On the spectrum of nested G-designs, where G has four non-isolated vertices or less

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Abstract

The spectrum problem for G-decompositions of λK_n that have a nesting was first considered in the case $G \cong K_3$ by C.J. Colbourn and M.J. Colbourn (1983) and by D.R. Stinson (1985). For $\lambda = 1$ and $G \cong C_m$ this problem was studied in many papers (see C.C. Lindner and C.A. Rodger, Chapter 8 in Contemporary Design Theory: a collection of surveys, Wiley 1992, and D.R. Stinson, Utilitas Math. **33** (1988) for more details and references). In this paper we generalize the nesting definition given by C.J. Colbourn and M.J. Colbourn [Ars Combin. **16** (1983), 27–34] and we study the spectrum problem in the case that G has four non-isolated vertices or less.

1 Introduction

Let λK_n be the complete multigraph on n vertices, where every edge is repeated λ times. If G is a graph, the multigraph λK_n is said to be *G-decomposable* if it is the union of edge-disjoint subgraphs of K_n, each of them isomorphic to G. This situation is denoted by $\lambda K_n \to G$; λK_n is also said to admit a *G-decomposition* $\Sigma = (V, \underline{B})$, where V is the vertex-set of λK_n and $\underline{B}$ is the edge-disjoint decomposition of λK_n into copies of G. Usually $\underline{B}$ is called the *block-set* of the G-decomposition and any $B \in \underline{B}$ is said to be a *block*.

A G-decomposition of λK_n, $\Sigma = (V, \underline{B})$, is also called a *$G$-design of order n, block-size $|V(G)|$ and index λ* [3]. A G-design $\Sigma^* = (V^*, \underline{B}^*)$ is said to be a *subdesign* of $\Sigma = (V, \underline{B})$ if $V^* \subseteq V$ and $\underline{B}^* \subseteq \underline{B}$. More generally, it is possible to define G-decompositions of λH, instead of λK_n, where H is any graph.

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A *path-design* $P(n,k,\lambda)$ is a G-design of order n, block-size k, index λ, where G is a *path* on k vertices, i.e. a graph having for vertices $x_1, x_2, \ldots, x_k$ and for edges all the pairs $\{x_i, x_{i+1}\}$, for every $i = 1, 2, \ldots, k-1$. Such a *path* will be denoted by $\langle x_1, x_2, \ldots, x_k \rangle$.

A *star-system* $S(n,m,\lambda)$ is a G-design of order n, block-size $m+1$, index λ, where G is a *star* with m terminal vertices, i.e. $G \cong S_m$ graph having $m+1$ vertices x' *(centre)*, $x_1, x_2, \ldots, x_m$ *(terminal)* and for edges all the pairs $\{x', x_i\}$, for every $i = 1, 2, \ldots, m$. Such a *star* will be denoted by $\langle x'; x_1, x_2, \ldots, x_m \rangle$.

An *m-cycle-system* $\mathrm{CS}(n,m,\lambda)$ is a G-design of order n, block-size m, index λ, where $G \cong C_m$, the cycle with m vertices.

A *Steiner triple system* $S_\lambda(2,3,v)$ is a C_3-design or also a K_3-design.

In the literature there are some definitions of *nesting* for G-designs, mainly, for $\lambda = 1$ and $G \cong C_m$.

Let $\Sigma = (V, C)$ be a C_m-design having order n and index $\lambda = 1$.

A *nesting* of the C_m-design Σ is a mapping $f : C \to V$ such that the set $\Pi = \{\{x, f(c)\} : c \in C, x \text{ vertex of } c\}$ is a partition of the edges of K_n. Observe that any nesting of a C_m-design produces an edge-disjoint decomposition of K_n into m-stars. It is clear that a nesting of an m-cycle-design of order n is equivalent to an *edge-disjoint decomposition* of $2K_n$ into *wheels* W_m having the additional property that for each pair of vertices x, y, one of the edges joining x to y is on the *rim* of a wheel and the other is the *spoke* of a wheel.

The spectrum problem for *m-cycle*-systems that have a *nesting* was first considered in the case where $m = 3$, i.e. for $S(2,3,v)$. This case was studied by C.J. Colbourn and M.J. Colbourn [1] and by C.C. Lindner and C.A. Rodger [4] who left 15 possible exceptions; Stinson [13] completed the spectrum. *Nested* 4-*cycle*-systems were studied by Stinson [14], while *nested* 5-*cycle*-systems were studied by Lindner and Rodger [4]. Further, general results have been obtained by Lindner, Rodger and Stinson [5].

The same definition of *nesting* can be given for G-designs, in which $G = (V(G), E(G))$ is not necessarily a cycle. A necessary condition is that

$$|V(G)| = |E(G)|.$$

Recently, Milici and Quattrocchi [10] have given the following definition.

Let $G = (V(G), E(G))$ be a graph and let $\Sigma = (V, \underline{B})$ be a *G-decomposition of* λK_n. A *nesting* of Σ is a triple $N = \{\Sigma, \Pi, F\}$, where $\Pi = (V(K_n), S)$ is a decomposition of λK_n into m-stars S_m and $F : \underline{B} \to S$ is a 1-1 mapping such that:

(i) for every $B \in \underline{B}$, the *centre* of the m-star $F(B)$ does not belong to $V(B)$; all the *terminal* vertices of $F(B)$ belong to $V(B)$;

(ii) for every pair $B_1, B_2 \in \underline{B}$, the graphs $B_1 \cup F(B_1), B_2 \cup F(B_2)$ are isomorphic.

A necessary condition is that $|V(G) \geq |E(G)|$. If $|V(G)| = |E(G)|$, this definition is equivalent to the previous.

In this paper we give the following definition of nesting of a G-design and we study the spectrum for all G-designs in which G is a graph having four non-isolated vertices, or less.

Let $G = (V(G), E(G))$, $H = (V(H), E(H))$ be two graphs and let $\Sigma = (V, \underline{B})$ be a *G-design of index* λ_1, briefly $\lambda_1 H \to G$. A *nesting* $N(G, H; \lambda_1, \lambda_2)$ of Σ is a triple (Σ, Π, F), where $\Pi = (V(H), S)$ is an *m-star-design of index* λ_2, briefly $\lambda_2 H \to S_m$, and $F : \underline{B} \to S$ is a bijection such that for every $B \in \underline{B}$:

(i) the centre of the m-star $F(B)$ does not belong to $V(B)$;

(ii) x is a terminal vertex of $F(B)$ if and only if x is a vertex of $V(B)$.

In what follows, when $H \cong K_n$, such a nesting will be denoted by $N = N(G, n; \lambda_1, \lambda_2)$. Observe that N is a *G^*-design* of order n, block-size $|V(G)| + 1$ and index $\lambda = \lambda_1 + \lambda_2$, where $G^* = G \cup S_{|V(G)|}$.

If $\lambda_1 = \lambda_2 = \lambda$, this definition is the same as given in [1], [7], [10].

Further:

- $(x_1, x_2, \ldots, x_n)$ will be a *cycle* C_n;
- $\langle x_1, x_2, \ldots, x_n\rangle$ will be a *path* P_n;
- $\langle y; x_1, x_2, \ldots, x_n\rangle$ will be a *star* S_n with *centre* y;
- $[y; x_1, x_2, \ldots, x_n]$ will be $P_n \cup S_n$, where $P_n = \langle x_1, x_2, \ldots, x_n\rangle$ and $S_n = \langle y; x_1, x_2, \ldots, x_n\rangle$;
- $(y; (x_1, x_2, \ldots, x_n))$ will be a *wheel* with centre y.

Example A nesting $N(P_3, 7; 2, 3)$ is given by

- the P_3-design $\Sigma = (V, \underline{B})$, having index $\lambda_1 = 2$ and order $v = 7$, so defined: $V = Z_7$ and $\underline{B} = \{\langle i, i+1, i+2\rangle, \langle i, i+2, i+4\rangle, \langle i, i+3, i+6\rangle \mid i \in Z_7\}$;
- the S_3-design $\Pi = (V, S)$, having index $\lambda_2 = 3$ and order $v = 7$, so defined: $V = Z_7$ and $S = \{\langle i+5; i, i+1, i+2\rangle, \langle i+3; i, i+2, i+4\rangle, \langle i+1; i, i+3, i+6\rangle \mid i \in Z_7\}$;
- $F(\langle i, i+1, i+2\rangle) = \langle i+5; i, i+1, i+2\rangle$,
  $F(\langle i, i+2, i+4\rangle) = \langle i+3; i, i+2, i+4\rangle$,
  $F(\langle i, i+3, i+6\rangle) = \langle i+1; i, i+3, i+6\rangle$.

Result 1: Observe that in the case $G \cong K_n$ this new definiton of nesting is the same as given by Kageyama and Miao [7], [8], [9].

Result 2: Note that if there exists an $N(G, n; \lambda_1, \lambda_2)$, then there exists also an $N(G, n; h\lambda_1, h\lambda_2)$. It is sufficient to repeat all the blocks h times.

Result 3: In what follows, when a G-design is defined on $Z_n = \{0, 1, 2, \ldots, n-1\}$, it is understood that all the sums in Z_n must be reduced mod n.

