The 4-choosability of planar graphs without 6-cycles

Wang Weifan

Department of Mathematics
Liaoning University
Shenyang 110036, P. R. China

Ko-Wei Lih

Institute of Mathematics
Academia Sinica
Nankang, Taipei 115, Taiwan
makwlih@sinica.edu.tw

Abstract

Let G be a planar graph without 6-cycles. We prove that G is 4-choosable.

1 Introduction

All graphs considered in this paper are finite, loopless and without multiple edges unless otherwise stated. Let G be a graph with the vertex set $V(G)$, the edge set $E(G)$, and the maximum degree $\Delta(G)$. A k-coloring of G is a mapping ϕ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge xy of G. The graph G is k-colorable if it has a k-coloring. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is k-colorable. The mapping L is said to be an *assignment* for the graph G if it assigns a list $L(v)$ of possible colors to each vertex v of G. If G has some k-coloring ϕ such that $\phi(v) \in L(v)$ for all vertices v, then we say that G is L-colorable or ϕ is an L-coloring of G. We call G *k-choosable* or *k-list colorable* if it is L-colorable for every assignment L satisfying $|L(v)| = k$ for all vertices v. An L-coloring ϕ of such an assignment L is also called a *k-list coloring*. The *choice number* or *list chromatic number* $\chi_l(G)$ of G is the smallest k such that G is k-choosable.

It follows from the definition that $\chi_l(G) \geq \chi(G)$. However, the inequality can be strict. For instance, $K_{3,3}$ is not 2-choosable. The concept of list-coloring was introduced by Vizing [7] and independently by Erdös, Rubin and Taylor [2]. In recent years, a number of interesting results about the choosability of planar graphs have been obtained. Alon and Tarsi [1] proved that every planar bipartite graph is

3-choosable. Thomassen [5] proved that every planar graph is 5-choosable, whereas Voigt [8] presented an example of a planar graph which is not 4-choosable. The smallest known 3-colorable non-4-choosable planar graph of order 63 was constructed by Mirzakhani [4]. In 1995, Thomassen [6] showed that every planar graph of girth at least 5 is 3-choosable. In the same year, Voigt [9] gave an example of a planar graph of girth 4 which is not 3-choosable. Lam, Xu and Liu [3] proved that every planar graph without 4-cycles is 4-choosable. Recently, we have proved in [10] and [11] that every planar graph either without 5-cycles or without two 3-cycles sharing a common vertex is 4-choosable. In this paper we will prove the following.

Theorem 1 *Every planar graph without* 6*-cycles is* 4*-choosable.*

Now we are going to introduce the notation used in this paper. A *plane* graph G is a particular drawing in the Euclidean plane of a certain planar graph. We denote its face set, order, and minimum degree by $F(G)$, $|G|$, and $\delta(G)$, respectively. Let $d_G(v)$ (or $d(v)$) denote the degree of v in G. Let $N_G(v)$ (or $N(v)$) denote the set of neighbors of the vertex v in G. For $f \in F(G)$, we use $b(f)$ to denote the closed boundary walk of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk in the clockwise order. The set of boundary vertices of f is occasionally denoted by $V(f)$. Let $\lambda_G(f)$ (or $\lambda(f)$ for short) denote the degree of a face f in G, i.e., the number of edge-steps in $b(f)$. A vertex (or a face) of degree k is called a *k-vertex* (or *k-face*). A face f of G is called a *simple* face if $b(f)$ forms a cycle. Obviously, when $\delta(G) \geq 2$, each k-face ($k \leq 5$) is a simple face. We say that two faces or cycles of a plane graph are *adjacent* if they share at least one common boundary edge. A vertex v is said to be *incident* to a face f, and vice versa, if v lies on the boundary of f. Let $F(v)$ denote the set of all faces that are incident to the vertex v. Furthermore, let $T(v), Q(v)$, and $P(v)$ denote, respectively, the set of 3-faces, the set of 4-faces, and the set of 5-faces that are incident to the vertex v. If $f \in F(G)$, let $T^*(f)$ denote the set of 3-faces that are adjacent to the face f. A *k-wheel* W_k, $k \geq 3$, is a plane graph of order $k+1$ obtained from a k-cycle $C_k = x_1x_2 \cdots x_kx_1$ by adding a new vertex x to the interior of C_k and joining x to every x_i, $1 \leq i \leq k$. The vertex x is called the *center* of W_k. A *k-fan* F_k is the plane graph $W_k - x_1x_k$. We call x the *root* of F_k. We also denote F_3 by K^*. Obviously, K^* is isomorphic to $K_4 - e$, where e is an edge of the complete graph K_4.

