 1 & 1 & 1 \\ \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0 & 0 & 0 & \ldots & 2 & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

$$A'_2 = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 2 & \ldots & 0 & 0 & 0 \\ \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 1 & 1 & 1 & \ldots & 2m-3 & 0 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 2m-2 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 2m-1 \end{bmatrix}.$$

In the second step, schedule the remaining rounds in which all games T_i vs. T_j will be played, with $i \in \{0, 1, \ldots, 2m-1\}$ and $j \in \{2m, 2m+1, \ldots, 4m-1\}$. These rounds will be scheduled in such a way that the corresponding partial appearance matrix M will be

$$M = [2mI|J]$$

where I is the identity matrix of order $2m$, and J is a matrix of all 1's of order $2m$. Clearly, we will have then $A = A' + M$.

In order to implement step one, we define a mapping π from the set of $4m$ teams into the set of $2m$ fields which assigns to every team its *preferred field.* With a few exceptions, each team will play more games at its preferred field than at any other field. The mapping π is given by

$$\pi(T_j) = P_j \ if \ j \in \{0, 1, \ldots, 2m-1\}$$

$$\pi(T_j) = P_{j-2m} \ if \ j \in \{2m, 2m+1, \ldots, 4m-1\}.$$

The two subtournaments with $2m$ teams will have GK_{2m} as the underlying 1-factorization. In the lower subtournament, the vertex T_0 is used as the fixed (or infinite) point; denote the round in which the game T_0 vs. T_s is played as round s. Every game in the lower subtournament is scheduled to the preferred field of the team with the smaller subscript: if the game T_a vs. T_b is played in round s and $a < b \leq 2m-1$ then it is scheduled to the field $\pi(T_a) = P_a$.

In the upper subtournament, the vertex T_{4m-1} is used as the fixed (or infinite) point, and round s is the round in which the game T_{4m-1} vs. T_{3m+s-1} is played. Every game in the upper subtournament is scheduled to the preferred field of the team with the larger subscript: if the game T_a vs. T_b is played in round s and $2m \leq a < b$ then it is scheduled to the field $\pi(T_b) = P_{b-2m}$.

One can see that in every round of the subtournaments there is exactly one game scheduled to each field. It follows from the construction that the teams T_0 and T_{4m-1} play all their games at their respective preferred fields. In general, the team T_j [T_{4m-j-1}, respectively], $0 \leq j \leq 2m-1$, plays $2m-1-j$ games at its preferred field and one game at each of the fields $P_0, P_1, \ldots, P_{j-1}$ [$P_{4m-j}, P_{4m-j-1}, \ldots, P_{4m-1}$, respectively]. Every field P_i is the preferred field of exactly two teams, namely T_i and T_{2m+i}, and so the partial appearance matrix after $2m-1$ rounds is precisely the matrix A'. Up to this point, the teams T_{2m-1} and T_{2m} have played no games at their respective preferred fields.

In the second step of our construction, we now use any 1-factorization of the complete bipartite graph $K_{2m,2m}$ (in effect, a latin square of order $2m$) with $V_1 = \{T_0, T_1, \ldots, T_{2m-1}\}$ and $V_2 = \{T_{2m}, T_{2m+1}, \ldots, T_{4m-1}\}$ as partite sets. Every game will now be scheduled to the preferred field of the team from V_1. The partial appearance matrix is clearly the matrix M.

All that remains to be done in order to complete the proof is to calculate the respective imbalances. The arithmetic presents no difficulties. □

We conclude this section with two recursive constructions for homogeneous tournament designs.

Theorem 2. *Suppose $\mathcal{T}$ is a field-homogeneous (team-homogeneous, respectively) $TD(n)$ with $(IT(\mathcal{T}), IF(\mathcal{T})) = (r, s)$, and suppose there exists a pair of orthogonal* *latin squares of order n. Then there exists a field-homogeneous (team-homogeneous, respectively) TD$(2n)$ $\mathcal{T}'$ with $(IT(\mathcal{T}'), IF(\mathcal{T}')) = (r', s')$ where $r' = 2r$ or $2r + 4n$, and $s' = 2s$ or $2s + 2n$.*

