# Weakly completable critical sets for proper vertex and edge colourings of graphs

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**Abstract**

A long-standing problem has been that of finding weakly completable critical sets for latin squares and, in particular, determining the smallest latin squares which have such sets. Here, we look at the analogous problem for graph colouring and determine the graphs of smallest size which possess weakly completable partial vertex or edge colourings.

## I. INTRODUCTION.

A long-standing problem concerns the sizes of critical sets in latin squares; that is, subsets of the cell entries which are completable to only one latin square and such that, if one member of the subset is removed, the unique completion property ceases to hold. The problem can be re-interpreted (in several ways) as a problem concerning completion of the colouring of the vertices or edges of a graph when the colouring of a subset of these items has been prescribed. (This was first pointed out by one of the present authors at a conference held in Beijing. See [K1] and [K2] for full details. Later, E.S.Mahmoodian drew attention to the same fact.)

A problem of particular difficulty has been that of finding so-called weakly completable critical sets in latin squares (despite their anticipated abundance) and so it seemed a good strategy to look at the corresponding problem for graphs: in particular, to find the graphs of smallest size which possess weakly completable partial vertex or edge colourings. In the present paper, we give a complete solution to the latter problem, our main results being those of Theorems 4.2 and 4.3, 5.2 and 5.3. (The consequences for the latin square problem are discussed in [B1] and [K3] and in a forthcoming paper of the first author.)

## II. DEFINITIONS.

Let $G = (V, E)$ be a given graph with vertex set $V$, edge set $E$ and with an assigned proper vertex (or edge) $r$-colouring $\Sigma$. (By a *proper r-colouring*, we mean a colouring with $r$ colours such that no two adjacent vertices (edges) have the same colour.) A subset $U$ of the vertices (or edges) of $G$ which, when coloured with the colours of $\Sigma$, has the property that the only assignment of colours to the remaining vertices (or edges) of $G$ which results in a proper $r$-colouring of $G$ is that of $\Sigma$ itself is said to be a *uniquely completable subset* of the vertices (or edges) relative to $\Sigma$.

If such a subset $U$ cannot be reduced by deletion of any one vertex (edge) to one of smaller cardinality which is again a uniquely completable subset of the vertices (edges) relative to $\Sigma$, then we say that it is a *critical set* of vertices (edges) relative to $\Sigma$.

A given graph $G$ may have critical sets of more than one size relative to an $r$-colouring $\Sigma$. A critical set of smallest cardinality is called a *minimal critical set* relative to $\Sigma$.

Note that these definitions implicitly assume that $r \geq X_v(G)$ or $X_e(G)$ according as the colouring under consideration is of vertices or of edges. [$X_v$, $X_e$ denote respectively chromatic number and chromatic index.]

The definitions may be illustrated by means of the following examples:

Example 2.1. The Petersen graph may be defined as the graph whose vertices are the ten unordered pairs *ab, ac, ad, ae, bc, bd, be, cd, ce, de* with a pair of vertices adjacent if and only if their letter pairs have no letter in common. Then the four vertices *ab, cd, bc, bd* form a minimal critical set of vertices for the proper vertex 3-colouring in which vertices *ab, ac, ad, ae* have colour 1, vertices *bc, bd, be* have colour 2, and vertices *cd, ce, de* have colour 3. (It is well known - and easily shown - that, up to isomorphism, the Petersen graph has only one proper 3-colouring.)

Example 2.2. A directed graph (digraph) has directed edges of the form $xy$ joining vertex $x$ to $y$ or, equivalently, y from x and is such that two edges are regarded as adjacent for edge-colouring if and only if there is a vertex to which or from which they are both directed.

Consider a 3-colouring $\Sigma$ of the edges of the complete digraph with loops, on three vertices $v_1$, $v_2$, $v_3$ for which each loop edge has colour $c_1$, the directed edges $v_2v_3$, $v_3v_1$, $v_1v_2$ have colour $c_2$ and the directed edges $v_3v_2$, $v_2v_1$, $v_1v_3$ have colour $c_3$. Relative to $\Sigma$, the sets of edges $v_1v_2$, $v_1v_3$, $v_2v_3$ and $v_1v_3$, $v_2v_2$ are both critical sets, the second being a minimal critical set. The set of edges $v_1v_3$, $v_2v_2$, $v_3v_1$ is uniquely completable but not critical.

Next we define for graph colourings, a concept analogous to that of weak completion of a latin square. (See, for example, [K3].)

Suppose that $X$ is a given subset of the vertices (or edges) of a graph $G = (V, E)$ whose members have been coloured in the colours prescribed by a given proper $r$-colouring $\Sigma$ of $G$. To try to complete the colouring, we look for an uncoloured vertex (edge) of $G$ such that those vertices (edges) which are adjacent to it and which are already coloured jointly use $r-1$ of the given $r$ colours. If such a vertex (edge) exists, we colour it with the remaining colour. We continue this process until no further such vertices (edges) can be found. The set of vertices (edges) of $G$ which have then been coloured forms the *strong partial completion* $S$ of $X$ in $G$ relative to $\Sigma$. If the strong partial completion of $X$ in $G$ is the vertex (edge) set of $G$ itself, $X$ is called a *strongly completable subset* of $V$ (or $E$) relative to $\Sigma$. A uniquely completable subset $X$ of $V$ (or $E$) relative to $\Sigma$ which is not a strongly completable subset is a *weakly completable subset* of $V$ (or $E$) relative to $\Sigma$.