2 The Proof

In order to obtain our main result, we need the following lemma.

Lemma 2 *Let G be a* 2*-connected plane graph without* 6*-cycles and $t \in V(G)$. If $d(v) \geq 4$ for all $v \in V(G) \setminus \{t\}$, then $G - t$ contains an induced K^* such that each of its vertices is of degree* 4 *in G.*

Proof. To prove by contradiction, we assume that there is a 2-connected plane graph G with vertex t that satisfies the following:

(a) $d(v) \geq 4$ for every $v \in V(G) \setminus \{t\}$;

(b) $G-t$ does not contain an induced K^* such that each of its vertices is of degree 4 in G;

(c) G does not contain 6-cycles. In particular, the following seven configurations are excluded from G:

(c1) a 6-face;

(c2) a 5-face adjacent to a 3-face;

(c3) two adjacent 4-faces sharing a single edge;

(c4) a 4-face adjacent to two non-adjacent 3-faces;

(c5) a 4-face having only one common edge with two adjacent 3-faces;

(c6) a 3-face adjacent to three mutually non-adjacent 3-faces;

(c7) a 5-fan.

The following identity is a straightforward consequence of Euler's formula.

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (\lambda(f) - 6) = -12.$$

To define a weight function w on $V(G) \cup F(G)$, we let $w(x) = 2d(x) - 6$ if $x \in V(G)$ and $w(x) = \lambda(x) - 6$ if $x \in F(G)$. Thus $\sum\{w(x) | \ x \in V(G) \cup F(G)\} = -12$. Now we are going to describe a discharging process that will redistribute the weight $w(x)$ to its neighboring elements while the total sum of weights is kept fixed. We use $\mathbf{W}(x \to y)$ to denote the amount transferred to an element y from an element x in the following rules. Furthermore, let $\mathbf{W}(x \to)$ and $\mathbf{W}(\to y)$ denote, respectively, the total amount transferred out of an element x and the total amount transferred into an element y. We call a vertex v of G an *improper* vertex if $d(v) = 4$, $|T(v)| = 1$, $|Q(v)| = 2$, and $|P(v)| = 1$.

Our discharging rules are as follows.

(R0) $\mathbf{W}(t \to f) = 2$ for every $f \in F(t)$.

For $v \in V(G) \setminus \{t\}$, we have $d(v) \geq 4$ by (a).

(R1) $d(v) = 4$. Since $w(v) = 2$ and $0 \leq |T(v)| \leq 4$, we consider the following subcases.

If $|T(v)| = 0$ or 4, we let $\mathbf{W}(v \to f) = 1/2$ for each $f \in F(v)$.

If $|T(v)| = 1$, then $|Q(v)| \leq 2$ by (c3). We let $\mathbf{W}(v \to f) = 1$ for the unique $f \in T(v)$, $\mathbf{W}(v \to f) = 1/2$ for every $f \in Q(v)$, and $\mathbf{W}(v \to f) = (2 - |Q(v)|)/2|P(v)|$ for every $f \in P(v)$ if $P(v) \neq \emptyset$.

