A classification of Laguerre near-planes of order four

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Abstract

We investigate Laguerre near-planes of order 4 and classify all such planes. We further develop a description of these planes in terms of a single map and characterise those Laguerre near-planes that can be extended to the Miquelian Laguerre plane of order 4.

1. Introduction and result

A finite *Laguerre plane of order* n where $n \geq 2$ is an integer consists of a set P of points, a set $\mathcal{C}$ of circles and a set $\mathcal{G}$ of generators (subsets of P) such that the following four axioms are satisfied:

(P) P contains $n(n+1)$ points.
(G) $\mathcal{G}$ partitions P and each generator contains n points.
(C) Each circle intersects each generator in precisely one point.
(J) Three points no two of which are on the same generator can be uniquely joined by a circle.

From this definition it readily follows that a Laguerre plane of order n has $n+1$ generators, that every circle contains exactly $n+1$ points and that there are n^3 circles. Labelling the generators from 1 to $n+1$ and the points on each generator from 1 to n and identifying each circle with the $(n+1)$-tuple $(c_1, \ldots, c_{n+1})$ where c_i is the unique point of the circle on generator i, we see that a Laguerre plane of order n corresponds to an orthogonal array of strength 3 on n symbols (levels), $n+1$ constraints and index 1, cf. [1].

All known models of finite Laguerre planes are of the following form. Let $\mathcal{O}$ be an oval in the Desarguesian projective plane $\mathcal{P}_2 = \mathrm{PG}(2, p^m)$, p a prime. Embed $\mathcal{P}_2$ into 3-dimensional projective space $\mathcal{P}_3 = \mathrm{PG}(3, p^m)$ and let v be a point of $\mathcal{P}_3$ not belonging to $\mathcal{P}_2$. Then P consists of all points of the cone with base $\mathcal{O}$ and vertex v except the point v. Circles are obtained by intersecting P with planes of $\mathcal{P}_3$ not passing through v. In this way one obtains an *ovoidal Laguerre plane of order* p^m. If the oval $\mathcal{O}$ one starts off with is a conic, one obtains the *Miquelian*

Laguerre plane of order p^m. All known finite Laguerre planes of odd order are Miquelian.

The *internal incidence structure* $\mathcal{A}_p$ at a point p of a Laguerre plane has the collection of all points not on the generator through p as point set and, as lines, all circles passing through p (without the point p) and all generators not passing through p. This is an affine plane, the *derived affine plane at p.* A circle K not passing through the point of derivation p induces an oval in the projective extension of the derived affine plane at p which intersects the line at infinity in the point corresponding to lines that come from generators of the Laguerre plane; in $\mathcal{A}_p$ one has a *parabolic curve.* (The derived affine planes of the Miquelian Laguerre planes are Desarguesian and the parabolic curves are parabolae whose axes are the verticals, i.e., the lines that come from generators of the Laguerre plane.) A Laguerre plane can thus be described in one derived affine plane $\mathcal{A}$ by the lines of $\mathcal{A}$ and a collection of parabolic curves. This planar description of a Laguerre plane, which is the most commonly used representation of a Laguerre plane, is then extended by the points of one generator where one has to adjoin a new point to each line and to each parabolic curve of the affine plane. It follows from [7] that every parabolic curve in a finite Desarguesian affine plane of odd order is in fact a parabola. Furthermore, using a simple counting argument it was shown in [2] that a finite Laguerre plane of odd order that admits a Desarguesian derivation is Miquelian.

The spatial description of an ovoidal Laguerre plane as the geometry of plane sections of an oval cone is related to the planar description in one derived plane by stereographic projection from one point of the cone onto a plane not passing through the point of projection. In this description all points of the Laguerre plane except the points on the generator through the point of projection are covered.

In this note we consider the restriction of a finite Laguerre plane to one of its derived affine planes. When verifying the axioms of a Laguerre plane in such a planar representation one always has to consider special cases involving the extra points. We now ask to what extend the description in a derived affine plane determines the Laguerre plane. A partial solution to this problem was given in [9] in the case of odd order and under the assumption that a point exists at which the internal incidence structure (defined in exactly the same way as for Laguerre planes) can be extended to a Desarguesian affine plane. To be more precise, a *Laguerre near-plane of order* $n \geq 3$ is an incidence structure of n^2 points, circles and generators satisfying the axioms (G), (C) and (J) from above. This definition extends the terminology for Minkowski near-planes and Möbius near-planes adopted in [5] and [8], respectively. Laguerre near-planes occur as special Laguerre semi-planes in [6] but have not been further investigated there. Also note that a Laguerre near-plane is not a restricted L1-space as defined in [11] since the restriction made in [11] on the number of points and lines in an internal incidence structure at a point is not satisfied.