2 Preliminary results

In this section we give some definitions and theorems useful to construct *nestings* of a G-design, i.e. *nested G-designs*. In some of them we will use pairwise balanced designs and group divisible designs.

Let X be a finite set of *points*, $\mathcal{C}$ a family of distinct subsets of X called *groups* which partition X, A a collection of subsets of X called *blocks*. Let v and λ be positive integers and K, M sets of positive integers. The triple $(X, \mathcal{C}, A)$ is a *group divisible design*, briefly a GDD, GD$[K, \lambda, M; v]$ if:

(c_1) $|X| = v$;

(c_2) $\{|C| \mid C \in \mathcal{C}\} \subseteq M$;

(c_3) $\{|B| \mid B \in \mathcal{B}\} \subseteq K$;

(c_4) $|C \cap B| \leq 1$, for every $C \in \mathcal{C}$, $B \in \mathcal{B}$;

(c_5) every pair $\{x, y\} \subseteq X$, such that x, y belong to distinct groups, is contained in exactly λ blocks of A.

If C contains t_i groups of size m_i, for $i = 1, 2, \ldots, s$, the GDD is said to have *group type* $m_1^{t_1} m_2^{t_2} \ldots m_s^{t_s}$. When $K = \{k\}$, we will write GD$[k, \lambda, M; v]$ instead of GD$[\{k\}, \lambda, M; v]$.

A GD$[K, \lambda, \{1\}; v]$ having group type 1^v is called a *pairwise balanced design* and is denoted by (X, A) or by (v, K, λ)-PBD. A (v, k, λ)-PBD is simply a K_k-design. For $\lambda = 1$, a $(v, k, 1)$-PBD is a (v, k)-PBD.

A GD$[k, 1, \{m\}; km]$ is called a *transversal design*, denoted by TD$[k, m]$; it is also called a k-GDD.

A (v, k, λ)-BIBD *(balanced incomplete block-design)* or an $S_\lambda(2, k, v)$ *(Steiner system of index λ)* is a pair $(V, \underline{B})$, where V is a finite v-set and $\underline{B}$ is a collection of k-subsets of V, called *blocks*, such that every 2-subset of V is contained in exactly λ blocks of $\underline{B}$.

A *parallel class* of a (v, k, λ)-BIBD $(V, \underline{B})$ is a set of blocks of $\underline{B}$ that partition V. A (v, k, λ)-BIBD is said to be *resolvable* and is denoted by (v, k, λ)-RBIBD if $\underline{B}$ can be partitioned into parallel classes.

A *near resolvable* $(v, k, k-1)$-BIBD, briefly a $(v, k, k-1)$-NRB, is a $(v, k, k-1)$-BIBD with the property that $\underline{B}$ can be partitioned into partial parallel classes missing a single $x \in V$ and every $x \in V$ is absent from exactly one class.

Theorem 2.1 [3]: *Let $G = (V(G), E(G))$ be a graph and let $\Sigma = (V, \underline{B})$ be a G-design of index λ_1. A necessary condition for the existence of a $N(G, n; \lambda_1, \lambda_2)$ is that $\lambda_1 |V(G)| = \lambda_2 |E(G)|$.*

The following two theorems are special cases of the Wilson fundamental construction for group divisible designs and other well-known theorems. So we will omit the proofs.

Theorem 2.2: *Let $\Sigma = (X, A)$ be a (n, K)-PBD, where $K = \{h_1, h_2, \ldots, h_t\}$, and let G be a graph. If, for every $h_i \in K$, there exists a nesting $N(G, h_i; \lambda_1, \lambda_2)$, then there exists a nesting $N(G, n; \lambda_1, \lambda_2)$.*

Theorem 2.3: *Let* $\Lambda = (X, P, A)$ *be a* k*-GDD of order* n*, where* $P = \{P_1, P_2, \dots, P_t\}$ *and* $|P_i| = n_i$*, and let* G *be a graph. If, for every* n_i*, there exists a nesting* $N(G, mn_i + w; \lambda_1, \lambda_2)$ *containing a sub-design* $N(G, w; \lambda_1, \lambda_2)$ *(where* $w = 0, 1$*) and there exists a nesting* $N(G, K_{m_1, m_2, \dots, m_k}; \lambda_1, \lambda_2)$ *(where* $m_1 = m_2 = \dots = m_k$*), then there exists a nesting* $N(G, mn + w; \lambda_1, \lambda_2)$*.*

We prove the following

Theorem 2.4: *Let* $G \cong P_3, P_4, S_3, K_4 - e$ *and suppose that there exist a nesting design* $N(G, v; \lambda_1, \lambda_2)$*, a nesting design* $N(G, w; \lambda_1, \lambda_2)$*, two orthogonal quasi-groups of order* $w - q$*, where* $q = 0$ *or* 1*. Then there exist nesting designs* $N(G, v(w-q) + q; \lambda_1, \lambda_2)$*.*

Proof: At first, consider two orthogonal quasigroups of order $w - q$ (they exist for every $w - q \neq 2, 6$); let $(Z_{w-q}, \circ)$, $(Z_{w-q}, *)$.

Let $G \cong P_3$.

If $(Z_v, \underline{B})$ is a nesting design $N(P_3, v; \lambda_1, \lambda_2)$, $T = \{\infty\}$ for $q = 1$ and $T = \emptyset$ for $q = 0$, then it is possible to define the design $N(P_3, v(w-q) + q; \lambda_1, \lambda_2)$ $(V, \underline{D})$ as follows:

i) for every $[x; a, b, c] \in \underline{B}$ put in $\underline{D}$ the blocks $[(x, i \circ j); (a, i), (b, j), (c, i)]$, $i, j \in Z_{w-q}$;

ii) for every $x \in Z_v$, put in $\underline{D}$ the blocks of a design $N(P_3, w; \lambda_1, \lambda_2)$ defined on $\{x\} \times Z_{w-q} \cup T$.

The same technique can be used in the cases $G \cong P_4, S_3$.

Let $G \cong K_4 - e$.

Using the same symbolism of the case above, it is possible to define the nesting design $(V, \underline{D})$ of order $v(w-q) + q$ as follows:

i) for every $\{x; a, b, (c, d)\} \in \underline{B}$ (c, d are the non-adjacent vertices) put in $\underline{D}$ the blocks: $\{(x, i \circ j); (a, j), (b, i * j), ((c, i), (d, i))\}$, $i, j \in Z_{w-q}$;

ii) for every $x \in Z_v$, put in $\underline{D}$ the blocks of a design $N(K_4 - e, w; \lambda_1, \lambda_2)$ defined on $\{x\} \times Z_{w-q} \cup T$.

Theorem 2.5 [3]: *If there exists a nesting* $N(C_m, n; 1, 1)$*, then there exists a nesting* $N(P_k, n; k-1, k)$*, for every integer* k *such that* $3 \leq k < m$*.*

Theorem 2.6: *For every* $k \geq 3$ *and for every* $n \geq 2k + 1$*,* n *odd, there exists a nesting* $N(P_k, n; k-1, k)$*.*

The statement follows from Theorem 2.5 and from the existence of a nesting-design $N(C_m, n; 1, 1)$ for all $n = 2m + 1$ and $m \geq 3$ [4].

Theorem 2.7: *If there exists a* $(v, k, k-1)$*-NRB, then there exists a nesting* $(v, k, k-1)$*-BIBD.*

Proof: Let $\Sigma = (V, \underline{B})$ be a $(v, k, k-1)$-NRB. Further, for every block $B \in \underline{B}$, if Π_B is the almost-parallel class containing B, $f(B)$ is the element of V which does not belong to its blocks. It is immediate to see that it is possible to obtain a nesting $(v, k, k-1)$-BIBD to associate each block B of $\underline{B}$ with the star having $f(B)$ as centre and the elements of B as terminal vertices.

3 $N(G, n; \lambda_1, \lambda_2)$ where G has $n \leq 3$ non-isolated vertices

If G has 2 non-isolated vertices, then $G \cong K_2 \cong P_2$.

It is known that the spectrum of the nesting designs $N(P_2, n; \lambda_1, \lambda_2)$ was completely determined by Kageyama and Miao [7].

Now, we study the spectrum of a nesting $N(G, n; \lambda_1, \lambda_2)$, where G has 3 non-isolated vertices. Two cases are possible: 1) $G \cong K_3$, 2) $G \cong P_3$.

3.1 $G \cong K_3$

It is well-known that the spectrum of the nesting designs $N(K_3, n; 1, 1)$ was completely determined by Stinson [13] and the results can be extended to designs $N(K_3, n; h, h)$, where $\lambda_1 = \lambda_2 = h \in N$, by a repetition of blocks.

3.2 $G \cong P_3$

From Theorem 2.1, necessary conditions for the existence of a nesting design $N(P_3, n; \lambda_1, \lambda_2)$ are: $n \geq 4$, $3\lambda_1 = 2\lambda_2$, i.e. $\lambda_1 = 2h$, $\lambda_2 = 3h$, $h \in N$.

Theorem 3.2.1: *If there exists a nesting $N(P_3, 4; 2h, 3h)$, then h is even.*

Proof: Suppose that (Σ, Π, F) is a nesting $N(P_3, 4; 2h, 3h)$. If x is a point of Π, T_x the number of blocks of Π containing x as a *terminal* vertex and C_x is the number of blocks of Π containing x as a *centre*, then

$$3C_x + T_x = 9h$$
$$C_x + T_x = 6h$$

From which $C_x = 3h/2$ and this implies h is even.

