On partitions into four distinct squares of equal parity

Michael D. Hirschhorn

School of Mathematics
University of New South Wales
Sydney NSW 2052
Australia

Abstract

We study $p_{4e}(n)$ and $p_{4o}(n)$, the number of partitions of n into four distinct even (four distinct odd) squares. The main result is that for $n \equiv 4 \pmod 8$, we have $p_{4o}(n) = p_{4e}(n) + p_{4e+}(n)$, where p_{4e+} is the number of partitions of n into four distinct strictly positive squares.

1. Introduction and statement of results

In recent work [1, 2], we studied $p_{4\square}(n)$, the number of partitions of n into four squares. We found the generating function for $p_{4\square}(n)$, and showed that the numbers $p_{4\square}(n)$ possess some (rare) arithmetic properties.

In this paper we make a study of what we will call $p_{4e}(n)$ and $p_{4o}(n)$, the number of partitions of n into four distinct **even** (respectively **odd**) squares. We shall also have reason to consider $p_{4e+}(n)$, where the squares are **strictly positive**.

The main result of this paper is that for $n \equiv 4 \pmod 8$,

$$p_{4o}(n) = p_{4e}(n) + p_{4e+}(n).$$

As a corollary we obtain the surprising result that for $n \equiv 4 \pmod 8$,

$$p_{4o}(n) \geq p_{4e}(n) \geq \frac{1}{2} p_{4o}(n).$$

Thus, if $p_{4o}(n) > 0$ (and so $n \equiv 4 \pmod 8$) then $p_{4e}(n) > 0$, a conjecture announced by R. Wm. Gosper at the Ramanujan Centenary Conference in 1987, the inspiration for this investigation.

We shall give two proofs of our theorem, one arithmetic, the other via generating functions.

2. The arithmetic proof

We shall show that if $n \equiv 4 \pmod 8$ there are two partitions of n into four distinct odd squares for each partition of n into four distinct even squares greater than zero, and one partition of n into four distinct odd squares for each partition of n into four distinct even squares one of which is zero.

Suppose $n \equiv 4 \pmod 8$ and

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

is a partition of n into four distinct even squares. Then without loss of generality

$$(x_1, x_2, x_3, x_4) \equiv (0,2,2,2) \text{ or } (2,0,0,0) \pmod 4,$$

$$x_1 + x_2 + x_3 + x_4 \equiv 2 \pmod 8$$

and

$$x_2 > x_3 > |x_4|.$$

(In the case $(x_1, x_2, x_3, x_4) \equiv (0,2,2,2) \pmod 4$, the condition $x_1+x_2+x_3+x_4 \equiv 2 \pmod 8$ determines the sign on x_4, (the sign of x_1 is undetermined); in the case $(x_1, x_2, x_3, x_4) \equiv (2,0,0,0) \pmod 4$ the condition determines the sign on x_1 (the sign of x_4 is undetermined).)

In the case $(x_1, x_2, x_3, x_4) \equiv (0,2,2,2) \pmod 4$, let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac12 & \frac12 & \frac12 & \frac12 \\ \frac12 & \frac12 & -\frac12 & -\frac12 \\ \frac12 & -\frac12 & \frac12 & -\frac12 \\ \frac12 & -\frac12 & -\frac12 & \frac12 \end{pmatrix} \begin{pmatrix} |x_1| \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} \frac12 & \frac12 & \frac12 & \frac12 \\ \frac12 & \frac12 & -\frac12 & -\frac12 \\ \frac12 & -\frac12 & \frac12 & -\frac12 \\ \frac12 & -\frac12 & -\frac12 & \frac12 \end{pmatrix} \begin{pmatrix} -|x_1| \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Then

$$n = y_1^2 + y_2^2 + y_3^2 + y_4^2 = {y_1'}^2 + {y_2'}^2 + {y_3'}^2 + {y_4'}^2,$$

$$y_1,\ y_2,\ y_3,\ y_4 \equiv 1 \pmod 4, \quad y_1',\ y_2',\ y_3',\ y_4' \equiv 1 \pmod 4,$$

$$y_1 > y_2 > y_3 > y_4, \quad y_1' > y_2' > y_3' > y_4'$$

are partitions of n into four distinct odd squares, different if and only if $x_1 \neq 0$.