## III. PRELIMINARY LEMMAS.

We shall make frequent use of the following Lemma 3.1. The statements and their proofs are valid both for vertex colourings and for edge colourings but, because the lemma will be used mostly in the section on edge colourings, we state it in terms of the latter.

*Lemma 3.1*: Let $W$ be a weakly completable subset of the edges of a simple graph

$G = (V, E)$ relative to an $r$-colouring $\Sigma$ of $E$ which uses colours $c_1, c_2, \dots, c_r$, say, where $r > 2$. Let $S$ be the strong partial completion of $W$ in $G$ relative to $\Sigma$ and let $T = E \setminus S$. Then,

(i) the set $S_T$ of edges of $S$ which are adjacent to edges of $T$ must jointly use at least $r-1$ colours and so, a fortiori, $|S| \geq r-1$;

(ii) for each edge $e \in T$, the edges of $S$ adjacent to $e$ cannot jointly use more than $r-2$ colours;

(iii) each edge $e \in T$ is adjacent to at least one other edge $e'$ of $T$: moreover, the colour assigned by $\Sigma$ to one (at least) such edge $e'$ must be distinct from those assigned to $e$ and to the edges of $S$ which are adjacent to $e$;

(iv) each edge $e \in T$ which is not assigned colour $c_i$ $(1 \leq i \leq r)$ by $\Sigma$ is adjacent to at least one edge (not necessarily in $T$) which is assigned colour $c_i$ by $\Sigma$;

*Proof*: Statement (i). In the contrary case, there are at least two colours, say $c_1$ and $c_2$, which are not used on any edge of $S_T$. If these colours both occur on the edges of $T$ when the colouring $\Sigma$ has been completed, a colouring completion alternative to $\Sigma$ exists in which $c_1$ and $c_2$ are exchanged on those edges of $T$ which use these two colours, while all other edges of $G$ have the same colours as before. This contradicts the hypothesis that $W$ is weakly completable. If, on the other hand, colour $c_1$ (and/or $c_2$) has not been used to colour any edge of $T$ when the colouring $\Sigma$ has been completed, a colouring completion alternative to $\Sigma$ exists in which some colour $c_i$ $(i \neq 1, 2)$ is exchanged for the colour $c_1$ (or $c_2$) on all those edges of $T$ which are assigned colour $c_i$ by $\Sigma$. Again, all other edges of $G$ (including those of $W$ and $S$) have the same colours as in $\Sigma$, and so the hypothesis that $W$ is uniquely completable is again contradicted.

Statement (ii). If the edges of $S$ adjacent to $e$ used $r$ colours, there would be no colour available for $e$. If they used $r-1$ colours, the colour of $e$ would be forced, implying that $e \in S$.

Statement (iii). Let edge $e \in T$. If no edge $e' \in T$ and adjacent to $e$ exists then, since the edges of $S$ adjacent to $e$ use at most $r-2$ colours, there is an alternative colour available for $e$ in the completion of the colouring of $W$ to a colouring of $G$, contrary to hypothesis.

Statement (iv). Suppose that $e \in T$, that $e$ is assigned a colour different from $c_i$ by $\Sigma$ and that no edge of $G$ adjacent to $e$ is assigned colour $c_i$ by $\Sigma$. Then colour $c_i$ is available for $e$ so there exist two different completions of $W$ to a colouring of $G$, contrary to hypothesis.

The following lemma, due to Mahmoodian et al[M1], will be used in Theorem 4.2 of the next section:

*Lemma* 3.2: Let $U$ be a uniquely completable set of vertices of a simple graph $G = (V, E)$ relative to an $r$-colouring $\Sigma$ of the vertices of $G$. Then:

$$|E| \geq (r-1)(|V| - |U|).$$

*Proof*: Let the colours be denoted by $c_1, c_2, \dots, c_r$ and let the number of vertices which have colour $c_i$ in $V$, $U$ (under the colouring $\Sigma$) be denoted by $v_i$, $u_i$ respectively. Let $G_{ij}$ be the subgraph of $G$ which is induced by the vertices of $G$ which are coloured $c_i$ or $c_j$ under $\Sigma$. Since $U$ must contain at least one vertex of each connected component of $G_{ij}$, the number of connected components of $G_{ij}$ is at most $u_i$+$u_j$. Now, the number of edges in each connected component $K$ is at least equal to one less than the number of vertices of $K$. Summing this over all the components of $G_{ij}$, we deduce that the total number of edges of $G_{ij}$ is at least $(v_i+v_j)-(u_i+u_j)$. Further, the vertices of $G$ which are coloured $c_i$ occur in $r-1$ of the subgraphs $G_{ij}$ since there are $r$ colours all together. Hence, taking the sum of the edges in all of the subgraphs $G_{ij}$, we find that $|E| \geq (r-1)(|V|-|U|)$ because each $v_i - u_i$ occurs $r-1$ times in the sum.