Theorem 3.2.2: *For every n prime, $n \geq 5$, there exists a nesting $N(P_3, n; 2, 3)$. Further, there exist $N(P_3, 6; 2, 3)$, $N(P_3, 8; 2, 3)$, $N(P_3, 10; 2, 3)$.*

Proof: Consider the following design, defined on Z_n and having the blocks:

$$[n+j-2; j, j+1, j+2]$$
$$[n+j-4; j, j+2, j+4]$$
$$\cdots\cdots$$
$$[n+j-2i; j, j+i, j+2i]$$
$$\cdots\cdots$$
$$[1; j, j+(n-1)/2, j+n-1] \quad \text{for every } j \in Z_5.$$

It is possible to verify that it is an $N(P_3, n; 2, 3)$.

Further, the following design, defined on Z_6 and having the blocks:

$$\begin{array}{llllll} [4;5,0,3], & [1;2,0,4], & [1;3,0,5], & [2;4,0,1], & [2;0,1,5], & [3;2,1,4], \\ [5;4,1,3], & [0;5,1,3], & [4;0,2,1], & [0;5,2,3], & [5;3,2,4], & [3;5,2,4], \\ [0;3,4,5], & [1;4,5,3], & [2;5,3,4] & & & \end{array}$$

is a nesting $N(P_3, 6; 2, 3)$.

The following design, defined on Z_8 and having the blocks:

$$\begin{array}{llllll} [2;0,1,4], & [2;0,1,5], & [1;0,2,6] & [3;2,0,7], & [4;3,0,5], & [1;0,3,6] \\ [3;0,4,5], & [2;0,4,6], & [3;0,5,6] & [4;0,6,7], & [5;0,6,7], & [6;0,7,2] \\ [7;1,2,3], & [7;1,2,4], & [6;5,1,3] & [4;1,3,6], & [5;1,4,6], & [0;7,1,6] \\ [4;7,1,6], & [6;2,3,7], & [3;7,2,5] & [7;5,2,6], & [5;2,4,7], & [1;5,3,4] \\ [2;5,3,4], & [0;7,4,5], & [0;6,5,7], & [1;5,7,3] & & \end{array}$$

is a nesting $N(P_3, 8; 2, 3)$.

The following design, defined on Z_{10} and having the blocks:

$$\begin{array}{llllll} [2;0,1,4], & [2;0,1,5], & [3;0,2,1], & [1;0,2,8], & [4;0,3,1], & [4;0,3,2], \\ [3;0,4,2], & [1;0,4,3], & [1;5,0,7], & [2;5,0,8], & [3;0,6,1], & [4;0,6,2], \\ [5;7,0,8], & [5;0,9,1], & [6;0,9,2], & [6;1,2,8], & [5;1,3,2], & [8;1,7,2], \\ [7;1,6,3], & [9;1,7,3], & [9;1,8,3], & [6;1,9,3], & [8;4,1,5], & [7;1,8,4], \\ [6;2,4,3], & [8;2,5,5], & [9;2,5,4], & [9;2,6,4], & [9;2,7,4], & [7;2,9,4], \\ [8;3,5,4], & [7;3,6,5], & [8;3,7,5], & [9;3,8,5], & [8;3,9,5], & [8;4,6,5], \\ [5;4,7,6], & [5;4,8,6], & [1;4,9,6], & [4;5,7,6], & [6;5,8,7], & [0;5,9,7], \\ [0;6,8,7], & [0;6,3,8], & [0;7,9,8] & & & \end{array}$$

is a nesting $N(P_3, 10; 2, 3)$.

Theorem 3.2.3: *There exists a nesting* $N(P_3, K_{2,2,2}; 2, 3)$.

Proof: Let $K_{2,2,2}$ be a 3-partite graph defined on $V = X \cup Y \cup Z$, where $X = \{x_0, x_1\}$, $Y = \{y_0, y_1\}$, $Z = \{z_0, z_1\}$ are the three stable sets which partition V. The following blocks:

$$\begin{array}{llll} [z_0; x_0, y_0, x_1], & [x_0; y_0, z_0, y_1], & [y_0; z_0, x_0, z_1], & [z_1; x_0, y_0, x_1], \\ [x_1; y_0, z_0, y_1], & [y_1; z_0, x_0, z_1], & [z_1; x_1, y_1, x_0], & [z_0; x_1, y_1, x_0], \\ [x_1; y_1, z_1, y_0], & [x_0; y_1, z_1, y_0], & [y_1; z_1, x_1, z_0], & [y_0; z_1, x_1, z_0] \end{array}$$

define a $N(P_3, K_{2,2,2}; 2, 3)$.

Theorem 3.2.4: *For every $n \geq 5$ there exists a $N(P_3, n; 2, 3)$, except possibly for $n = 12, 14, 16, 20, 22, 28, 68, 98, 124$.*

Proof: Since there exists a PBD(n) having blocks of size 5,6,7 ([2], p. 208), from Theorem 2.2 and Theorem 3.2.2 it follows that there exists a nesting $N(P_3, n; 2, 3)$ of order $n \geq 5$, with possible exceptions for $n = 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34, 68, 69, 93, 94, 98, 99, 104, 108, 109, 114, 124$.

From Theorem 2.6 and Theorem 3.2.2, the list of possible exceptions can be reduced to: $12, 14, 16, 18, 20, 22, 24, 28, 32, 34, 68, 94, 98, 104, 108, 114, 124$.

Since there exist 3-GDD of type 3^3, 4^3, 4^4, 5^1 and 3^4, 3^{14} and 5, 9^6 and 3 ([2], p. 189), from Theorem 2.3 and Theorem 3.2.3 it follows that the list of possible exceptions becomes: $12, 14, 16, 20, 22, 28, 68, 98, 104, 108, 124$.

From Theorem 2.4, for $(v, w) = (8, 13), (6, 18)$, there exist $N(P_3, n; 2, 3)$ for $n = v.w = 104, 108$.

Now, we examine the spectrum of nesting $N(P_3, n; \lambda_1, \lambda_2)$ for $\lambda_1 = 4,\ \lambda_2 = 6$.

Theorem 3.2.5: *There exist $N(P_3, 4; 4, 6)$, $N(P_3, 12; 4, 6)$, $N(P_3, 14; 4, 6)$.*

Proof: Consider the following design, defined on Z_3 and having the blocks:

$$\begin{array}{llllll} [0;1,2,3], & [0;1,3,2], & [0;2,1,3], & [1;0,2,3], & [1;0,3,2], & [1;2,0,3], \\ [2;0,1,3], & [2;0,3,1], & [2;1,0,3], & [3;0,1,2], & [3;0,2,1], & [3;1,0,2]. \end{array}$$

It is a nesting $N(P_3, 4; 4, 6)$.

Further, since there exists a 3-GDD of type 2^3 ([2], p. 189), the existence of a nesting $N(P_3, 12; 4, 6)$ follows from Theorem 2.3.

Finally, consider the following design, defined on $Z_{13} \cup \{\infty\}$ and having the blocks:

$$\begin{array}{lll} [j; j+1, j+3, j+2], & [j; j+7, j+4, j+8], & [j+1; j, j+5, j+11], \\ [j+7; \infty, j, j+5], & [j+8; \infty, j, j+6], & [j+6; j, \infty, j+1], \\ [\infty; j, j+5, j+11] & \text{for every } j \in Z_{13}. & \end{array}$$

It is a nesting $N(P_3, 14; 4, 6)$.

Theorem 3.2.6: *For every $n \geq 4$ there exists a $N(P_3, n; 4, 6)$.*

Proof: From Theorem 3.2.4, by a repetition of blocks, and from Theorem 3.2.5, it follows that there exists a nesting $N(P_3, n; 4, 6)$ for every $n \geq 4$, except possibly for $n = 16, 20, 22, 28, 68, 98, 124$. Since there exists a PBD(n) having blocks of size 4, 5, 6 ([2], p. 206), from Theorem 2.2 the existence of $N(P_3, n; 4, 6)$ follows in all the other cases.

Collecting together the results obtained, we can formulate the following.

Corollary 3.2 *The necessary conditions for the existence of a nesting design $N(P_3, n; \lambda_1, \lambda_2)$ are: $3\lambda_1 = 2\lambda_2$, $n \geq 4$. These conditions are also sufficient except in the following cases:*

i) $n = 4$ *and* $\lambda_1 \equiv 2 \bmod 4$, $\lambda_2 \equiv 3 \bmod 6$ *(effective exceptions);*

ii) $n = 12, 14, 16, 20, 22, 28, 68, 98, 124$, *when* $\lambda_1 \equiv 2 \bmod 4$, $\lambda_2 \equiv 3 \bmod 6$ *(possible exceptions).*

REMARK: Note that if it is possible to delete some exception in Corollary 3.2.*ii)*, for a pair λ_1^*, λ_2^*, giving a solution for it, then the same case can be considered solved for any $\lambda_1 = k\lambda_1^*$, $\lambda_2 = k\lambda_2^*$, $k \in N$. So, the number of exceptions in Corollary 3.2 is exactly 9 and not *infinite*.