Clearly, there are n generators, every circle contains exactly n points and there are n^3 circles. Like for Laguerre planes we see that a Laguerre near-plane of order n corresponds to an orthogonal array of strength 3 on n symbols, n constraints and index 1. By [1], an orthogonal array of strength 3, n symbols, size n^3 and index 1 can have at most $n+2$ constraints. In fact, if n is odd one has at most $n+1$ constraints. In terms of orthogonal arrays the question is whether such an array on n constraints can be extended to one on $n+1$ constraints and if so in how many essentially different ways.

One obviously obtains a Laguerre near-plane of order n by deleting a generator from a Laguerre plane of order n. In the case of the Miquelian Laguerre plane we obtain in this way the *parabola model*: its circles are graphs of polynomials of degree at most 2, that is,

$$\mathcal{C} = \{\{(u, au^2 + bu + c) \mid u \in \mathbb{F}_q\} \mid a, b, c \in \mathbb{F}_q\}$$

where $\mathbb{F}_q$ denotes the Galois field of order q. The axiom (J) is readily verified in this model.

Conversely, it is not clear how to extend circles in order to construct a Laguerre plane from a Laguerre near-plane since all circles have the same length. Even worse, if an extension exists, it may not be unique.

In [9] all Laguerre near-planes of order at most seven, except order 4 are covered. Furthermore, an example was given that a Laguerre near-plane of even order may be extended in more than one way to a Laguerre plane of the same order. This basically is due to the fact that in even order one can replace a point of an oval in a projective plane by its nucleus and again obtain an oval. Moreover, Laguerre near-planes of order 4 were used in [8] to construct Möbius near-planes of order 4. Also note that the case of order 4 stands out in that the derived incidence structure at a point of a Laguerre near-plane of order 4 may not extend to an affine plane, see [9] and [4].

In this paper we investigate Laguerre near-planes of order 4. We develop a representation of such planes in terms of a single map. We determine, up to isomorphism, all Laguerre near-planes of order 4. The results obtained in this note can be summarized as follows.

Theorem. *Let $f : \mathbb{F}_4^3 \to \mathbb{F}_4$ where $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, $\omega^2 = \omega + 1$, denotes the Galois field of order 4 be a map such that for each $x_0, y_0, z_0 \in \mathbb{F}_4$ the functions $x \mapsto f(x, y_0, z_0)$, $y \mapsto f(x_0, y, z_0)$ and $z \mapsto f(x_0, y_0, z)$ are permutations of $\mathbb{F}_4$. Such a map describes a Laguerre near-plane $\mathcal{L}(f)$ of order 4 as follows. The point set is $\mathbb{F}_4 \times \mathbb{F}_4$ and generators are the verticals $\{c\} \times \mathbb{F}_4$ for $c \in \mathbb{F}_4$. Circles are of the form*

$$\{(1, x), (\omega, y), (\omega^2, z), (0, f(x, y, z))\}$$

for $x, y, z \in \mathbb{F}_4$. Conversely, every Laguerre near-plane of order 4 can be uniquely described in this way by such a map.

A Laguerre near-plane $\mathcal{L}(f)$ can be uniquely extended to the Miquelian Laguerre plane of order 4 by adjoining the points of one generator if and only if one of the following holds.

(1) *$f + f(0, 0, 0)$ is additive;*
(2) *the circle set $\{(x, y, z, f(x, y, z)) \mid x, y, z \in \mathbb{F}_4\}$ (i.e., the graph of f) forms an affine subspace of $\mathbb{F}_4^4$ over the Galois field $\mathbb{F}_2$ of order 2;*