If $|T(v)| = 2$, then $|Q(v)| = |P(v)| = 0$ by (c2), (c4), and (c5). We let $\mathbf{W}(v \to f) = 1$ for every $f \in T(v)$.

If $|T(v)| = 3$, then $|Q(v)| = |P(v)| = 0$. We let $\mathbf{W}(v \to f) = 2/3$ for every $f \in T(v)$.

(R2) $d(v) = 5$. Then $w(v) = 4$ and $0 \leq |T(v)| \leq 3$ by (c7).

If $|T(v)| = 0$, we let $\mathbf{W}(v \to f) = 4/5$ for every $f \in F(v)$.

If $|T(v)| = 1$, we let $\mathbf{W}(v \to f) = 4/3$ for the unique $f \in T(v)$, $\mathbf{W}(v \to f) = 2/3$ for every $f \in Q(v) \cup P(v)$.

If $|T(v)| = 2$, then it follows from (c2), (c3), (c4), and (c5) that both $|Q(v)|$ and $|P(v)|$ are ≤ 1 and $|P(v)| = 1$ implies $|Q(v)| = 0$. We let $\mathbf{W}(v \to f) = 4/3$

for every $f \in T(v)$, $\mathbf{W}(v \to f) = 1/2$ for every $f \in Q(v)$, and $\mathbf{W}(v \to f) = (8 - 3|Q(v)|)/6|P(v)|$ for every $f \in P(v)$ if $P(v) \neq \emptyset$.

If $|T(v)| = 3$, then $|Q(v)| = |P(v)| = 0$. We let $\mathbf{W}(v \to f) = 4/3$ for every $f \in T(v)$.

(R3) $d(v) \geq 6$. Then $0 \leq |T(v)| \leq d(v) - 2$ by (c7). We let $\mathbf{W}(v \to f) = 3/2$ for every $f \in T(v)$, $\mathbf{W}(v \to f) = 1$ for every $f \in Q(v)$, and $\mathbf{W}(v \to f) = 1/2$ for every $f \in P(v)$.

(R4) For every face $f \in F(G)$ with $\lambda(f) \geq 7$, we let $\mathbf{W}(f \to f') = (\lambda(f) - 6)/|T^*(f)|$ for every $f' \in T^*(f)$ if $T^*(f) \neq \emptyset$.

The following straightforward claims summarize the consequences of the discharging rules (R0) to (R4).

Claim 1. *For every vertex $v \in V(G)$ and every face $f \in T(v) \cup Q(v)$, we have $\mathbf{W}(v \to f) \geq 1/2$.*

Claim 2. *Let $v \in V(G) \setminus \{t\}$ and $f \in P(v)$. If v is an improper vertex, then $\mathbf{W}(v \to f) = 0$; otherwise, $\mathbf{W}(v \to f) \geq 1/2$.*

Claim 3. *If $f \in F(G)$ with $\lambda(f) \geq 7$ and $f' \in T^*(f)$, then $\mathbf{W}(f \to f') \geq 1/7$.*

Let $w'(x)$ denote the final weight function when the discharging is complete. We are now going to show that $w'(v) = w(v) - \mathbf{W}(v \to) \geq 0$ for every $v \in V(G) \setminus \{t\}$.

Let $v \in V(G) \setminus \{t\}$. Thus $d(v) \geq 4$ by (a). If $4 \leq d(v) \leq 5$, (R1) and (R2) imply that $w'(v) \geq 0$. Assume that $d(v) \geq 6$. It suffices to show that $\mathbf{W}(v \to) \leq w(v) = 2d(v) - 6$.