Another result which will be useful later is due to Mahdian and Mahmoodian [M2] who have recently introduced a new concept related to unique completability of a graph $G = (V, E)$. If there exists a set $S = \{L_1, L_2, \ldots, L_n\}$ ($n= |V|$) of lists of size at least two (i.e. $|L_i| \geq 2$) and associated one-to-one with the vertices of $G$ for which the colours for one and only one proper colouring of the vertices of $G$ can be selected, then clearly there exists such a set of lists of size *exactly* two (we simply remove from each of our lists all but one of the colours which are not used in the proper colouring). In this case, $G$ is said to be *uniquely 2-list colourable*.

*Lemma* 3.3: A graph is uniquely 2-list colourable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph.

*Proof*: See [M2].

*Corollary*: A graph with less than four vertices is not uniquely 2-list colourable. Moreover, up to isomorphism, the only graph on four vertices which is uniquely 2-list colourable is that formed by removing an edge from the complete graph on four vertices.

## IV. VERTEX COLOURINGS.

*Theorem* 4.1: Let $G$ be a simple graph and $W$ a weakly completable subset of its vertex set $V$ relative to an $r$-colouring $\Sigma$. Let $S$ be the strong partial completion of $W$ in $G$ and let $T = V \setminus S$. Then $|T| \geq 4$.

*Proof*: Let $G' = (T, E')$ be the subgraph of $G$ induced by the vertices of $T$. Then $G'$ is uniquely 2-list colourable, the appropriate list of colours for each vertex $v$ of $T$ being those of $\Sigma$ which are not used by $\Sigma$ to colour the vertices of $S$ which are adjacent to $v$ (by Lemma 3.1(ii)). Hence the result of the theorem follows from the Corollary to Lemma 3.3.

*Theorem* 4.2: Let $G$ be a simple graph and $W$ a weakly completable subset of its vertex set $V$ relative to an $r$-colouring $\Sigma$ of $V$ ($r \geq 3$). Then $G$ has at least $r+3$ vertices and $4(r-1)$ edges.

*Proof*: Let $S$ be the strong partial completion of $W$ in $G$ and let $T = V \setminus S$. Then $T$ has at least four vertices by Theorem 4.1. Also, $|S| \geq r-1$ by Lemma 3.1(i) and so

$$|V| = |S|+|T| \geq (r-1)+4 = r+3.$$

Since $S$ is a uniquely completable subset of the vertices of the graph $G$ relative to the $r$-colouring $\Sigma$, it follows from Lemma 3.2 that

$$|E| \geq (r-1)(|V|-|S|) = (r-1)|T| \geq 4(r-1).$$

*Theorem* 4.3: For every $r \geq 3$, there is at least one connected graph with $r+3$ vertices and $4(r-1)$ edges for which a weakly completable subset of vertices relative to an $r$-colouring exists.

*Proof*: We prove the result constructively. For $r \geq 3$, let $\Gamma_r = (V_r, E_r)$ be the graph with $V_r = \{1, 2, ..., r+3\}$ and $E_r$ such that
$E_3 = \{[1, 2], [1, 3], [1, 4], [2, 4], [2, 5], [3, 4], [4, 5], [5, 6]\}$ and, for $r \geq 4$,
$E_r = E_{r-1} \cup \{[1, r+3], [2, r+3], [4, r+3], [5, r+3]\}$. (Hence $|V_r| = r+3$ and $|E_r| = 4(r-1)$.)
We illustrate this graph in Fig. 4.1 for the cases $r = 3$, 4, 5 and 6.