(3) *f is of degree at most 3.*

Up to isomorphism, there are precisely five Laguerre near-planes of order 4. These planes are described by the maps

$$
\begin{aligned}
f_0(x, y, z) =& x + y + z,\\
f_1(x, y, z) =& (x^2 + x)(y^2 + y) + (y^2 + y)(z^2 + z) + (x^2 + x)(z^2 + z) + x + y + z,\\
f_2(x, y, z) =& (x^2 + x)(z^2 + z) + x + y + z,\\
f_3(x, y, z) =& (x^2 + x)(y^2 + y)(z^2 + z) + x + y + z,\\
f_4(x, y, z) =& (x^2 + \omega^2 x)(y^2 + \omega y)(z^2 + \omega z) + (x^2 + \omega^2 x)(y^2 + \omega^2 y)\\
&+ (x^2 + \omega^2 x)(z^2 + \omega^2 z) + (y^2 + \omega y)(z^2 + \omega z) + x + y + z.
\end{aligned}
$$

Note that a Laguerre plane of order 4 is Miquelian so that the Laguerre near-planes described by f_1, f_2, f_3 or f_4 cannot be extended to Laguerre planes, that is, the corresponding orthogonal arrays are maximal. The Laguerre near-plane $\mathcal{L}(f_0)$ extends to the Miquelian Laguerre plane of order 4 and this plane in turn can be extended by adding one generator through the nucleus of the conic over which the Miquelian Laguerre plane is constructed in 3-dimensional projective space $\mathcal{P}_3$, that is, one has the geometry of plane intersections of a cone over an hyperoval with planes of $\mathcal{P}_3$ not passing through the vertex of the cone. This leads to orthogonal array with 6 constraints.

Replacing in axiom (J) in the definition of a Laguerre near-plane the number of points by k one obtains an interpolating system of rank k. By the above theorem there are essentially five interpolating systems of rank 3 over $\mathbb{F}_4$. Clearly there is only one interpolating system of rank 4 over $\mathbb{F}_4$. An interpolating system of rank 2 corresponds to an affine plane of order 4, so that there is only one interpolating system of rank 2 over $\mathbb{F}_4$.

We deal with Laguerre near-planes of order 4 exclusively and sometimes omit order 4 when speaking of Laguerre near-planes.

2. A representation of Laguerre near-planes of order 4

We denote by $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, $\omega^2 = \omega + 1$, the Galois field of order 4. Then the point set of a Laguerre near-plane of order 4 can be identified with $\mathbb{F}_4 \times \mathbb{F}_4$ and generators being the verticals $\{c\} \times \mathbb{F}_4$.

Since ω is a generator of the multiplicative group of $\mathbb{F}_4$, the non-zero elements of $\mathbb{F}_4$ can be written in the form ω^i for $i = 0, 1, 2$. We use the notation $\omega^\infty = 0$ and

$$I = \{0, 1, 2, \infty\}.$$

Then

$$\mathbb{F}_4 = \{\omega^i \mid i \in I\}.$$

2.1. A representation of Laguerre near-planes of order 4 in terms of a single map. Each circle has four points and is uniquely determined by these points. Each circle is therefore described by some $(c_0, c_1, c_2, c_\infty) \in \mathbb{F}_4^4$ as

$$C_{c_0,c_1,c_2,c_\infty} = \{(1, c_0), (\omega, c_1)(\omega^2, c_2), (0, c_\infty)\} = \{(\omega^i, c_i) \mid i \in I\}.$$

There are 256 vectors in $\mathbb{F}_4^4$ and 64 of them describe circles as above. We denote the collection of all circle describing vectors again by $\mathcal{C}$. In this representation axioms (G) and (C) of a Laguerre near-plane are clearly satisfied. Axiom (J) is equivalent to saying that for any three mutually distinct $i_1, i_2, i_3 \in I$ and any three $y_1, y_2, y_3 \in \mathbb{F}_4$ there is a unique solution $(c_0, c_1, c_2, c_\infty)$ in $\mathcal{C}$ such that $c_{i_k} = y_k$ for $k = 1, 2, 3$. In particular, for $i_k = k - 1$ we obtain that c_∞ is a function f of c_0, c_1 and c_2, that is, circles in $\mathcal{C}$ are represented by vectors of the form $(x, y, z, f(x, y, z))$ for $x, y, z \in \mathbb{F}_4$ where $f : \mathbb{F}_4^3 \to \mathbb{F}_4$ is some map. Hence, we can write $\mathcal{C}$ in the form

$$\mathcal{C} = \{(x, y, z, f(x, y, z)) \mid x, y, z \in \mathbb{F}_4\}$$

and $\mathcal{C}$ can then be viewed as a hypersurface in $\mathbb{F}_4^4$.