If $d(v) = 6$, then $w(v) = 6$ and $|T(v)| \leq 4$. When $|T(v)| = 0$, $\mathbf{W}(v \to f) \leq 1$ for every $f \in F(v)$, hence $\mathbf{W}(v \to) \leq 6$. When $|T(v)| = 1$, we have $|Q(v)| \leq 3$, $|P(v)| \leq 3$, and $|Q(v)| + |P(v)| \leq 5$. Thus $\mathbf{W}(v \to) \leq 11/2$. When $|T(v)| = 2$, we have $|Q(v)| + |P(v)| \leq 3$, hence $\mathbf{W}(v \to) \leq 6$. When $|T(v)| = 3$, we have $|Q(v)| + |P(v)| \leq 1$, hence $\mathbf{W}(v \to) \leq 11/2$. When $|T(v)| = 4$, obviously $|Q(v)| + |P(v)| = 0$, hence $\mathbf{W}(v \to) = 6$.

If $d(v) = 7$, then $w(v) = 8$ and $|T(v)| \leq 5$. When $|T(v)| \leq 2$, then $\mathbf{W}(v \to) \leq 8$ by (R3). When $|T(v)| = i$ for $i = 3, 4, 5$, we have $|Q(v)| + |P(v)| \leq 5 - i$, hence $\mathbf{W}(v \to) \leq 5 + i/2$.

If $d(v) = 8$, then $w(v) = 10$ and $|T(v)| \leq 6$. When $|T(v)| \leq 4$, we have $\mathbf{W}(v \to) \leq 10$. When $|T(v)| = 5$, we have $|Q(v)| + |P(v)| \leq 2$, hence $\mathbf{W}(v \to) \leq 19/2$. When $|T(v)| = 6$, we have $|Q(v)| + |P(v)| = 0$, hence $\mathbf{W}(v \to) \leq 9$.

Finally suppose $d(v) \geq 9$. Clearly, $|T(v)| \leq d(v) - 2$. If $|T(v)| \leq d(v) - 3$, then $\mathbf{W}(v \to) \leq 3(d(v) - 3)/2 + 3 = 2d(v) - 6 - (d(v) - 9)/2 \leq 2d(v) - 6 = w(v)$. If $|T(v)| = d(v) - 2$, we have $|Q(v)| = |P(v)| = 0$. So $\mathbf{W}(v \to) = 3(d(v) - 2)/2 = 2d(v) - 6 - (d(v) - 6)/2 \leq 2d(v) - 6 = w(v)$.

Now we are going to compute $w'(f)$ for $f \in F(G)$. If $\lambda(f) = 6$, then $w'(f) = w(f) = 0$. If $\lambda(f) \geq 7$, then $w'(f) \geq 0$ by (R4). If $\lambda(f) = 4$, then $w(f) = -2$. It follows from Claim 1 that $w'(f) \geq 0$.

Suppose $\lambda(f) = 5$ and $f = [u_1 u_2 u_3 u_4 u_5]$. Hence $w(f) = -1$. If the vertex t is incident to f, then $w'(f) \geq 1$ by (R0). Otherwise, $d(u_i) \geq 4$ for all $i = 1, 2, \ldots, 5$ by (a). We assert that at most two of u_i's are improper vertices, hence $w'(f) \geq 1/2$ by Claim 2. Suppose on the contrary that there were at least three improper boundary vertices of f. Then two of them, say u_1 and u_2, are adjacent. Let $N(u_1) =$

$\{v_1, v_2, u_2, u_5\}$ and $N(u_2) = \{w_1, w_2, u_3, u_1\}$. By the definition and (c2), $[u_1v_1v_2] \in T(u_1)$, $[u_2w_1w_2] \in T(u_2)$, and $[u_1v_2w_1u_2] \in Q(u_1)$. A 6-cycle $u_1v_1v_2w_1w_2u_2u_1$ is thus produced, which contradicts (c).

Finally let $\lambda(f) = 3$, hence $w(f) = -3$. If $t \in V(f)$, then $\mathbf{W}(t \to f) = 2$ by (R0) and $\mathbf{W}(v \to f) \geq 1/2$ for each $v \in V(f) \setminus \{t\}$ by Claim 1. Therefore $w'(f) \geq 0$. So assume that $t \notin V(f)$. If f receives at least 1 from each of its boundary vertices, then $w'(f) \geq 0$. Suppose that $\mathbf{W}(v \to f) < 1$ for some $v \in V(f)$. According to (R0) to (R4), this happens only in two cases.