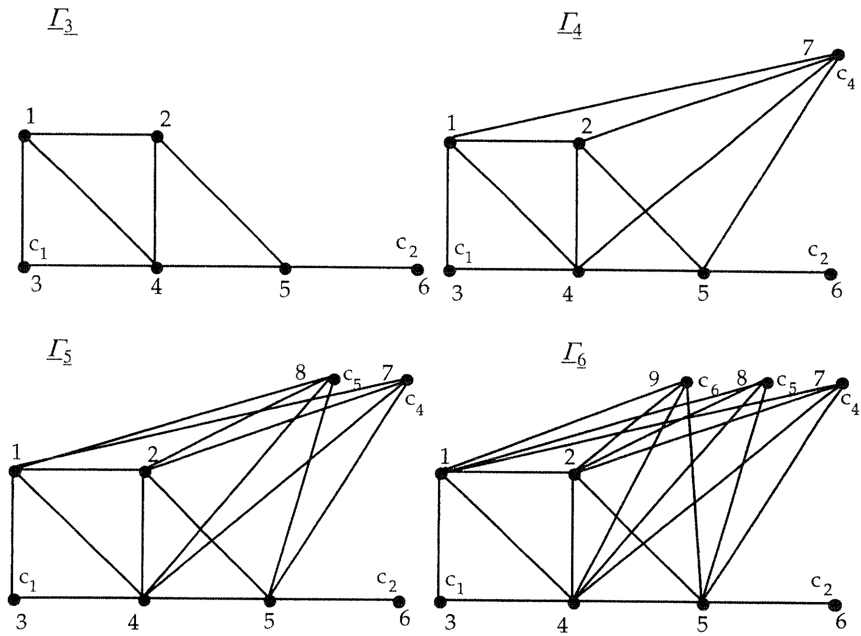

Fig. 4.1.

Let $\Sigma_r$ be an $r$-colouring of $\Gamma_r$ (using colours $c_1, c_2, ..., c_r$) which assigns colour $c_1$ to vertex 3, colour $c_2$ to vertex 6 and, if $r \geq 4$, for each $k$ such that $7 \leq k \leq r+3$, colour $c_{k-3}$ to vertex $k$. Let $W_3 = \{3, 6\}$ and, for $r \geq 4$, $W_r = W_{r-1} \cup \{r+3\}$. Then $W_r$ is not a strongly completable subset of $V_r$ relative to $\Sigma_r$ since, after the assignment of the colours of $\Sigma_r$ to the vertices of $W_r$, each of the uncoloured vertices (i.e. 1, 2, 4 and 5) is such that those coloured vertices which are adjacent to it use, between them, no more than $r-2$ colours. (The fact that this is true for $r = 3$ implies that it is true for $r = 4$, $r = 5$ successively, and so on.)

However $W_r$ is a uniquely completable subset of $V_r$ relative to $\Sigma_r$. This can be seen as follows: After the vertices of $W_r$ have been coloured in the colours of $\Sigma_r$, the pair of adjacent vertices (1, 4) must be coloured $(c_2, c_3)$ or $(c_3, c_2)$. In either case, vertex 2 must be coloured $c_1$. This implies that vertex 5 must be coloured $c_3$ and so the pair (1, 4) can only be coloured $(c_3, c_2)$. It follows that $W_r$ is a weakly completable subset of $V_r$ relative to the 3-colouring $\Sigma_r$ thus defined.

*Note*: We thank the referee for pointing out that "it is not an accident" that, for each $r$, the graph $\Gamma_r$ and its $r$-colouring $\Sigma_r$ used to prove Theorem 4.3 are such that the subgraph of $\Gamma_r$ induced by the vertex set $T_r = V_r \backslash W_r$ (as defined therein) is the complete graph on four vertices with an edge removed. By the Corollary to Lemma 3.3, this is the only graph with less than five vertices which is uniquely 2-list colourable.

## V. EDGE COLOURINGS.

Next, we obtain theorems analogous to Theorems 4.2 and 4.3 for edge-colourings.

We first prove a lemma and we shall again suppose that the colours prescribed by $\Sigma$ are $c_1, c_2, \ldots, c_r$.

*Lemma* 5.1: Let $G$ be a simple graph and $W$ a weakly completable subset of its edge set $E$ relative to an $r$-colouring $\Sigma$ ($r \geq 3$). Let $S$ be the strong partial completion of $W$ in $G$ and let $T = E \setminus S$. Suppose, if possible, that such a graph exists with less than $2r$ edges. Then the colouring given by $\Sigma$ is such that there exists exactly one edge with the property that it is the only edge of $G$ to be assigned a particular colour. Moreover, that edge is in $S$ and every edge of $T$ is adjacent to it. Hence all the other colours are assigned to exactly two edges and $|E| = 2r-1$.

*Proof*: Let $r \geq 3$ and suppose that $|E| < 2r$. If every colour occurs on at least two edges of $G$ then $|E| \geq 2r$, so this is not the case. Thus, there exists at least one colour, say $c_1$, which is assigned by $\Sigma$ to at most one edge of $G$. But, from Lemma 3.1(iv) and the observation that $T$ must contain at least one edge, it follows that $\Sigma$ assigns every one of the colours $c_1, c_2, \ldots, c_r$ to at least one edge of $G$. We conclude that there is exactly one edge, say $[v_1, v_2] = e_{12}$ which is assigned the colour $c_1$.

If $e_{12} \in T$, then, by Lemma 3.1(ii), there exists a colour $c'$ distinct from $c_1$ and from the colours assigned by $\Sigma$ to the edges of $S$ which are adjacent to $e_{12}$ and there is at least one edge $e'$ of $T$ adjacent to $e_{12}$ which is assigned this colour $c'$ by $\Sigma$. But, in that case, since no edge except $e_{12}$ has colour $c_1$, an alternative edge colouring $\Sigma$ of $G$ exists in which $e_{12}$ is given the colour $c'$ and the edge or edges adjacent to it which had colour $c'$ are given the colour $c_1$ instead. This contradiction to the unique completability of the edge colouring of $W$ shows that $e_{12} \notin T$. Hence, every edge in $T$ is coloured with a colour which occurs on more than one edge of $G$.