This remark is valid also in all the following sections.

4 $N(G, n; \lambda_1, \lambda_2)$ where G has 4 non-isolated vertices

In this section we study the spectrum of a nesting G-design $N(G, n; \lambda_1, \lambda_2)$, where G is a graph with 4 non-isolated vertices. The possible cases are:
1) $G \cong K_4$, 2) $G \cong K_4 - e$, 3) $G \cong K_3 + e$, 4) $G \cong C_4$, 5) $G \cong P_4$, 6) $G \cong S_3$, 7) $G \cong 2P_2$.

Observe that $n \geq 5$, necessarily, and that the cases 3), 4) have already been studied.

4.1 $G \cong K_4$

For the necessary conditions we have the following theorem.

Theorem 4.1.1: *If there exists a nesting design* $N(K_4, n; \lambda_1, \lambda_2)$, *then the parameters* n, λ_1, λ_2 *must satisfy one of the following conditions:*

1) $\lambda_1 = 3h$, $\lambda_2 = 2h$, $n \equiv 1 \bmod 4$, $n \geq 5$, *for any positive odd integer* h;

2) $\lambda_1 = 3h$, $\lambda_2 = 2h$, $n \equiv 1 \bmod 2$, $n \geq 5$, *for any positive integer* $h \equiv 2 \bmod 4$;

3) $\lambda_1 = 3h$, $\lambda_2 = 2h$, $n \geq 5$, *for any positive integer* $h \equiv 0 \bmod 4$.

Proof: From Theorem 2.1, it follows that $2\lambda_1 = 3\lambda_2$, $n \geq 5$. Let $N = (\Sigma, \Pi, F)$ be a nesting $N(K_4, n; 3h, 2h)$. If x is a point of N, denote by M_x the number of blocks of Σ containing x and by C_x the number of blocks of Π containing x as *centre*. It follows that:

$$M_x = h(n-1),$$
$$4C_x + M_x = 2h(n-1),$$

hence $C_x = h(n-1)/4$. From this,

1) if h is an odd number, necessarily $n \equiv 1 \bmod 4$;

2) if h is an even number and $h \equiv 2 \bmod 4$, necessarily $n \equiv 1 \bmod 2$;

3) if $h \equiv 0 \bmod 4$, n can be any integer, $n \geq 5$.

Theorem 4.1.2: *There exists a nesting $N(K_4, n; 3, 2)$ if and only if $n \equiv 1$ mod 4.*

Proof: $\Rightarrow$ Immediate from Theorem 4.1.1, *1)*.

$\Leftarrow$ Since a $(n, k, k-1)$-NRB exists if and only if $n \equiv 1 \bmod k$ ([2], p. 88,91), the statement follows from Theorem 2.7.

Theorem 4.1.3: *For every $n \in N$, n prime, $n \geq 5$, there exists a nesting $N(K_4, n; 6, 4)$.*

Proof: Let n be a prime number, $n \geq 5$. Let $\Sigma = (Z_n, B)$ be the K_4-design having the following blocks:

$$B_{i,j} = \{x_{i,j,1} = j,\ x_{i,j,2} = j + i,\ x_{i,j,3} = j + 2i,\ x_{i,j,4} = j + 3i\},$$
$$\text{for every } j \in Z_n,\ i = 1, 2, \ldots, (n-1)/2.$$

We can verify that Σ has index $\lambda_1 = 6$. Observe that the differences between two vertices of $B_{j,i}$ are: $i, i, i, 2i, 2i, 3i$. Further, for $i = 1, 2, .., (n-1)/2$, $2i$ and $3i$ cover all the possible differences, respectively. So, if x, y are two vertices of Σ, $x < y$, $y - x = i$, $\{x, y\}$ is contained in exactly six blocks of Σ.

Now, consider the S_4-design $\Pi = (Z_n, S)$ having the following blocks:

$$S_{i,j} = \langle y_{i,j} = n - 2i + j; x_{i,j,1} = j, x_{i,j,2} = j + i, x_{i,j,3} = j + 2i, x_{i,j,4} = j + 3i \rangle,$$
$$\text{for every } j \in Z_n,\ i = 1, 2, \ldots, (n-1)/2.$$

Since n is prime, then $n - 2i + j \notin \{j, j + i, j + 2i, j + 3i\}$.

We can verify that Π has index $\lambda_2 = 4$. The differences between the *centre* and the other vertices of $S_{j,i}$ are: $n - 2i, n - 3i, n - 4i, n - 5i$, which are equivalent to: $2i, 3i, 4i, 5i$.

Since n is prime, for $i = 1, 2, \ldots, (n-1)/2$ each of them describes the set of all the possible differences. So, if x, y are two vertices of Π, $x < y$, $y - x = i$, $\{x, y\}$ is contained in exactly four blocks of Π.

If $F : B \to S$ is a mapping such that $F(B_{i,j}) = S_{i,j}$, then $N = (\Sigma, \Pi, F)$ is a nesting $N(K_4, n; 6, 4)$.

Theorem 4.1.4: *There exists a nesting $N(K_4, n; 6, 4)$ if and only if $n \equiv 1$ mod 2, except possibly for $n = 15, 27, 39, 75, 87, 135, 183, 195$.*

Proof: $\Rightarrow$ From Theorem 4.1.1. *2)*, for $h = 1$, directly.

$\Leftarrow$ Observe that if for any n there exists a nesting $N(K_4, n; 3, 2)$, then for this n there exists also a nesting $N(K_4, n; 6, 4)$. Further, for every admissible $n \equiv 1$ mod 2, there exists a PBD(n) having blocks of size $5, 7, 9$ ([2], p. 208), with some possible exceptions.

Collecting together Theorem 4.1.2, Theorem 4.1.3, Theorem 2.2, and also the possible exceptions, the existence of a nesting $N(K_4, n; 6, 4)$ is proven for $n \equiv 1$ mod 2, $n \neq 15, 27, 39, 51, 75, 87, 95, 99, 111, 115, 119, 135, 143, 183, 195, 243, 411$.

From Theorem 2.4, since there exist pairs of $N(K_4, n; 6, 4)$ of order n_1, n_2 such that $(n_1, n_2) = (5, 19), (9, 11), (5, 23), (7, 17), (11, 13), (9, 27)$, existence follows for $n = n_1.n_2 = 95, 99, 115, 119, 143, 243$; further, since there exist pairs of $N(K_4, n; 6, 4)$ of order n_1, n_2 such that $(n_1, n_2) = (5, 11), (11, 11), (41, 11)$, existence follows for $n = n_1.(n_2 - 1) + 1 = 51, 111, 411$. This part of the statement is now proved.

Theorem 4.1.5: *There exists a nesting $N(K_4, 6; 12, 8)$ and a nesting $N(K_4, 8; 12, 8)$.*

Proof: Consider the following design, defined on Z_6 and having the blocks:

$$\begin{array}{lllll}
\{0;1,2,3,4\}, & \{0;1,2,4,5\}, & \{0;1,3,4,5\}, & \{1;2,3,4,5\}, & \{1;0,3,4,5\}, \\
\{1;0,2,3,4\}, & \{1;0,2,3,5\}, & \{2;0,1,3,4\}, & \{2;0,1,4,5\}, & \{3;0,1,2,5\}, \\
\{3;0,2,4,5\}, & \{4;0,1,2,3\}, & \{4;0,1,3,5\}, & \{4;1,2,3,5\}, & \{5;0,1,2,4\}, \\
\{5;0,1,2,3\}, & \{3;0,1,2,4\}, & \{4;0,1,2,5\}, & \{5;0,1,3,4\}, & \{2;0,1,3,5\}, \\
\{3;0,1,4,5\}, & \{4;0,2,3,5\}, & \{5;0,2,3,4\}, & \{1;0,2,4,5\}, & \{2;0,3,4,5\}, \\
\{5;1,2,3,4\}, & \{3;1,2,4,5\}, & \{0;1,2,3,5\}, & \{2;1,3,4,5\}, & \{0;2,3,4,5\}.
\end{array}$$

It is possible to verify that this is a nesting $N(K_4, 6; 12, 8)$.

Consider the following design, defined on $Z_7 \cup \{\infty\}$ and having the blocks:

$$\begin{array}{ll}
\{j;\infty, j+1, j+2, j+3\}, & \{j;\infty, j+1, j+3, j+5\}, \\
\{j;\infty, j+1, j+4, j+5\}, & \{j;\infty, j+1, j+2, j+4\}, \\
\{\infty; j, j+1, j+2, j+4\}, & \{j; j+1, j+2, j+3, j+5\}, \\
\{j; j+2, j+3, j+4, j+6\}, & \{j; j+4, j+5, j+6, j+1\}, \\
 & \text{for every } j \in Z_7.
\end{array}$$

It is possible to verify that this is a nesting $N(K_4, 8; 12, 8)$.

Theorem 4.1.6: *There exists a nesting $N(K_4, n; 12, 8)$ for every $n \geq 5$, except possibly for $n = 10, 12, 14, 15, 16, 18, 20, 22, 24, 27, 28, 32, 34$.*

Proof: Observe that for every admissible $n \in N$ there exists a PBD(n) having blocks of size $5, 6, 7, 8, 9$ ([2], p. 209), with possible exceptions for $n = 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34$.