Similarly, in the case $(x_1, x_2, x_3, x_4) \equiv (2,0,0,0) \pmod 4$, let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac12 & \frac12 & \frac12 & \frac12 \\ \frac12 & \frac12 & -\frac12 & -\frac12 \\ \frac12 & -\frac12 & \frac12 & -\frac12 \\ \frac12 & -\frac12 & -\frac12 & \frac12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ |x_4| \end{pmatrix},$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} \frac12 & \frac12 & \frac12 & \frac12 \\ \frac12 & \frac12 & -\frac12 & -\frac12 \\ \frac12 & -\frac12 & \frac12 & -\frac12 \\ \frac12 & -\frac12 & -\frac12 & \frac12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -|x_4| \end{pmatrix}.$$

Then

$$n = y_1^2 + y_2^2 + y_3^2 + y_4^2 = {y'_1}^2 + {y'_2}^2 + {y'_3}^2 + {y'_4}^2,$$

$$y_1,\ y_2,\ y_3,\ y_4 \equiv 1 \pmod 4,\quad y'_1,\ y'_2,\ y'_3,\ y'_4 \equiv 1 \pmod 4,$$

$$y_1 > y_2 > y_3 > y_4,\quad y'_1 > y'_2 > y'_3 > y'_4$$

are partitions of n into four distinct odd squares, different if and only if $x_4 \neq 0$. Conversely, suppose $n \equiv 4 \pmod 8$ and

$$n = y_1^2 + y_2^2 + y_3^2 + y_4^2$$

is a partition of n into four distinct odd squares. Then without loss of generality

$$y_1,\ y_2,\ y_3,\ y_4 \equiv 1 \pmod 4 \text{ and } y_1 > y_2 > y_3 > y_4.$$

If we now let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

then

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

$$(x_1, x_2, x_3, x_4) \equiv (0,2,2,2) \text{ or } (2,0,0,0) \pmod 4,$$

$$x_1 + x_2 + x_3 + x_4 \equiv 2 \pmod 8$$

and

$$x_2 > x_3 > |x_4|.$$

This establishes the (one–to–one–or–two) correspondence described earlier between the partitions of n into four distinct even squares and the partitions of n into four distinct odd squares. ■

An example

Consider $n = 420$. The partitions of 420 into four distinct even squares are

$$420 = 20^2 + 4^2 + 2^2 = 16^2 + 12^2 + 4^2 + 2^2 = 16^2 + 10^2 + 8^2 = 14^2 + 12^2 + 8^2 + 4^2.$$

Corresponding to these we have, respectively,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 20 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \\ -7 \\ -11 \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 16 \\ 12 \\ \pm 4 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 17 \\ 1 \\ -3 \\ -11 \end{pmatrix}, \quad \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \\ 1 \\ -15 \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 16 \\ 8 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 17 \\ 9 \\ 1 \\ -7 \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -14 \\ 12 \\ 8 \\ \pm 4 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ -11 \\ -15 \end{pmatrix}, \quad \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -7 \\ -19 \end{pmatrix}.$$

Thus

$$\begin{aligned} 420 &= 20^2 + 4^2 + 2^2 = 16^2 + 12^2 + 4^2 + 2^2 = 16^2 + 10^2 + 8^2 = 14^2 + 12^2 + 8^2 + 4^2 \\ &= 19^2 + 7^2 + 3^2 + 1^2 = 17^2 + 11^2 + 3^2 + 1^2 = 17^2 + 9^2 + 7^2 + 1^2 \\ &= 15^2 + 13^2 + 5^2 + 1^2 = 15^2 + 11^2 + 7^2 + 5^2 = 13^2 + 11^2 + 9^2 + 7^2, \end{aligned}$$

$p_{4o}(420) = 6,\ p_{4e}(420) = 4,\ p_{4e+}(420) = 2$, and $p_{4o}(420) = p_{4e}(420) + p_{4e+}(420)$.