If there is an edge $e''$ which is in $T$ and is not adjacent to $e_{12}$, we can obtain a proper edge colouring of $G$ distinct from $\Sigma$ by replacing the colour assigned by $\Sigma$ to $e''$ by the colour $c_1$ of $e_{12}$. This contradicts the unique completability of the colouring assigned by $\Sigma$ to the edges of $W$. We conclude that, if $|E| < 2r$, the colouring given by $\Sigma$ is such that there exists at least one edge $e_{12}$ which is the only edge of $G$ to be assigned a particular colour, every such edge is in $S$, and every edge of $T$ is adjacent to each such edge.

Suppose that there is a second such edge $e^*$. Then depending on whether or not it is adjacent to $e_{12}$, we can suppose without loss of generality that $e^* = [v_1, v_3]$ or $[v_3, v_4]$. But if $e^* = [v_1, v_3]$, the only possible edges in $T$ are $[v_2, v_3]$ and those incident to $v_1$ and, if $e^* = [v_3, v_4]$, the only possible edges in $T$ are $[v_1, v_3]$, $[v_2, v_3]$, $[v_1, v_4]$ and $[v_2, v_4]$. In each case all of the blocks of the line graph of the subgraph $X$ of $G$ induced by the edges of $T$ must be a cycle, a complete graph, or a complete bipartite graph. However, this line graph of $X$ is uniquely 2-list colourable by Lemma 3.1(ii) (the appropriate list of colours for each vertex $v$ corresponding to an edge of $T$ being those of $\Sigma$ which are not used by $\Sigma$ to colour the edges of $S$ which are adjacent to the edge corresponding to $v$), contradicting the Corollary to Lemma 3.3. Therefore there is only one such edge $e_{12}$ and the lemma follows.

*Theorem* 5.2: Let $G$ be a simple graph and $W$ a weakly completable subset of its edge set $E$ relative to an $r$-colouring $\Sigma\,(r \geq 3)$. Then $|E| \geq 2r$.

*Proof*: We first prove that, if $r = 3$, $G$ has at least six edges. We then use induction on the number $r$ of colours. We use $S$ to denote the strong partial completion of $W$ in $G$.

Suppose that $r = 3$ and that $G$ has less than six edges. Then the line graph $L(G)$ of G has fewer than six vertices and the subset of vertices of $L(G)$ which correspond to the edges of $W$ form a weakly completable subset of vertices of $L(G)$ relative to the vertex 3-colouring of $L(G)$ which corresponds to $\Sigma$ for $G$ (since vertices of $L(G)$ are adjacent if and only if edges of $G$ are adjacent). But, by Theorem 4.2, a graph (connected or not) for which a weakly completable subset of vertices relative to a vertex 3-colouring exists cannot have less than six vertices or less than eight edges. This is a contradiction, so $r = 3 \Rightarrow |E| \geq 6$.

We now take as an induction hypothesis that the theorem is true for all $r \leq k-1$, where $k \geq 4$. In particular, we assume that, for every simple graph $G' = (V', E')$ which has a weakly completable subset $W'$ of its edge-set $E'$ relative to a $(k-1)$-colouring $\Sigma_{k-1}$, we have $|E'| \geq 2(k-1)$.

If the theorem is false for $r = k$, then there exists a simple graph $G = (V, E)$ such that $|E| < 2k$ and such that there is a weakly completable subset $W$ of its edge-set $E$ relative to some $k$-colouring $\Sigma_k$ of $G$. By Lemma 5.1, $|E| = 2k-1$ and there is exactly one edge $[v_1, v_2] = e_{12}$ which is the only edge of $G$ to be assigned a particular colour $c_1$, this edge is in $S$, and every edge of $T$ is adjacent to it. All the other colours ($c_2$, $c_3$, ... , $c_k$) are assigned to exactly two edges.