From Theorem 2.2, Theorem 4.1.4, Theorem 4.1.5, there exists a nesting $N(K_4, n; 12, 8)$ of order $n \geq 5$, except possibly for $n = 10, 12, 14, 15, 16, 18, 20, 22, 24, 27, 28, 32, 34$.

Collecting together the results obtained, we can formulate the following.

Corollary 4.1 *The necessary conditions for the existence of a nesting design $N(K_4, n; \lambda_1, \lambda_2)$ [Theorem 4.1.1] are also sufficient with the possible exceptions of $n = 10, 12, 14, 15, 16, 18, 20, 22, 24, 27, 28, 32, 34$, when $\lambda_1 \equiv 0$ mod 12 and $\lambda_2 \equiv 0$ mod 8.*

4.2 $G \cong K_4 - e$

From Theorem 2.1, necessary conditions for the existence of a nesting design $N(K_4 - e, n; \lambda_1, \lambda_2)$ are: $n \geq 5$, $4\lambda_1 = 5\lambda_2$, i.e. $\lambda_1 = 5h$, $\lambda_2 = 4h$, $h \in N$.

Many results can be obtained from 4.1), by deleting an edge in the blocks of Σ.

Recall that we indicate the graph $K_4 - e$ by $\{a, b, (c, d)\}$ where c, d are the non-adjacent vertices, and $S_4 \cup (K_4 - e)$ by $\{x; a, b, (c, d)\}$, where x is the centre of the star.

Theorem 4.2.1: *There exists a nesting $N(K_4 - e, n; 5, 4)$ for every prime integer $n \in N$, $n \geq 5$.*

Proof: Let $\Sigma' = (Z_n, B')$ be the $(K_4 - e)$-design obtained from $\Sigma = (Z_n, B)$, the K_4-design of index $\lambda_1 = 6$ defined in Theorem 4.1.3, by deleting in every block $B_{i,j} \in B$ the edge $\{x_{i,j,1}, x_{i,j,4}\}$. So, Σ' has the following blocks:

$$B'_{i,j} = B_{i,j} - \{x_{i,j,1}, x_{i,j,4}\},\ B_{i,j} \in B.$$

Since the difference between the endpoints of the deleted edge is $3i$ (see Theorem 4.1.3) and n is prime, then for $i = 1, 2, \ldots, (n-1)/2$ the value $3i$ covers all the possible differences $1, 2, \ldots, (n-1)/2$ between two vertices of Z_n. So, Σ' has index $\lambda'_1 = 5$.

If $\Pi = (Z_n, S)$ is the same S_4-design defined in Theorem 4.1.3 and $F(B'_{i,j}) = S_{i,j}$, then $N = (\Sigma', \Pi, F)$ is a nesting $N(K_4 - e, n; 5, 4)$.

Theorem 4.2.2: *There exists a nesting $N(K_4 - e, 9; 5, 4)$ of order 9.*

Proof: Consider the design, defined on Z_9 and having the following blocks:

$$\{j-1; j, j+2, (j+1, j+3)\},\quad \{j-2; j, j+4, (j+2, j+6)\},$$
$$\{j-2; j, j+6, (j+1, j+3)\},\quad \{j+5; j, j-1, (j+3, j+4)\},$$
$$\text{for every } j = 0, 1, 2, \ldots, n-1.$$

It is possible to verify that this is a nesting $N(K_4 - e, 9; 5, 4)$.

Theorem 4.2.3: *There exists a nesting $N(K_4 - e, n; 5, 4)$ for every $n \equiv 1 \bmod 2$, $n \geq 5$, with possible exceptions for $n = 15, 27, 33, 39, 75, 87, 93, 183, 195$.*

Proof: Observe that for every admissible $n \equiv 1$ mod 2 there exist PBD(n) having blocks of size $5, 7, 9$ ([2], p. 208), with the following possible exceptions for $n = 11, 13, 15, 17, 19, 23, 27, 29, 31, 33, 39, 43, 51, 59, 71, 75, 83, 87, 93, 95, 99, 107, 111, 113, 115, 119, 131, 135, 139, 143, 167, 173, 179, 183, 191, 195, 243, 283, 411, 563$.

From Theorem 2.2, Theorem 4.2.1 and Theorem 4.2.2, there exists a nesting $N(K_4 - e, n; 5, 4)$ for the same values of n, deleting all prime numbers.

So, the possible exceptions are:
$n = 15, 27, 33, 39, 51, 75, 87, 93, 95, 99, 111, 115, 119, 143, 183, 195, 243, 411$.

From Theorem 2.4, since there exist pairs of $N(K_4 - e, n; 5, 4)$ of order n_1, n_2 such that $(n_1, n_2) = (5, 19), (9, 11), (5, 23), (7, 17), (11, 13)$, existence follows for $n = n_1.n_2 = 95, 99, 115, 119, 143$; further, since there exist pairs of $N(K_4 - e, n; 5, 4)$ of order n_1, n_2 such that $(n_1, n_2) = (5, 11), (5, 23), (11, 23), (41, 11)$, existence follows for $n = n_1.(n_2 - 1) + 1 = 51, 111, 243, 411$.

REMARK: Note that, in this case, the sufficiency for the existence of a nesting design $N(K_4 - e, n; \lambda_1, \lambda_2)$ is proved (apart from a few cases) only for odd orders n. For even order n, we are able to solve the problem of the existence only for $n = 6$ in the next Theorem 4.2.4.

We remark that the problem is open for any even n, $n \geq 8$.

Theorem 4.2.4: *Nesting designs $N(K_4 - e, 6; 5, 4)$ of order 6 do not exist.*

Proof: Suppose that there exists a nesting $N(K_4 - e, 6; 5, 4)$ of order 6. If, for a point x:

- M indicates the number of blocks of the $(K_4 - e)$-design in which x is adjacent to all the other vertices of the block;
- T indicates the number of blocks in which x is adjacent to two vertices of the block;
- C indicates the number of the blocks of the S_4-design in which x is the centre;

then necessarily

$$3M + 2T = 25$$
$$4C + M + T = 20$$

from which

$$C = \frac{M + 15}{8}$$
$$T = \frac{25 - 3M}{2}$$

and this implies $M = 1$ *and* $C = 2$, $T = 11$.
But this is not possible for a nesting-design with 15 blocks.

4.3 $G \cong K_3 + e$

From Theorem 2.1, it follows that $\lambda_1 = \lambda_2$.

The spectrum of $N(K_3 + e, n; 1, 1)$ was studied by S. Milici and G. Quattrocchi in [11].

4.4 $G \cong C_4$

From Theorem 2.1, it follows that $\lambda_1 = \lambda_2$.

The spectrum of $N(C_4, n; 1, 1)$ was studied by C.C. Lindner and D.R. Stinson [6] and by S. Milici and G. Quattrocchi [11] and the results can be extended to designs $N(C_4, n; h, h)$, where $\lambda_1 = \lambda_2 = h \in N$, by a repetition of blocks.

4.5 $G \cong P_4$

From Theorem 2.1, necessary conditions for the existence of a nesting design $N(P_4, n; \lambda_1, \lambda_2)$ are: $n \geq 5$, $4\lambda_1 = 3\lambda_2$, i.e. $\lambda_1 = 3h$, $\lambda_2 = 4h$, $h \in N$.

At first, we prove the existence in some particular cases.

Theorem 4.5.1: *There exist nesting designs $N(P_4, 5; 3, 4)$, $N(P_4, 6; 3, 4)$, $N(P_4, 8; 3, 4)$, $N(P_4, 9; 3, 4)$.*

Proof: Consider the following design, defined on Z_5 and having the blocks:

$$[j; j+1, j+2, j+3, j+4],\ [j; j+2, j+4, j+1, j+3] \text{ for every } j = 0, 1, 2, 3, 4.$$

We can verify that this is a nesting $N(P_4, 5; 3, 4)$.

The following design is defined on Z_6 and its blocks are:

$$\begin{array}{lllll} [6;1,3,2,4], & [4;2,1,3,5], & [5;6,1,2,3], & [3;1,4,5,2], & [6;1,5,4,3], \\ [1;2,4,5,3], & [5;1,4,6,2], & [2;1,6,4,3], & [1;2,4,6,3], & [3;1,5,6,2], \\ [2;1,6,5,3], & [4;2,5,6,3], & [5;1,4,3,2], & [4;1,2,6,3], & [6;2,5,1,3]. \end{array}$$

We can verify that it is a nesting $N(P_4, 6; 3, 4)$.

The following design is defined on Z_8 and its blocks are:

$$\begin{array}{lllll} [4;0,2,3,1], & [3;1,2,0,6], & [3;0,2,1,6], & [5;4,0,1,3], & [2;6,0,1,5], \\ [6;0,3,4,1], & [7;0,3,4,1], & [4;0,3,2,7], & [2;1,0,7,5], & [5;0,4,6,1], \\ [1;4,0,7,3], & [6;0,7,4,3], & [7;0,5,3,2], & [3;0,5,4,2], & [6;0,5,4,2], \\ [1;0,6,4,7], & [5;3,1,2,7], & [0;1,5,4,2], & [6;1,7,5,2], & [4;1,6,5,2], \\ [2;1,7,6,3], & [7;1,4,6,3], & [0;4,7,1,5], & [1;2,6,5,3], & [7;2,6,5,3], \\ [0;2,6,7,3], & [5;2,7,6,3], & [4;2,5,7,3]. & & \end{array}$$

We can verify that it is a nesting $N(P_4, 8; 3, 4)$.