Proof. If A is the appearance matrix of $\mathcal{T}$, construct a TD$(2n)$ $\mathcal{T}'$ having appearance matrix B of the form

$$B = \begin{bmatrix} A & J \\ J & A \end{bmatrix}$$

as follows: schedule two identical subtournaments isomorphic to $\mathcal{T}$, one for the teams $T_1, \ldots, T_{2n}$, the other for the teams $T_{2n+1}, \ldots, T_{4n}$; all $2n-1$ rounds of the first subtournament are played on fields $P_1, \ldots, P_n$, while those of the second subtournament are played on fields $P_{n+1}, \ldots, P_{2n}$. The remaining $2n$ rounds of $\mathcal{T}'$ are then scheduled as follows: let $L = (l_{ij}), M = (m_{ij})$ be two MOLS(n), with the elements of L being $1, \ldots, n$, and the elements of M being $n+1, \ldots, 2n$. The game T_i vs. T_j, $i \in \{1, \ldots, n\}, j \in \{n+1, \ldots, 2n\}$ is then scheduled in round s to the field P_r if $(l_{rs}, m_{rs}) = (i, j)$. The orthogonality of L and M ensures that during the last $2n$ rounds, each team plays exactly one game on each field, and thus the appearance matrix of the constructed TD$(2n)$ is indeed B, as claimed. It is also easily seen that if $\mathcal{T}$ is field-homogeneous [team-homogeneous, respectively] then $\mathcal{T}'$ is field-homogeneous [team-homogeneous, respectively] as well, with the field imbalance $I_F(P_i)$ [team imbalance $I_T(P_j)$, respectively] increased by one if $\mathcal{T}$ was a TD$^*(n)$; otherwise, it remains the same. □

Corollary 3. *If $\mathcal{T}$ is a field-uniform TD(n) then the TD$(2n)$ constructed in Theorem 2 is also field-uniform.*

Proof. Each field graph of the new TD$(2n)$ is obtained from a field graph of $\mathcal{T}$ by appending a pendant edge at every vertex of the latter. □

The next construction produces from a (field- and/or team-) homogeneous TD(k) a homogeneous TD(kn) for any odd n, $n \geq 1$.

Construction. Suppose we are given a (team- and/or field-) homogeneous tournament design $\mathcal{T}$ with $2k$ teams $T_1, T_2, \ldots, T_{2k}$ and k fields $F_1, F_2, \ldots, F_k$ whose appearance matrix is $S = (s_{ij})$. We want to extend this tournament to a homogeneous tournament design $\mathcal{T}^*$ with $2k(2m+1)$ teams $T_1^i, T_2^i, \ldots, T_{2k}^i$, $i = 0, 1, \ldots, 2m$ and $k(2m+1)$ fields $F_1^i, F_2^i, \ldots, F_k^i$, $i = 0, 1, \ldots, 2m$. First we schedule $2m+1$ subtournaments with $2k$ teams each such that we "copy" $\mathcal{T}$ into $2m+1$ homogeneous tournament designs $\mathcal{T}^0, \mathcal{T}^1, \ldots, \mathcal{T}^{2m}$ with teams $T_1^i, T_2^i, \ldots, T_{2k}^i$ and fields $F_1^i, F_2^i, \ldots, F_k^i$ for each $i = 0, 1 \ldots, 2m$.

We define the appearance matrix A of the tournament $\mathcal{T}^*$ as a block matrix in which the rows are indexed by the fields $F_1^0, F_2^0, \ldots, F_k^0, F_1^1, \ldots, F_k^{2m}$ and the columns are indexed by the teams $T_1^0, T_2^0, \ldots, T_{2k}^0, T_1^1, \ldots, T_{2k}^{2m}$. An entry $a_{e,f}^{b,c}$ then denotes the number of games played by the team T_f^c on the field F_e^b. The *auxiliary* appearance matrix A' of the TD $\mathcal{T}^*$ after the subtournaments $\mathcal{T}^0, \mathcal{T}^1, \ldots, \mathcal{T}^{2m}$ will have been scheduled will be a block matrix in which every diagonal block is a copy of the matrix S. That is to say, we have $a_{l,j}^{i,i} = s_{l,j}$ for $l = 1, 2, \ldots, k; j = 1, 2, \ldots, 2k; i = 0, 1, \ldots, 2m$. All other entries of A' will be zeros.