Case 1. $d(v) = 4$ and $|T(v)| = 3$. We call v a $\frac{2}{3}$-*bad* vertex because $\mathbf{W}(v \to f) = 2/3$ by (R1). Let v_1, v_2, v_3, and v_4 denote the neighbors of v arranged around v in the clockwise order. Then $f \in T(v) = \{[vv_1v_2], [vv_2v_3], [vv_3v_4]\}$.

Case 2. $d(v) = 4$ and $|T(v)| = 4$. We call v a $\frac{1}{2}$-*bad* vertex because $\mathbf{W}(v \to f) = 1/2$ by (R1). Let v_1, v_2, v_3, and v_4 denote the neighbors of v arranged around v in the clockwise order. Then $f \in T(v) = \{[vv_1v_2], [vv_2v_3], [vv_3v_4], [vv_4v_1]\}$.

We call a vertex v of G *bad* if it is either a $\frac{2}{3}$-bad vertex or a $\frac{1}{2}$-bad vertex. If a 3-face $[xyz]$ of G has two bad boundary vertices, say x and y, then $N(x) \cup N(y) \cup \{x, y\}$ induces a subgraph containing a 6-cycle. It follows that every 3-face of G is incident to at most one bad vertex.

Claim 4. *Let $v \in V(G) \setminus \{t\}$ be a bad vertex with $T(v) \cap F(t) = \emptyset$ and let $f \in T(v)$. If $f' \in F(G) \setminus T(v)$ is adjacent to f, then $\lambda(f') \geq 7$.*

We only prove the case when v is a $\frac{2}{3}$-bad vertex. The other case can be handled in an analogous manner. Let f' denote a face in $F(G) \setminus T(v)$ that is adjacent to f. Obviously, $\lambda(f') \neq 6$. Assume that $\lambda(f') = 3$. Since $d(v) \geq 4$ for every $v \in V(G) \setminus \{t\}$, there is $y \in V(f') \setminus \{v_1, v_2, v_3, v_4\}$. A 6-cycle containing v, y, v_1, v_2, v_3, and v_4 exists in G, contradicting (c). Similar contradictions can be derived if $\lambda(f')$ is either 4 or 5. The proof of Claim 4 is complete.

When v is either a $\frac{2}{3}$-bad or $\frac{1}{2}$-bad vertex, we write $\mathbf{W}(\to T(v)) = \sum\{\mathbf{W}(\to f) | f \in T(v)\}$ and $w(T(v)) = \sum\{w(f) | f \in T(v)\}$. We are going to show that $\mathbf{W}(\to T(v)) + w(T(v)) \geq 0$.

First assume that v is $\frac{2}{3}$-bad. Then $w(T(v)) = -9$ and all v_i's are not bad vertices by the foregoing argument. If t lies on the boundary of some face in $T(v)$, i.e., $t \in \{v_1, v_2, v_3, v_4\}$, we have two subcases in view of the symmetry between v_1, v_2 and v_3, v_4. If $t = v_1$, then $T(v)$ receives exactly 2 from each of t and v, at least 2 from each of v_2 and v_3, and at least 1 from v_4. Hence $\mathbf{W}(\to T(v)) \geq 9$. If $t = v_2$, then $T(v)$ receives 4 from t, 2 from v, at least 2 from v_3, and at least 1 from each of v_1 and v_4. Consequently, $\mathbf{W}(\to T(v)) \geq 10$.