Consider the following design, defined on Z_8 and having the blocks:

$$\begin{array}{ll} [j+4; j, j+1, j+2, j+3], & [j+7; j, j+2, j+4, j+6], \\ [j+4; j, j+3, j+6, j+1], & [j+1; j, j+4, j+8, j+5] \\ & \text{for every } j = 0,1,2,3,4,5,6,7,8. \end{array}$$

We can verify that this is a nesting $N(P_4, 9; 3, 4)$.

Theorem 4.5.2: *There exists a nesting $N(P_4, n; 3, 4)$, for every $n \in N$, n prime, $n \geq 5$.*

Proof: For $n = 5$, the existence is proved in Theorem 4.5.1. Let $n \geq 7$, n prime.

Let $\Sigma^* = (Z_n, B^*)$ be the P_4-design obtained from $\Sigma = (Z_n, B)$, the K_4-design of index $\lambda_1 = 6$ defined in Theorem 4.1.3, by deleting in every block $B_{i,j} \in B$ the edges:

$$e_{i,j,13} = \{x_{i,j,1}, x_{i,j,3}\},\ e_{i,j,24} = \{x_{i,j,2}, x_{i,j,4}\},\ e_{i,j,14} = \{x_{i,j,1}, x_{i,j,4}\}.$$

So, Σ^* has the following blocks:

$$B^*_{i,j} = B_{i,j} - (e_{i,j,13} + e_{i,j,24} + e_{i,j,14}), \quad B_{i,j} \in B.$$

The differences between the endpoints of the deleted edges $e_{i,j,13}, e_{i,j,24}, e_{i,j,14}$ are: $2i, 2i, 3i$, respectively, while the differences between the endpoints of the remaining edges are: i, i, i. Further, since n is prime, for every $i = 1, 2, \ldots, (n-1)/2$ the values $2i, 2i, 3i, i, i, i$ assume all the possible values of the differences between two vertices of Z_n (see Theorem 4.1.3). Therefore Σ^* has index $\lambda^* = 3$.

If $\Pi = (Z_n, S)$ is the same S_4-design defined in Theorem 4.1.3 and $F(B^*_{i,j}) = S_{i,j}$, then $N^* = (\Sigma^*, \Pi, F)$ is a nested-design $N(P_4, n; 3, 4)$.

Theorem 4.5.3: *There exist nestings $N(P_4, K_{2,2,2}; 3, 4)$, $N(P_4, K_{2,2,2,2}; 3, 4)$.*

Proof: Let $K_{2,2,2}$ be the 3-partite complete graph defined on $V = X \cup Y \cup Z$, where $X = \{1, 4\}$, $Y = \{2, 5\}$, $Z = \{3, 6\}$ partition V in stable sets. The following blocks:

$$\begin{array}{llll} [3; 2, 1, 5, 4], & [1; 2, 3, 5, 6], & [2; 3, 1, 6, 4], & [3; 5, 1, 2, 4], \\ [1; 5, 3, 2, 6], & [2; 6, 1, 3, 4], & [6; 1, 5, 4, 2], & [4; 3, 5, 6, 2], \\ [5; 1, 6, 4, 3], & [6; 1, 2, 4, 5], & [4; 3, 2, 6, 5], & [5; 1, 3, 4, 6], \end{array}$$

define a $N(P_4, K_{2,2,2}; 3, 4)$.

Now, let $K_{2,2,2,2}$ be the 4-partite complete graph defined on

$$V' = L \cup M \cup N \cup P,$$

where $L = \{0, 4\}$, $M = \{1, 5\}$, $N = \{2, 6\}$, $P = \{3, 7\}$ partition V' in stable sets. The following blocks:

$$\begin{array}{ll} [j+7; j, j+1, j+2, j+4], & [j+3; j, j+2, j+4, j+1], \\ [j+5; j, j+3, j+6, j+7] & \text{for every } j \in Z_8, \end{array}$$

define a $N(P_4, K_{2,2,2,2}; 3, 4)$.

Theorem 4.5.4: *There exists a nesting $N(P_4, n; 3, 4)$, for every $n \in N$, $n \geq 5$, with the following possible exceptions: $n = 10, 12, 14, 16, 20, 22, 28, 34$.*

Proof: For every admissible n, there exists a PBD(n) having blocks of size $5, 6, 7, 8, 9$ ([2], p. 209), with possible exceptions for $n = 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34$. From Theorem 2.2, Theorem 2.7, Theorem 4.5.1 and Theorem 4.5.2, it follows that there exists a nesting $N(P_4, n; 3, 4)$ for the same values of n. From Theorem 4.5.2 and Theorem 2.7, the previous list can be reduced by deleting all n odd. Since there exist 3-GDD of type 3^3, 4-GDD of type 3^4, 4^4 ([2], p. 189–190), from Theorem 2.3, Theorem 4.5.1 and Theorem 4.5.3, the existence of nesting $N(P_4, n; 3, 4)$ follows, also for $n = 18, 32, 24$ and this completes the proof.

Collecting together the results obtained, we can formulate the following.

Corollary 4.5 *The necessary conditions for the existence of a nesting design $N(P_4, n; \lambda_1, \lambda_2)$ are $4\lambda_1 = 3\lambda_2$, $n \geq 5$. These conditions are also sufficient for every $n \geq 5$, with the possible exceptions of $n = 10, 12, 14, 16, 20, 22, 28, 34$.*

4.6 $G \cong S_3$

In what follows, given an $S_3 = \langle y; a, b, c\rangle$ and an $S_4 = \langle x; y, a, b, c\rangle$, we denote $S_3 \cup S_4 = \langle x; \langle y; a, b, c\rangle\rangle$.

For the necessary conditions we have the following theorem.

Theorem 4.6.1: *If there exists a nesting design $N(S_3, n; \lambda_1, \lambda_2)$, then the parameters n, λ_1, λ_2 must satisfy one of the following conditions:*

1) $\lambda_1 = 3h$, $\lambda_2 = 4h$, $n \equiv 1 \bmod 2$, $n \geq 5$, for any positive odd integer h;

2) $\lambda_1 = 3h$, $\lambda_2 = 2h$, $n \geq 5$, for any positive integer $h \equiv 0 \bmod 2$.

Proof: From Theorem 2.1, it follows that: $4\lambda_1 = 3\lambda_2$, $n \geq 5$.

Let $N = (\Sigma, \Pi, F)$ be a nesting $N(S_3, n; 3h, 4h)$. Consider a point x of N. If C_x, Ω_x, T_x are respectively the number of blocks containing x as a *centre* in a star of Π, the number of blocks containing x as a *centre* in a star of Σ and the number of blocks containing x as a *terminal* vertex always in a star of Σ, then:

$$3\Omega_x + T_x = 3h(n-1)$$
$$4C_x + \Omega_x + T_x = 4h(n-1)$$

It follows that:

$$4C_x - 2\Omega_x = h(n-1);$$

hence $h.(n-1)$ is an even number and if h is odd, $n \equiv 1 \bmod 2$.

Theorem 4.6.2: *There exists a nesting $N(S_3, n; 3, 4)$, for every $n \in N$, n prime, $n \geq 5$.*

Proof: Consider the S_3-design $\Sigma'' = (Z_n, B'')$, having for blocks the following 3-stars:

$$B''_{i,j} = \langle j+i; j, j+2i, j+3i\rangle, \text{ for every } j = 0, 1, 2, \ldots, n-1, \; i = 1, 2, \ldots, (n-1)/2,$$

where the values of i represent all the possible differences between two distinct vertices $x, y \in Z_n$. We can verify that Σ'' has index $\lambda''_1 = 3$. Consider that for every pair $x, y \in Z_n$, $x < y$, the difference $y - x$ can be: $1, 2, \ldots, (n-1)/2$, and that in the edges of a block $B''_{i,j}$ these differences are: $i, i, 2i$.

It follows that any difference $\delta = y - x = 1, 2, \ldots, (n-1)/2$ appears in the following blocks of B'': $B''_{\delta,j}$, $B''_{\delta,j}$, $B''_{(n-\delta)/2j}$; so, the pair $\{x, y\}$ is contained in exactly 3 blocks of Σ. Observe that every block $B''_{i,j}$ of Σ'' is contained in the block $B_{i,j}$ of the K_4-design Σ, defined in Theorem 4.1.3 and having index $\lambda_1 = 6$.

If $\Pi = (Z_n, S)$ is the S_4-design defined in Theorem 4.1.3 and $F(B''_{j,i}) = S_{i,j}$, then $N'' = N(\Sigma'', \Pi, F)$ is a nested-design $N(S_3, n; 3, 4)$.