Now we have to schedule the remaining games T_f^b vs. T_d^c where $b \neq c$ and $d, f = 1, 2, \ldots, k$. To do this, we proceed in several steps. First we find a decomposition of the complete graph K_{2m+1} with vertices $x^0, x^1, \ldots, x^{2m}$ into m Hamiltonian cycles $C_{2m+1}^1, C_{2m+1}^2, \ldots, C_{2m+1}^m$. Suppose that one of these is the cycle $C_{2m+1}^1 = x^0, x^1, \ldots, x^{2m}, x^0$ (for convenience, we repeat here the initial vertex x^0). Then we construct the *lexicographic product* (or the *composition*) $K_{2m+1}[2K_1]$ of the complete graph K_{2m+1} and the graph $2K_1$. This means, in effect, that we "blow up" every vertex x^i into a pair of vertices x_1^i, x_2^i and replace every edge x^ix^j by four edges $x_1^ix_1^j, x_1^ix_2^j, x_2^ix_1^j, x_2^ix_2^j$. In this way we have replaced each cycle C_{2m+1}^i by the graph $C_{2m+1}^i[2K_1]$. We now want to decompose each such graph into two cycles of length $4m+2$ each. This can be done as follows: one of the cycles will be $x_1^0x_1^1x_1^2\ldots x_1^{2m}x_2^0x_2^{2m}x_2^{2m-1}x_2^1x_1^0$, and the other one will be the cycle $x_2^0x_2^1x_1^2x_2^3\ldots x_2^{2m-1}x_1^{2m}x_1^0x_2^{2m}x_1^{2m-1}\ldots x_1^3x_2^2x_1^1x_2^0$. For decompositions of other graphs $C_{2m+1}^j[2K_1]$ we proceed in exactly the same manner, always using as "initial" vertices the vertices x_1^0 and x_2^0. (In the next step we will assign an orientation to all edges of the cycles, thus the choice of the initial vertices is essential.) This means that the first cycle will be $C = x_1^0x_1^{j_1}x_1^{j_2}\ldots x_1^{j_{2m}}x_2^0x_2^{j_{2m}}x_2^{j_{2m-1}}\ldots x_2^{j_1}x_1^0$ and the other cycle will be $C' = x_2^0x_2^{j_1}x_1^{j_2}x_2^{j_3}\ldots x_2^{j_{2m-1}}x_1^{j_{2m}}x_1^0x_2^{j_{2m}}x_1^{j_{2m-1}}\ldots x_1^{j_3}x_2^{j_2}x_1^{j_1}x_2^0$.

Now we determine an orientation of the cycles. Each of the cycles C will consist of two directed paths with initial vertex x_1^0 and terminal vertex x_2^0. On the other hand, the cycles C' will consist of two directed paths with initial vertex x_2^0 and terminal vertex x_1^0. Then we decompose each cycle into two (directed) 1-factors. In this way we get a factorization of the graph $K_{2m+1}[2K_1]$ into $4m$ (directed) 1-factors.

Assume for a moment that $k = 1$ and therefore each subtournament $\mathcal{T}^i$ consists of just two teams, T_1^i and T_2^i. We now schedule the remaining games as follows. Each directed 1-factor of the graph $K_{2m+1}[2K_1]$ will correspond to one round. If there is a directed edge $x_e^ax_f^b$ (where x_e^a is the initial vertex and x_f^b is the terminal vertex), then the game is scheduled to the field F_1^a. One can check that the rounds are scheduled correctly because each directed 1-factor contains exactly one edge with initial vertex x_e^j: either an edge $x_1^{j_l}x_f^{j_{l+1}}$ or $x_2^{j_l}x_{f+1}^{j_{l+1}}$ (where $f \in \{1,2\}$) but not both. It follows from the construction that a team $T_e^{j_l}, e \in \{1,2\}$ plays exactly $2m$ games at its "preferred field" $F_1^{j_l}$. Furthermore, the team $T_e^{j_l}$ plays two games at each of the m fields $F_1^{j_{l-1}}$ such that the directed edges $x_1^{j_{l-1}}x_e^{j_l}$ and $x_2^{j_{l-1}}x_e^{j_l}$ appear in the 1-factorization. On the other hand, there are m fields at which the team T_e^j does not play any game - these fields correspond to the terminal vertices of the edges $x_1^{j_l}x_e^{j_{l+1}}$ and $x_2^{j_l}x_e^{j_{l+1}}$, as these games are scheduled to the field $F_1^{j_l}$.

Matrix M is then defined as follows: $m_{1,1}^{i,i} = m_{1,2}^{i,i} = 2m$ for every $i = 0, 1, \ldots, 2m$; further, $m_{1,1}^{i,j_{l+1}} = m_{1,2}^{i,j_{l+1}} = 2$, and, at the same time, $m_{1,1}^{i,j_{l-1}} = m_{1,2}^{i,j_{l-1}} = 0$ exactly when $i = j_l$ in one of the Hamiltonian cycles of the original graph K_{2m+1}. One can easily verify that each column then contains exactly the entries of the vector $(2m, 2, 0, 2, 0, \dots, 2, 0)$, and therefore the tournament is team homogeneous. Similarly, each row contains the entries of the vector $(2m, 2m, 2, 0, 2, 0, \dots, 2, 0)$ and therefore the tournament is also field-homogeneous.