Now let $c_i$ be any one of the colours which $\Sigma_k$ assigns to two edges of $G$ and let $G' = (V', E')$ be the graph which is obtained from $G$ by deleting the pair $Z$ of edges of $G$ which are assigned colour $c_i$ by $\Sigma_k$. Then, since $|E| = 2k-1$, we have $|E'| = 2k-3$. Let $W' = S \setminus (S \cap Z)$. Clearly, $W'$ can be completed to a $(k-1)$-colouring of $G'$ in which all the edges of $E'$ have the colours assigned to them by the $k$-colouring $\Sigma_k$ of $G$. If $W'$ could be completed to an alternative $(k-1)$-colouring of $G'$ then $S$ could be completed to an alternative $k$-colouring of $G$ by adjoining to $G'$ the two edges of $Z$ and giving each of them the colour $c_i$. This contradicts the hypothesis that the colouring of $W$ (and so $S$) assigned by $\Sigma_k$ is uniquely completable (weakly) to a $k$-colouring of $G$. We conclude that the colouring of $W'$ using the colours assigned by $\Sigma_k$ has a unique completion to a $(k-1)$-colouring of $G'$. Since $|E'| = 2k-3 < 2(k-1)$, this completion cannot be weak, so it must be strong. Thus, since $E' \setminus W'$ consists of those edges of $E \setminus W$ which are not assigned colour $c_i$ by $\Sigma_k$ and this is non-empty by Lemma 3.1(iii), there exists an edge $e \in E' \setminus W' \subset T$ which is adjacent to edges of $W'$ which jointly are assigned (by $\Sigma_k$) $k-2$ of the $k-1$ colours distinct from $c_i$. If $e$ is adjacent in $G$ to one of the edges of $S$ assigned the colour $c_i$ by $\Sigma_k$, then the colour of $e$ is forced in $G$ (as well as in $G'$). This implies that $e \in S \cap E' = W'$, contradicting the fact that $e \in E' \setminus W'$. So $e$ is not adjacent in $G$ to any edge of $S$ which is assigned the colour $c_i$ by $\Sigma_k$.

This argument can be repeated for each colour $c_i$ except $c_1$. Hence, for each colour $c_i$ ($i = 2, 3, \ldots, k$), there exists an associated edge $e(c_i) \in T$ which is not adjacent to any edge of $S$ which has the colour $c_i$ under $\Sigma_k$ but which is adjacent to edges of $S$ to which $\Sigma_k$ assigns every colour except that of itself and the colour $c_i$. There are now two possibilities. *Case* 1: Every colour appears on some edge of $S$; or *Case* 2: There is no edge in $S$ of colour $c_j$ and all the associated edges $e(c_2), e(c_3), \ldots, e(c_k)$ except $e(c_j)$ have the colour $c_j$. We shall show that Case 2 cannot occur but first we make a deduction valid for both cases.

Suppose, if possible, that the edges $e(c_i)$ and $e(c_j)$, $j \neq i$, coincide: say $e(c_i) = e(c_j) = e^*$, and that the edge $e^*$ is assigned the colour $c_l$ by $\Sigma_k$. Then the edge $e^* = e(c_i)$ is adjacent to edges of $S$ which are assigned $k-2$ colours distinct from $c_i$ and $c_l$ and also the edge $e^* = e(c_j)$ is adjacent to edges of $S$ which are assigned $k-2$ colours distinct from $c_j$ and $c_l$. This is absurd since there are only $k$ colours all together. We conclude that the $k-1$ ($> 2$) edges $e(c_i)$, $i = 2, 3, \ldots, k$, are all distinct.

We now return to Case 2: that is, we suppose that all edges $e(c_i)$, $i \neq j$, are coloured with the same colour $c_j$. Then, since $c_j$ is assigned to only two edges in total, $k = 4$ and the only colours are $c_1, c_2, c_3, c_4$ of which colour $c_1$ is assigned to just one edge (which is in $S$ and such that every edge of $T$ is adjacent to it). We may suppose that no edge of $S$ has colour $c_4$ and that both of the edges $e(c_1)$ and $e(c_2)$ have that colour. Since $|S| \geq k-1 = 3$ by Lemma 3.1(i) and $|E| = 2k-1 = 7$ by hypothesis and Lemma 5.1, $|T| \leq 4$. Now, the line graph of the subgraph of $G$ induced by the edges of $T$ is uniquely 2-list colourable (the appropriate list of colours for each vertex $v$ corresponding to an edge of $T$ being those of $\Sigma$ which are not used by $\Sigma$ to colour the edges of $S$ which are adjacent to the edge corresponding to $v$). However, this line graph has at most four vertices so, by the corollary to Lemma 3.3, the only possibility is that it is isomorphic to the complete graph $K_4$ minus an edge (shown in Fig. 5.1(a)). Since the unique edge of $G$ which has colour $c_1$ is adjacent to every edge of $T$, the line graph of $G$ must contain the graph shown in Fig. 5.1(b), where vertex $v$ corresponds to the edge of $G$ which has colour $c_1$. But by a result of Beineke (see Theorem 8.4 of [H1], page 74), the latter graph cannot be contained in the line graph of any graph. This contradiction shows that Case 2 cannot occur.

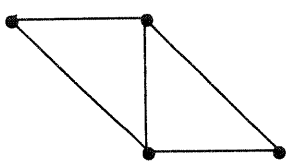
Fig. 5.1(a).

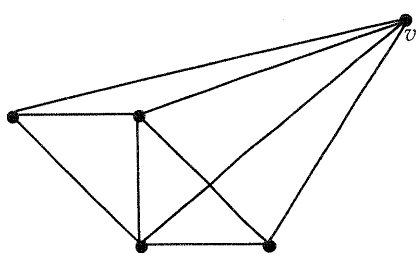

Fig.5.1(b).