Theorem 4.6.3: *There exist nesting $N(S_3, 9; 3, 4)$, $N(S_3, 15; 3, 4)$.*

Proof: The following design is defined on Z_9 and has the blocks:

$$\langle j;\langle j+1;j+2,j+3,j+4\rangle\rangle,\quad \langle j;\langle j+1;j+5,j+6,j+7\rangle\rangle,$$
$$\langle j;\langle j+6;j+4,j+7,j+8\rangle\rangle,\quad \langle j;\langle j+1;j+2,j+4,j+6\rangle\rangle,$$
$$\text{for every } j=0,1,\ldots,8.$$

We can verify that this is a nesting $N(S_3, 9; 3, 4)$.

The following design is defined on Z_{15} and has the blocks:

$$\langle j;\langle j+2;j+1,j+3,j+4\rangle\rangle \quad \langle j;\langle j+1;j+5,j+8,j-3\rangle\rangle$$
$$\langle j;\langle j-4;j+9,j-5,j-2\rangle\rangle \quad \langle j;\langle j-3;j+2,j-6,j+7\rangle\rangle$$
$$\langle j;\langle j-2;j+5,j+4,j+6\rangle\rangle \quad \langle j;\langle j+4;j+1,j+7,j+9\rangle\rangle$$
$$\langle j;\langle j-1;j+8,j+3,j+5\rangle\rangle \quad \text{for every } j=0,1,2,\ldots,14.$$

We can verify that this is a nesting $N(S_3, 15; 3, 4)$.

Theorem 4.6.4: *There exists a nesting $N(S_3, n; 3, 4)$ if and only if $n \equiv 1$ mod 2, $n \geq 5$, except possibly for $n = 15, 27, 39, 75, 87, 135, 183, 195$.*

Proof: $\Rightarrow$ Necessarily, $n \equiv 1$ mod 2. It follows from Theorem 4.6.1. *1)*.

$\Leftarrow$ For every admissible n, $n \equiv 1$ mod 2, there exist PBD(n) having blocks of size $5, 7, 9$ ([2], p. 208), with the possible exceptions of $n = 11, 13, 15, 17, 19, 23, 27, 29, 31, 33, 39, 43, 51, 59, 71, 75, 83, 87, 93, 95, 99, 107, 111, 113, 115, 119, 131, 135, 139, 143, 167, 173, 179, 183, 191, 195, 243, 283, 411, 563$.

From Theorem 2.2, Theorem 4.6.2 and Theorem 4.6.3, the existence of a nesting design $N(S_3, n; 3, 4)$ follows for the same values of n. From Theorem 4.6.2 and Theorem 4.6.3, the previous list can be reduced by deleting all n prime and also $n = 15$.

From Theorem 2.4, since there exist pairs of $N(S_3, n; 3, 4)$ of order n_1, n_2 such that $(n_1, n_2) = (5, 15), (5, 19), (9, 11), (5, 23), (7, 17), (9, 15), (11, 13), (13, 15)$, the existence for $n = n_1.n_2 = 75, 95, 99, 115, 119, 135, 143, 195$ follows.

From Theorem 2.4, since there exist pairs of $N(S_3, n; 3, 4)$ of order n_1, n_2 such that $(n_1, n_2) = (5, 11), (23, 5), (11, 11), (13, 15), (11, 23), (41, 11)$ it follows the existence also for $n = n_1.(n_2 - 1) + 1 = 51, 93, 111, 183, 243, 411$.

This part of the statement is so proved.

Theorem 4.6.5: *i) Nesting designs $N(S_3, 6; 6, 8)$ of order 6 do not exist.*
ii) There exists a nesting $N(S_3, 8; 6, 8)$ of order 8.

Proof: *i)* Suppose that there exists a nesting $N(S_3, 6; 6, 8)$ of order 6. If, for a point x:

- C indicates the number of blocks of the S_3-design in which x is the centre of the star;

- T indicates the number of blocks of the S_3-design in which x is a terminal of the star;

- Ω indicates the number of the blocks of the S_4-design in which x is the centre of the star;

then necessarily

$$3C + T = 30$$
$$4\Omega + C + T = 40$$

from which

$$\Omega = \frac{C+5}{2}, \; T = 30 - 3C$$

and this is not possible, because the number of blocks is equal to 20.

ii) Consider the following design, defined on $Z_7 \cup \{\infty\}$ and having the blocks:

$$\begin{array}{ll} \langle j+3; \langle j; j+1, j+2, j+6\rangle\rangle, & \langle j+1; \langle j; j+2, j+3, j+5\rangle\rangle, \\ \langle j+6; \langle j; j+1, j+3, j+4\rangle\rangle, & \langle \infty; \langle j; j+1, j+2, j+3\rangle\rangle, \\ \langle j; \langle j+1; \infty, j+2, j+4\rangle\rangle, & \langle j+5; \langle j+1; \infty, j+3, j+4\rangle\rangle, \\ \langle j; \langle j+1; \infty, j+2, j+3\rangle\rangle, & \langle j; \langle \infty; j+1, j+2, j+3\rangle\rangle, \\ & \text{for every } j \in Z_7. \end{array}$$

It is possible to verify that this is a nesting $N(S_3, 8; 6, 8)$.

Theorem 4.6.6: *There exists a nesting* $N(S_3, n; 6, 8)$ *for every* $n \geq 5$, $n \neq 6$, *except possibly for* $n = 10, 12, 14, 16, 18, 20, 22, 24, 26, 27, 28, 30, 32, 33, 34, 38, 39, 42, 44, 46, 52, 60, 94, 96, 98, 100, 102, 104, 106, 108, 110, 116, 138, 140, 142, 146, 150, 154, 156, 158, 162, 166, 170, 172, 174, 206, 228$.

Proof: Observe that for every admissible $n \in N$ there exists a PBD(n) having blocks of size 5, 7, 8, 9 ([2], p. 208), with a set of possible exceptions. The statement follows from Theorem 4.6.5, Theorem 2.2 and Theorem 4.6.4.

Collecting together the results obtained, we can formulate the following.

Corollary 4.6 *The necessary conditions for the existence of a nesting design* $N(S_3, n; \lambda_1, \lambda_2)$ *[Theorem 4.6.1] are also sufficient except possibly for:*

i) $n = 15, 27, 39, 75, 87, 135, 183, 195$, *when* $n \equiv 1$ *mod* 2, $\lambda_1 \equiv 3$ *mod* 6, $\lambda_2 \equiv 4$ *mod* 8;

ii) $n = 10, 12, 14, 16, 18, 20, 22, 24, 26, 27, 28, 30, 32, 33, 34, 38, 39, 42, 44, 46, 52, 60, 94, 96, 98, 100, 102, 104, 106, 108, 110, 116, 138, 140, 142, 146, 150, 154, 156, 158, 162, 166, 170, 172, 174, 206, 228$, *when* $\lambda_1 \equiv 0$ *mod* 6, $\lambda_2 \equiv 0$ *mod* 8.

4.7 $G \cong 2P_2$

In what follows, if $2P_2$ is a graph with edges $\{a, b\}, \{c, d\}$ and S_4 is a 4-star having terminal vertices a, b, c, d and centre x, then the graph $2P_2 + S_4$ will be indicated by $\langle x; (a, b), (c, d)\rangle$.

For the necessary conditions we have the following theorem.

Theorem 4.7.1: *If there exists a nesting design* $N(2P_2, n; \lambda_1, \lambda_2)$, *then the parameters* n, λ_1, λ_2 *must satisfy one of the following conditions:*

1) $\lambda_1 = h$, $\lambda_2 = 2h$, $n \equiv 1$ *mod* 4, $n \geq 5$, *for any positive odd integer* h*;*

2) $\lambda_1 = h$, $\lambda_2 = 2h$, $n \equiv 1$ *mod* 2, $n \geq 5$, *for any positive integer* $h \equiv 2$ *mod* 4*;*

3) $\lambda_1 = h$, $\lambda_2 = 2h$, $n \geq 5$, *for any positive integer* $h \equiv 0$ *mod* 4.

Proof: From Theorem 2.1, it follows that: $2\lambda_1 = \lambda_2$, $n \geq 5$.
Let $N = (\Sigma, \Pi, F)$ be a nesting $N(2P_2, n; h, 2h)$. If x is a point of N and T_x is the number of blocks of Σ containing x, $C_x$the number of blocks of Π containing x as a *centre*, then:

$$T_x = h(n-1)$$
$$4C_x + T_x = 2h(n-1).$$

It follows that $C_x = h(n-1)/4$, hence $h(n-1) \equiv 0$ mod 4. This implies *1),2),3)*.

Theorem 4.7.2: *There exists a nesting* $N(2P_2, n; 1, 2)$ *if and only if* $n \equiv 1$ *mod* 4, $n \geq 5$.
Proof: $\Rightarrow$ Necessity follows from Theorem 4.7.1.*1)*.
$\Leftarrow$ Let $n \equiv 1$ mod 4, $n \geq 5$, $Z' = Z_n \cup \{\infty\}$ and let $\Phi = \{F_1, F_2, \ldots, F_n\}$ be a 1-factorization defined on Z'. Without loss of generality, we can suppose that the 1-factor F_i contains the pair $\{i, \infty\}$. Observe that, if $k = |F_i - \{\{i, \infty\}\}|$, then $k \equiv 0$ mod 2. So, let $F_i - \{\{i, \infty\}\} = \{\{x_{i,1}, y_{i,1}\}, \{x_{i,2}, y_{i,2}\}, \ldots, \{x_{i,k-1}, y_{i,k-1}\}, \{x_{i,k}, y_{i,k}\}\}$, for every $i = 1, 2, \ldots, n$. Then, we can define the design N, having the blocks:

$$\langle i; (x_{i,1}, y_{i,1}), (x_{i,2}, y_{i,2})\rangle, \quad \ldots \quad \langle i; (x_{i,k-1}, y_{i,k-1}), (x_{i,k}, y_{i,k})\rangle, \quad \text{for each } i = 1, 2, \ldots, n$$

We can verify that N is a nesting design $N(2P_2, n; 1, 2)$.