We use a modification of this idea in the general case when $k \geq 1$. However, in this case we need one more step. We "blow up" the graph $K_{2m+1}[2K_2]$ again and replace each vertex by k independent vertices to obtain the graph $K_{2m+1}[2kK_2]$. We replace each original vertex x_1^j by vertices $x_1^j, x_2^j, \dots, x_k^j$ and the vertex x_2^j by vertices $x_{k+1}^j, x_{k+2}^j, \dots, x_{2k}^j$ for every $j = 0, 1, \dots, 2m$. Then we choose a fixed 1-factorization of the graph $K_{k,k}$ into factors $E_0, E_1, \dots, E_{k-1}$ to determine a 1-factorization of the graph $K_{2m+1}[2kK_2]$. If the partite sets of the graph $K_{k,k}$ are $\{y_1, y_2, \dots, y_k\}$ and $\{z_1, z_2, \dots, z_k\}$ then the factor E_p contains the edges $y_1z_{1+p}, y_2z_{2+p}, \dots, y_kz_{k+p}$ where the subscripts are taken mod k with the proviso that we write k instead of 0. From each 1-factor H_t of $K_{2m+1}[2K_2]$ we construct k 1-factors $H_{t,0}, H_{t,1}, \dots, H_{t,k-1}$ of $K_{2m+1}[2kK_2]$ as follows. Suppose that a directed edge $x_e^a x_f^b$ appears in a factor H_t of $K_{2m+1}[2K_2]$. Then the factor $H_{t,q}$ contains all directed edges $x_u^a x_{v+q}^b$ where $u = (e-1)k + p$, $v = (f-1)k + p$ and $p = 1, 2, \dots, k$. That means that any factor H_t of $K_{2m+1}[2K_2]$ yields k factors $H_{t,q}$ of $K_{2m+1}[2kK_2]$ such that each edge $x_e^a x_f^b$ of the factor H_t is replaced in $H_{t,q}$ by a fixed "copy" of the factor E_q of $K_{k,k}$.

Similarly to the procedure above we again assign preferred fields to teams. If an edge $x_e^a x_f^b$ appears in a factor $H_{t,q}$ then the game between T_e^a and T_f^b is scheduled to the field F_e^a (if $a \leq k$) or F_e^{a-k} (if $a > k$) in the round $(t-1)k + q$. Using the same arguments as in the case $k = 1$, we can prove that each column of the matrix M contains exactly the entries of the vector $(2km, 2, 0, 2, 0, \dots, 2, 0, \dots, 0)$ and each row contains exactly the entries of the vector $(2km, 2km, 2, 0, 2, 0, \dots, 2, 0, \dots, 0)$, and therefore the TD is also team- and field-homogeneous.

More precisely, the matrix M is a block matrix defined as follows: $m_{e,f}^{i,i} = 2km$ for every $i = 0, 1, \dots, 2m; e = 1, 2, \dots, k; f = 1, 2, \dots, k$; further, $m_{e,f}^{i,j_{l+1}} = 2$ and $m_{e,f}^{i,j_{l-1}} = 0$ for every $e = 1, 2, \dots, k; f = 1, 2, \dots, 2k$ exactly when $i = j_l$ in one of the Hamiltonian cycles of the original graph K_{2m+1}. We have assumed that the subtournaments $\mathcal{T}^0, \mathcal{T}^1, \dots, \mathcal{T}^{2m}$ are field- and/or team-homogeneous with the same appearance matrix S.
Therefore the auxiliary appearance matrix A' of the tournament $\mathcal{T}^*$ after the subtournaments $\mathcal{T}^0, \mathcal{T}^1, \dots, \mathcal{T}^{2m}$ had been scheduled is a block matrix in which every diagonal block is a copy of the matrix S; that is, $a_{l,j}^{i,i} = s_{l,j}$ for $l = 1, 2, \dots, k; j = 1, 2, , \dots, 2k; i = 0, 1, \dots, 2m$. All other entries of A' are zeros. Hence the matrix $A = A' + M$ clearly is the appearance matrix of a field and/or team- homogeneous tournament design.

5. Conclusion

Admittedly, the measures of imbalance proposed and discussed in this article are somewhat crude, and certainly not the only ones possible. Nevertheless, even the determination of the sets $S_T(n)$ and $S_F(n)$, as well as $S(n)$, is a challenging problem. As a further step, one may want to consider further measures of imbalance, as well as several measures of imbalance simultaneously, somewhat in the spirit of [F] where consideration of several bias categories with respect to balance has been proposed.

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