Hence we are left with Case 1: that is, for each colour $c_i$ ($i = 2, 3, \ldots, k$), there exists an edge in $S$ of that colour and, hence, there is at most one edge in $T$ of each colour. Since the $k-1$ edges $e(c_i)$, $i = 2, 3, \ldots, k$, are each assigned one of these $k-1$ colours (none of these edges is $e_{12}$, the only one coloured $c_1$, since it is in $S$), each of the colours

$c_i$, $i = 2, 3, \ldots, k$, is assigned to exactly one of them and there can be no other edges in $T$.

But, in that case, it is clear that an edge-colouring of $G$ exists in which the colours of $S$ are as assigned by $\Sigma_k$ and the colours of the edges of $T$ are such that the edge $e(c_i)$ has the colour $c_i$ (for $i = 2, 3, \ldots, k$). Thus $S$ is not uniquely completable either strongly or weakly to the colouring $\Sigma_k$ of $G$.

Thus, in both cases, we have a contradiction to the unique completability of the colouring of $W$. We conclude that $|E| \geq 2k$ and so, by induction on the integer $k$, the theorem is proved.

*Theorem* 5.3: For every $r \geq 3$, there is at least one connected graph with $2r$ edges for which a weakly completable subset of edges relative to an $r$-colouring exists.

*Proof*: We prove the result by example. For $r = 3, 4, 5, 6$ the graphs $\Delta_r$ which we construct have 5, 5, 5, 6 vertices respectively. For each $r \geq 7$, we use an iterative procedure to construct a suitable graph $\Delta_r$ with $r-1$ vertices. We illustrate the graphs $\Delta_r$ and the colours which are to be assigned to the appropriately-sized weakly completable subsets of their edges in Fig. 5.2 and Fig. 5.3 following.

For $r \in \{3, 4, 5\}$, let $\Delta_r = (V_r, E_r)$ where $V_r = \{1, 2, 3, 4, 5\}$ and $E_r$ is defined as follows: $E_3 = \{[1, 2], [1, 3], [2, 3], [2, 4], [3, 5], [4, 5]\}$, $E_4 = E_3 \cup \{[1, 4], [2, 5]\}$ and $E_5 = E_4 \cup \{[1, 5], [3, 4]\}$ so that $|E_r| = 2r$. For $r \in \{3, 4, 5\}$, let $\Sigma_r$ be an $r$-colouring of $\Delta_r$ (using colours $c_1, c_2, \ldots, c_r$) which assigns colour $c_1$ to edge [2, 4], colour $c_2$ to edge [4, 5] and, if $r \geq 4$, colour $c_4$ to edges [1, 4] and [2, 5] and, if $r = 5$, colour $c_5$ to edges [1, 5] and [3, 4].

Let $T = \{[1, 2], [1, 3], [2, 3], [3, 5]\}$ and $W_r = E_r \setminus T$. Then $W_r$ is not a strongly completable subset of $E_r$ relative to $\Sigma_r$ since, after the assignment of the colours of $\Sigma_r$ to the edges of $W_r$, each of the uncoloured edges (i.e. each edge in $T$) is such that those coloured edges which are adjacent to it use, between them, no more than $r-2$ colours. (The fact that this is true for $r = 3$ implies that it is true for $r = 4$ and $r = 5$ successively.)

However $W_r$ is a uniquely completable subset of $E_r$ relative to $\Sigma_r$. We may see this as follows: After the edges of $W_r$ have been coloured in the colours of $\Sigma_r$, the pair of adjacent edges ([1, 2], [2, 3]) must be coloured $(c_2, c_3)$ or $(c_3, c_2)$. In either case, edge [1, 3] must be $c_1$. This implies that edge [3, 5] must be coloured $c_3$ and so the pair ([1, 2], [2, 3]) can only be coloured $(c_3, c_2)$. It follows that $W_r$ is a weakly completable subset of $E_r$ relative to the $r$-colouring $\Sigma_r$ thus defined.

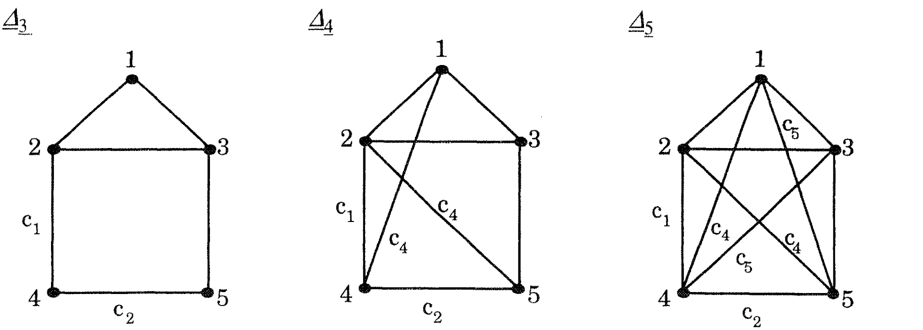


Fig. 5.2.