Suppose that $t \notin \{v_1, v_2, v_3, v_4\}$. The planarity of G implies that $v_1v_3 \notin E(G)$ or $v_2v_4 \notin E(G)$. Without loss of generality, we suppose that $v_1v_3 \notin E(G)$. If the degree of every v_i is 4, then $\{v, v_1, v_2, v_3\}$ induces a configuration that contradicts (b). We may first suppose that $d(v_2) \geq 5$. By (R1) to (R3), $T(v)$ receives at least 1 from v_1, at least 8/3 from v_2, at least 2 from v_3, at least 1 from v_4, and exactly 2 from v. Moreover, if $f \in T(v)$ and $f' \in F(G) \setminus T(v)$ are adjacent, then $\lambda(f') \geq 7$ and $\mathbf{W}(f' \to f) \geq 1/7$ by Claims 3 and 4. Therefore $\mathbf{W}(\to T(v)) \geq 197/21 > 9$. Next, we suppose that $d(v_1) \geq 5$. By (R2) and (R3), $T(v)$ receives at least 4/3 from v_1. Thus $\mathbf{W}(\to T(v)) \geq 190/21 > 9$.

Next assume that v is $\frac{1}{2}$-bad; then $w(T(v)) = -12$. If $t \in \{v_1, v_2, v_3, v_4\}$, say $t = v_1$, then $T(v)$ receives 4 from t, 2 from v, and at least 2 from each of v_2, v_3, and v_4. It is easy to see that $\mathbf{W}(\to T(v)) \geq 12$.

Suppose that $t \notin \{v_1, v_2, v_3, v_4\}$. Let $f_{i,i+1}$ denote the face of G that shares the edge $v_i v_{i+1}$ with the 3-face $[vv_iv_{i+1}]$, where the indices are taken modulo 4. By (c) and Claim 4, $\lambda(f_{i,i+1}) \geq 7$. If there exist distinct j and k such that $d(v_j) \geq 6$ and $d(v_k) \geq 5$, then $T(v)$ receives at least 3 from v_j, at least $8/3$ from v_k, at least 2 from each of v and v_i, $i \neq j, k$, and at least $4/7$ from the $f_{i,i+1}$, all together. Thus $\mathbf{W}(\to T(v)) \geq 257/21 > 12$. If there are at least three v_k's such that $d(v_k) \geq 5$, then $\mathbf{W}(\to T(v)) \geq 88/7 > 12$. If $d(v_1) = d(v_2) = 5$ and $d(v_3) = d(v_4) = 4$, then $|T^*(f_{23})| \leq \lambda(f_{23}) - 1$, $|T^*(f_{41})| \leq \lambda(f_{41}) - 1$, and $|T^*(f_{34})| \leq \lambda(f_{34}) - 2$ since f_{34} is adjacent to both f_{23} and f_{41}. Thus $\mathbf{W}(f_{23} \to [vv_2v_3]) \geq (\lambda(f_{23}) - 6)/(\lambda(f_{23}) - 1) \geq 1/6$. Similarly, $\mathbf{W}(f_{41} \to [vv_4v_1]) \geq 1/6$, $\mathbf{W}(f_{34} \to [vv_3v_4]) \geq (\lambda(f_{34}) - 6)/(\lambda(f_{34}) - 2) \geq 1/5$, and $\mathbf{W}(f_{12} \to [vv_1v_2]) \geq 1/7$. Therefore $\mathbf{W}(\to T(v)) \geq 1261/105 > 12$. If $d(v_1) = d(v_3) = 5$ and $d(v_2) = d(v_4) = 4$, then $|T^*(f_{i,i+1})| \leq \lambda(f_{i,i+1}) - 1$ for all i. In this case, $\mathbf{W}(\to T(v)) \geq 12$. Finally, let $d(v_1) \geq 5$ and $d(v_i) = 4$ for $i = 2, 3, 4$. If v_2 and v_4 are adjacent, then at least one of v_1 and v_3 is a cut vertex. This contradicts the 2-connectedness assumption about G. If v_2 and v_4 are non-adjacent, then $\{v, v_2, v_3, v_4\}$ induces a configuration that contradicts (b).