3. The generating function proof

Let $x_i,\ i = 0, \cdots, N$ be arbitrary. It is easy to verify by expanding the expression on the right that

$$(1) \quad 24 \sum_{i>j>k>l} x_i x_j x_k x_l = \left(\sum_{i=0}^{N} x_i\right)^4 - 6\left(\sum_{i=0}^{N} x_i\right)^2 \sum_{i=0}^{N} x_i^2 + 3\left(\sum_{i=0}^{N} x_i^2\right)^2 + 8\sum_{i=0}^{N} x_i \sum_{i=0}^{N} x_i^3 - 6\sum_{i=0}^{N} x_i^4.$$

(The author discovered this formula recently, but feels that it is unlikely to be new.)

If in (1) we set $x_i = q^{(2i+1)^2}$, let $N \to \infty$ and use the fact that

$$\sum_{i\geq 0} q^{(2i+1)^2} = q\psi(q^8)$$

where

$$\psi(q) = \sum_{i\geq 0} q^{(i^2+i)/2},$$

we obtain

$$(2)\quad \sum_{n\geq 0} p_{4o}(n)q^n = \frac{1}{24}\left(q^4\psi(q^8)^4 - 6q^4\psi(q^8)^2\psi(q^{16}) + 3q^4\psi(q^{16})^2 + 8q^4\psi(q^8)\psi(q^{24}) - 6q^4\psi(q^{32})\right)$$

or,

$$\sum_{n\equiv 4 \pmod 8} p_{4o}(n)q^n = \frac{1}{24}\left(q^4\psi(q^8)^4 - 6q^4\psi(q^8)^2\psi(q^{16}) + 3q^4\psi(q^{16})^2 + 8q^4\psi(q^8)\psi(q^{24}) - 6q^4\psi(q^{32})\right)$$

since the series in (2) contains only powers of q which are 4 modulo 8.
If in (1) we set $x_i = q^{(2i)^2}$, let $N \to \infty$ and use the fact that

$$\sum_{i\geq 0} q^{(2i)^2} = \frac{1}{2}(\phi(q^4)+1)$$

where

$$\phi(q) = \sum_{-\infty}^{\infty} q^{i^2},$$

we obtain, after simplification,

(3)

$$\sum_{n\geq 0} p_{4e}(n)q^n = \frac{1}{384}\left(\phi(q^4)^4 + 4\phi(q^4)^3 - 6\phi(q^4)^2 + 12\phi(q^4) - 15 + 12\phi(q^8)^2 + 12\phi(q^8) + 32\phi(q^{12}) - 48\phi(q^{16}) - 12\phi(q^4)^2\phi(q^8) - 24\phi(q^4)\phi(q^8) + 32\phi(q^4)\phi(q^{12})\right).$$

If in (1) we set $x_i = q^{(2i+2)^2}$, let $N \to \infty$ and use the fact that

$$\sum_{i\geq 0} q^{(2i+2)^2} = \frac{1}{2}(\phi(q^4)-1),$$

we obtain

$$(4)\quad \sum_{n\geq 0} p_{4e+}(n)q^n = \frac{1}{384}\left(\phi(q^4)^4 - 4\phi(q^4)^3 + 18\phi(q^4)^2 - 60\phi(q^4) + 105 + 12\phi(q^8)^2 - 36\phi(q^8) - 32\phi(q^{12}) - 48\phi(q^{16}) - 12\phi(q^4)^2\phi(q^8) + 24\phi(q^4)\phi(q^8) + 32\phi(q^4)\phi(q^{12})\right).$$

It follows from (3) and (4) that

$$\sum_{n\geq 0}(p_{4e}(n) + p_{4e+}(n))q^n = \frac{1}{192}\left(\phi(q^4)^4 + 6\phi(q^4)^2 - 24\phi(q^4) + 45 + 12\phi(q^8)^2 - 12\phi(q^8) - 48\phi(q^{16}) - 12\phi(q^4)^2\phi(q^8) + 32\phi(q^4)\phi(q^{12})\right).$$