Theorem 4.7.3: *There exists a nesting* $N(2P_2, n; 2, 4)$ *if and only if* $n \equiv 1$ *mod* 2, $n \geq 5$.
Proof: $\Rightarrow$ Necessity follows from Theorem 4.7.1.*2)*.
$\Leftarrow$ Let $n \equiv 1$ mod 2, $n \geq 5$. So: i) $n \equiv 1$ mod 4, or ii) $n \equiv 3$ mod 4. In case i), we obtain the same results of Theorem 4.7.2, by a repetition of blocks.

Examine the case ii). Thus: $n \equiv 3$ mod 4, $n \geq 7$. Let $\Phi' = \{F_1, F_2, \ldots, F_n\}$, $\Phi'' = \{G_1, G_2, \ldots, G_n\}$ be two 1-factorizations, defined on $Z' = Z_n \cup \{\infty\}$, such that $F_i \cap G_i = \{\{i, \infty\}\}$, for each $i = 1, \ldots, n$. If $F_i - \{\{i, \infty\}\} = \{\{x_{i,1}, x_{i,2}\}, \{x_{i,3}, x_{i,4}\}, \ldots, \{x_{i,k-1}, x_{i,k}\}\}$, $G - \{\{i, \infty\}\} = \{\{y_{i,1}, y_{i,2}\}, \{y_{i,3}, y_{i,4}\}, \ldots, \{y_{i,k-1}, y_{i,k}\}\}$, then $k \equiv 2$ mod 4.

Then, we can define the design N, having the following blocks:

$$\langle i; (x_{i,1}, x_{i,2}), (x_{i,3}, x_{i,4})\rangle, \qquad \langle i; (y_{i,1}, y_{i,2}), (y_{i,3}, y_{i,4})\rangle,$$
$$\ldots\ldots \qquad \ldots\ldots$$
$$\langle i; (x_{k-5}, x_{k-4}), (x_{k-3}, x_{k-2})\rangle, \quad \langle i; (y_{k-5}, y_{k-4}), (y_{k-3}, y_{k-2})\rangle,$$
$$\langle i; (x_{k-1}, x_k), (y_{k-1}, y_k)\rangle, \qquad \text{for every } i = 1, 2, \ldots, n.$$

We can verify that N is a nesting $N(2P_2, n; 2, 4)$.

Theorem 4.7.4: *There exist nestings* $N(2P_2, 6; 4, 8)$, $N(2P_2, 8; 4, 8)$.

Proof: Consider the following design, defined on $Z_5 \cup \{\infty\}$ and having the blocks:

$$\begin{array}{ll} \langle\infty; (j+1, j+4), (j+2, j+3)\rangle, & \langle j; (\infty, j+3), (j+1, j+4)\rangle, \\ \langle j; (\infty, j+1), (j+2, j+3)\rangle, & \langle j; (\infty, j+2), (j+1, j+4)\rangle, \\ \langle j; (\infty, j+4), (j+2, j+3)\rangle, & \langle j; (j+1, j+4), (j+2, j+3)\rangle \\ & \text{for every } j \in Z_5. \end{array}$$

We can verify that this is a nesting $N(2P_2, 6; 4, 8)$.

Consider the following design, defined on $Z_7 \cup \{\infty\}$ and having the blocks:

$$\begin{array}{ll} \langle\infty; (j+1, j+2), (j+3, j+5)\rangle, & \langle j; (\infty, j+1), (j+2, j+4)\rangle, \\ \langle j; (\infty, j+3), (j+2, j+4)\rangle, & \langle j; (\infty, j+5), (j+2, j+4)\rangle, \\ \langle j; (\infty, j+2), (j+1, j+4)\rangle, & \langle j; (j+1, j+2), (j+3, j+6)\rangle, \\ \langle j; (j+1, j+2), (j+3, j+6)\rangle, & \langle j; (j+1, j+2), (j+3, j+6)\rangle, \\ & \text{for every } j \in Z_7. \end{array}$$

We can verify that this is a nesting $N(2P_2, 8; 4, 8)$.

Theorem 4.7.5: *There exists a nesting* $N(2P_2, n; 4, 8)$ *for every* $n \in N$, $n \geq 5$.

Proof: For n odd and $n = 6, n = 8$, the statement follows from Theorem 4.7.3, by a repetition of blocks, and from Theorem 4.7.4.

Let $n \geq 10$, n even. Further, let N be the nesting $N(P_2, n-1; 1, 2)$, defined on $Z_{n-1}$by the blocks $[j; j+i, j+2i]$, where $j = 0, 1, 2, \ldots, n-1$, $i = 1, 2, \ldots, (n-1)/2$, and $[x; y_1, y_2]$ indicates $\langle x; y_1, y_2\rangle \cup \langle y_1, y_2\rangle$. Starting from N, it is possible to define a nested-design $N(2P_2, n; 4, 8)$ on $Z_{n-1} \cup \{\infty\}$, as follows.

1) Suppose $n \equiv 2 \bmod 4$. Then, for every $j \in Z_{n-1}$:

- repeat every block $[j; j+i, j+2i]$ of N four times:
  $[j; j+i, j+2i]^{(1)}, [j; j+i, j+2i]^{(2)}$,
  $[j; j+i, j+2i]^{(3)}, [j; j+i, j+2i]^{(4)}$;
- define, for $u = 1, 2, 3, 4$ and $i = 5, 7, \ldots, (n-2)/4$ (i odd) :
  $\langle j; (j+i, j+2i), (j+i+1, j+2i+2)\rangle^{(u)} =$
  $[j; j+i, j+2i]^{(u)} \cup [j; j+i+1, j+2i+2)]^{(u)}$
- define, for $u = 1, 2$:

$$\langle j; (j+1, j+2), (j+4), (j+8)\rangle^{(u)} = [j; j+1, j+2]^{(u)} \cup [j; j+4, j+8]^{(u)},$$
$$\langle j; (j+2, j+4), (j+3), (j+6)\rangle^{(u)} = [j; j+2, j+4]^{(u)} \cup [j; j+3, j+6]^{(u)}.$$

- define:
  $\langle j; (j+4, j+8), (j+3, j+6)\rangle^{(34)} = [j; j+4, j+8]^{(3)} \cup [j; j+3, j+6]^{(4)}$

– delete all the remaining blocks of N and define the following:
$\langle\infty;(j+1,j+2),(j+4,j+8)\rangle$,
$\langle j;(\infty,j+1),(j+2,j+4)\rangle$, $\langle j;(\infty,j+2),(j+3,j+6)\rangle$,
$\langle j;(\infty,j+8),(j+2,j+4)\rangle$, $\langle j;(\infty,j+4),(j+1,j+2)\rangle$.

It is possible to verify that this collection of blocks defines a nested-design $N(2P_2,n;4,8)$.

2) Suppose $n\equiv 0 \bmod 4$.

– repeat every block of N *four* times, using the symbolism of *1)*;
– define:

$$\langle j;(j+1,j+2),(j+3,j+6)\rangle^{(12)} = [j;j+1,j+2]^{(1)} \cup [j;j+3,j+6]^{(2)}$$
$$\langle j;(j+1,j+2),(j+4,j+8)\rangle^{(2)} = [j;j+1,j+2]^{(2)} \cup [j;j+4,j+8]^{(2)}$$
$$\langle j;(j+2,j+4),(j+5,j+10)\rangle^{(2)} = [j;j+2,j+4]^{(2)} \cup [j;j+5,j+10]^{(2)}$$

– define:
$\langle j;(j+i,j+2i),(j+i+1,j+2i+2)\rangle^{(u)} = [j;j+i,j+2i]^{(u)}\cup[j;j+i+1,j+2i+2]^{(u)}$
for every i even and
$i=2,4,\dots,(n-2)/2$ if $u=1$
$i=6,8,\dots,(n-2)/2$ if $u=2$
$i=4,6,\dots,(n-2)/2$ if $u=3,u=4$

– delete all the remaining blocks of N and define the following:
$\langle\infty;(j+1,j+2),(j+3,j+6)\rangle$
$\langle j;(\infty,j+1),(j+2,j+4)\rangle, \langle j;(\infty,j+6),(j+1,j+2)\rangle$
$\langle j;(\infty,j+2),(j+3,j+6)\rangle, \langle < j;(\infty,j+3),(j+2,j+4)\rangle$

It is possible to verify that this collection of blocks defines a nesting $N(2P_2,n;4,8)$.

Collecting together the results obtained we can formulate the following.

Corollary 4.7 *The necessary conditions for the existence of a nesting design* $N(2P_2,n;\lambda_1,\lambda_2)$ *[Theorem 4.7.1] are always sufficient.*

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