It follows from the above argument that

$$\sum\{w'(x) \mid x \in (V(G) \cup F(G)) \setminus \{t\}\} \geq 0.$$

However, we note that $w'(t) = 2d(t) - 6 - 2|F(t)| \geq 2d(t) - 6 - 2d(t) = -6$ by (R0). Therefore,

$$\sum\{w'(x) \mid x \in V(G) \cup F(G)\} \geq -6.$$

Since the total sum of weights was kept fixed during the discharging procedure, the following obvious contradiction is produced.

$$-12 = \sum\{w(x) \mid x \in V(G) \cup F(G)\} = \sum\{w'(x) \mid x \in V(G) \cup F(G)\} \geq -6.$$

□

Corollary 3 *Let G be a plane graph without 6-cycles and $\delta(G) \geq 4$. Then G contains an induced K^* such that each of its vertices is of degree 4 in G.*

Proof. If G is 2-connected, the result follows immediately from Lemma 2. In fact, we may choose any vertex of G as the specific vertex t. Otherwise, let B be a block of G that contains a unique cut vertex, say t, of G. Since B is 2-connected and $d_B(v) \geq 4$ for all $v \in V(B) \setminus \{t\}$, $B - t$ contains an induced K^* such that each of its vertices is of degree 4 in B by Lemma 2. Noting that $d_G(v) = d_B(v)$ for all $v \in V(K^*)$, K^* is a desired induced subgraph of G. □

Now we are ready to prove our main theorem. Every subgraph of a planar graph without 6-cycles is also a planar graph without 6-cycles. Every subgraph of a k-list colorable graph is also k-list colorable. These simple facts are essential in carrying out the induction in the following proof.

Proof of Theorem 1. We use induction on $|G|$. If $|G| \le 4$, the theorem is trivially true. Assume that it holds for all planar graphs without 6-cycles of order less than k. Let G be a planar graph without 6-cycles and $|G| = k \ge 5$. Let L denote an assignment for G such that $|L(v)| = 4$ for all $v \in V(G)$. If $\delta(G) \le 3$, let u be a vertex of minimum degree in G. By the induction hypothesis, $G - u$ is L-colorable. Obviously, we can extend any L-coloring of $G-u$ to an L-coloring of G. If $\delta(G) \ge 4$, then G contains an induced K^* such that each of its vertices x, x_1, x_2, x_3 is of degree 4 in G by Corollary 3. Let $G' = G - \{x, x_1, x_2, x_3\}$. By the induction hypothesis, G' has an L-coloring ϕ. For $v \in V(K^*)$, let $S(v)$ denote the set of colors that are used on $N_G(v) \setminus V(K^*)$ under ϕ. Thus $|S(v)| \le d_G(v) - d_{K^*}(v)$. Define an assignment $L'(v) = L(v) \setminus S(v)$ for every $v \in V(K^*)$. Obviously, $|L'(x_i)| \ge |L(x_i)| - |S(x_i)| \ge 2$ for $i = 1$ and 3; both $|L'(x)|$ and $|L'(x_2)|$ are at least 3. If $|L'(x)| = 4$, we color x_1, x_3, x_2, and x successively. If $|L'(x)| = 3$ and $L'(x_1) \cap L'(x_3) \neq \emptyset$, we first color x_1 and x_3 with the same color, then color x and x_2. If $|L'(x)| = 3$ and $L'(x_1) \cap L'(x_3) = \emptyset$, then there is some color $\alpha \in (L'(x_1) \cup L'(x_3)) \setminus L'(x)$, say $\alpha \in L'(x_1)$. We color x_1 with α, then color x_2, x_3, and x successively. We succeeded in obtaining an L'-coloring of K^*. Therefore G is L-colorable. □

It should be noted that 4-choosability in Theorem 1 can not be strengthened to 3-choosability. There exist infinitely many planar graphs without 6-cycles that are not 3-choosable. Two simple examples are K_4 and $K_5 - e$.

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