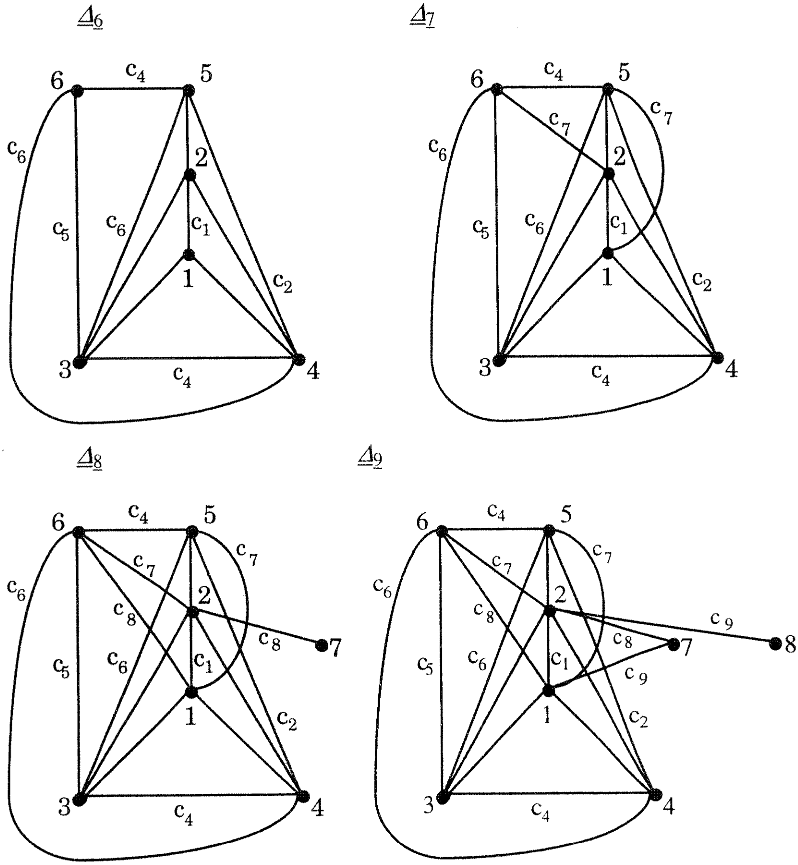


Fig. 5.3.

For $r = 6$, let $\Delta_6 = (V_6, E_6)$, where $V_6 = \{1, 2, \ldots, 6\}$ and $E_6 = \{[1, 2], [1, 3], [1, 4], [2, 3], [2, 4], [2, 5], [3, 4], [3, 5], [3, 6], [4, 5], [4, 6], [5, 6]\}$. For $r \geq 7$, let $\Delta_r = (V_r, E_r)$, where $V_r = \{1, 2, \ldots, r-1\}$ and $E_r$ is defined as follows: $E_r = E_{r-1} \cup \{[1, r-2], [2, r-1]\}$, so that again $|E_r| = 2r$.

For each $r \geq 6$, let $\Sigma_r$ be an $r$-colouring of $\Delta_r$ (using colours $c_1, c_2, \ldots, c_r$) which assigns colour $c_1$ to edge [1, 2], colour $c_2$ to edge [4, 5], colour $c_4$ to edges [3, 4] and [5, 6], colour $c_5$ to edge [3, 6], colour $c_6$ to edges [3, 5] and [4, 6] and, if $r \geq 7$, for each $k$ such that $7 \geq k \geq r$, colour $c_k$ to edges $[1, k-2]$ and $[2, k-1]$.

Let $T = \{[1, 3], [1, 4], [2, 3], [2, 4], [2, 5]\}$ and let $W_r = E_r \setminus T$. Then $W_r$ is not a strongly completable subset of $E_r$ relative to $\Sigma_r$ since, after the assignment of the colours of $\Sigma_r$ to the edges of $W_r$, each of the uncoloured edges (that is, each edge in $T$) is such that those coloured edges which are adjacent to it use, between them, no more than $r-2$ colours. (The fact that this is true for $r = 6$ implies that it is true for $r = 7$, $r = 8$ successively, and so on.)

However $W_{r'}$ is a uniquely completable subset of $E_r$ relative to $\Sigma_{r'}$. We may see this as follows: After the edges of $W_{r'}$ have been coloured in the colours of $\Sigma_{r'}$ , the pair of adjacent edges ([2, 4], [2, 5]) must be coloured ($c_3$, $c_5$) or ($c_5$, $c_3$). In either case, edge [2, 3] must have colour $c_2$ and so edge [1, 3] must have colour $c_3$. This means that edge [1, 4] must be coloured $c_5$ and that the pair ([2, 4], [2, 5]) can only be coloured ($c_3$, $c_5$). It follows that $W_{r'}$ is a weakly completable subset of $E_r$ relative to the $r$-colouring $\Sigma_{r'}$ thus defined.

ACKNOWLEDGEMENT.

The authors wish to thank the referee for drawing to their attention the recent paper [M2] which enabled them to simplify several of the above proofs.

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(Received 29/2/2000)