We now extract the terms in which $n \equiv 4 \pmod 8$. We use the fact that

$$\phi(q) = \phi(q^4) + 2q\psi(q^8)$$

to write

$$\begin{aligned}\sum_{n\geq 0}(p_{4e}(n) + p_{4e+}(n))q^n = \frac{1}{192}\big(&(\phi(q^{16}) + 2q^4\psi(q^{32}))^4 + 6(\phi(q^{16}) + 2q^4\psi(q^{32}))^2 \\ &-24(\phi(q^{16}) + 2q^4\psi(q^{32})) + 45 + 12\phi(q^8)^2 - 12\phi(q^8) \\ &-48\phi(q^{16}) - 12(\phi(q^{16}) + 2q^4\psi(q^{32}))^2\phi(q^8) \\ &+32(\phi(q^{16}) + 2q^4\psi(q^{32}))(\phi(q^{48}) + 2q^{12}\psi(q^{96}))\big).\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{n\equiv 4 \pmod 8} (p_{4e}(n) + p_{4e+}(n))q^n \\ &\quad= \frac{1}{24}\big(q^4\phi(q^{16})^3\psi(q^{32}) + 4q^{12}\phi(q^{16})\psi(q^{32})^3 + 3q^4\phi(q^{16})\psi(q^{32}) - 6q^4\psi(q^{32}) \\ &\qquad -6q^4\phi(q^8)\phi(q^{16})\psi(q^{32}) + 8q^4\phi(q^{48})\psi(q^{32}) + 8q^{12}\phi(q^{16})\psi(q^{96})\big) \\ &\quad= \frac{1}{24}\big(q^4\phi(q^8)^2\phi(q^{16})\psi(q^{32}) - 6q^4\phi(q^8)\phi(q^{16})\psi(q^{32}) + 3q^4\phi(q^{16})\psi(q^{32}) \\ &\qquad +8q^4\psi(q^8)\psi(q^{24}) - 6q^4\psi(q^{32})\big).\end{aligned}$$

Here we have used the facts that

$$\phi(q^2)^2 + 4q\psi(q^4)^2 = \phi(q)^2$$

and

$$\phi(q^6)\psi(q^4) + q\phi(q^2)\psi(q^{12}) = \psi(q)\psi(q^3).$$

So our theorem is equivalent to the identity

$$\psi(q)^4 - 6\psi(q)^2\psi(q^2) + 3\psi(q^2) = \phi(q)^2\phi(q^2)\psi(q^4) - 6\phi(q)\phi(q^2)\psi(q^4) + 3\phi(q^2)\psi(q^4).$$

This is straightforward, since, with $(q)_\infty = \prod_{n\geq 1}(1 - q^n)$,

$$\begin{aligned}&\psi(q) = \frac{(q^2)_\infty^2}{(q)_\infty}, \quad \phi(q) = \frac{(q^2)_\infty^5}{(q)_\infty^2(q^4)_\infty}, \\ &\phi(q)^2\phi(q^2)\psi(q^4) = \frac{(q^2)_\infty^{10}}{(q)_\infty^4(q^4)_\infty^4}\cdot\frac{(q^4)_\infty^5}{(q^2)_\infty^2(q^8)_\infty^2}\cdot\frac{(q^8)_\infty^2}{(q^4)_\infty} = \frac{(q^2)_\infty^8}{(q)_\infty^4} = \psi(q)^4,\end{aligned}$$

$$\begin{aligned}\phi(q)\phi(q^2)\psi(q^4) &= \frac{(q^2)_\infty^5}{(q)_\infty^2(q^4)_\infty^2}\cdot\frac{(q^4)_\infty^5}{(q^2)_\infty^2(q^8)_\infty^2}\cdot\frac{(q^8)_\infty^2}{(q^4)_\infty}\\ &= \frac{(q^2)_\infty^3(q^4)_\infty^2}{(q)_\infty^2} = \frac{(q^2)_\infty^4}{(q)_\infty^2}\cdot\frac{(q^4)_\infty^2}{(q^2)_\infty}\\ &= \psi(q)^2\psi(q^2)\end{aligned}$$

and

$$\phi(q^2)\psi(q^4) = \frac{(q^4)_\infty^5}{(q^2)_\infty^2(q^8)_\infty^2}\cdot\frac{(q^8)_\infty^2}{(q^4)_\infty} = \frac{(q^4)_\infty^4}{(q^2)_\infty^2} = \psi(q^2)^2. \blacksquare$$

References

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