Furthermore, choosing a different set of mutually distinct indices i_1, i_2, i_3 axiom (J) shows that for fixed $x_0, y_0, z_0 \in \mathbb{F}_4$ the functions $f_{y_0,z_0} : x \mapsto f(x, y_0, z_0)$, $f_{x_0,z_0} : y \mapsto f(x_0, y, z_0)$ and $f_{x_0,y_0} : z \mapsto f(x_0, y_0, z)$ are permutations of $\mathbb{F}_4$.

Conversely, every map f with this property describes a circle set $\mathcal{C}$ and thus defines a Laguerre near-plane of order 4. We denote this plane by $\mathcal{L}(f)$.

2.2. An alternative description. Each circle determines a unique polynomial of degree at most 3, that is, each circle is described by some $(c_3, c_2, c_1, c_0) \in \mathbb{F}_4^4$ as

$$C'_{c_3,c_2,c_1,c_0} = \{(u, c_3u^3 + c_2u^2 + c_1u + c_0) \mid u \in \mathbb{F}_4\}.$$

In order to describe how this description of circles relates to the preceding one let u_1, u_2, u_3, u_4 be the four elements of $\mathbb{F}_4$. Each polynomial

$$p_{u_4}(X) = (X - u_1)(X - u_2)(X - u_3)$$

vanishes at u_1, u_2 and u_3 and has value 1 at u_4 because $(u_4 - u_1)(u_4 - u_2)(u_4 - u_3)$ equals the product of all non-zero elements in $\mathbb{F}_4$. Expanding we explicitly have the following four polynomials

$$\begin{aligned} p_0(X) =& X^3 + 1, \\ p_1(X) =& X^3 + X^2 + X, \\ p_\omega(X) =& X^3 + \omega X^2 + \omega^2 X, \\ p_{\omega^2}(X) =& X^3 + \omega^2 X^2 + \omega X. \end{aligned}$$

Using these four polynomials, then

$$C_{c_0,c_1,c_2,c_\infty} = \{(u, c_0p_1(u) + c_1p_\omega(u) + c_2p_{\omega^2}(u) + c_\infty p_0(u)) \mid u \in \mathbb{F}_4\}.$$

Expanding $c_0p_1(u) + c_1p_\omega(u) + c_2p_{\omega^2}(u) + c_\infty p_0(u)$ in powers of u yields

$$C_{c_0,c_1,c_2,c_\infty} = C'_{c_0+c_1+c_2+c_\infty, c_0+c_1\omega+c_2\omega^2, c_0+c_1\omega^2+c_2\omega, c_\infty}.$$

In the parabola model of a Laguerre near-plane of order 4, all circles are graphs of polynomials of degree at most 2. Thus $c_0 + c_1 + c_2 + c_\infty = 0$ for all circles in this plane, that is, a describing map f as in 2.1 is $f(x, y, z) = x + y + z$.

Lemma 2.3. *The parabola model of a Laguerre near-plane of order 4 can be represented in the form $\mathcal{L}(f)$ for $f(x,y,z) = x+y+z$.*

With every map $f : \mathbb{F}_4^3 \to \mathbb{F}_4$ we can associate a unique polynomial in X, Y and Z of degree at most 3 in each of the three variables, i.e.,

$$f(X,Y,Z) = \sum_{ijk=0}^{3} a_{ijk} X^i Y^j Z^k$$

for some $a_{ijk} \in \mathbb{F}_4$. Each of the above restricted maps then is described by a polynomial in one variable of degree at most 3. However, there are only a few such polynomials that define permutations of $\mathbb{F}_4$, cf. [3] , §7.

Lemma 2.4. *A polynomial $p(X) = \sum_{i=0}^{3} a_i X^i$ of degree at most 3 over $\mathbb{F}_4$ defines a permutation of $\mathbb{F}_4$ if and only if $a_3 = 0$ and either $a_2 = 0$ or $a_1 = 0$. The latter condition is equivalent to $a_2^3 + a_1^3 = 1$.*

Proof. The polynomial $p(X)$ defines a permutation of $\mathbb{F}_4$ if and only if the evaluation map $p : \mathbb{F}_4 \to \mathbb{F}_4 : x \mapsto p(x)$ is one-to-one. Since translations and homotheties are permutations, we may assume that the leading coefficient of $p(X)$ is 1 and that the constant term equals 0.

Suppose that $p(X)$ has degree 3 so that $p(X) = X^3 + a_2X^2 + a_1X$. If $a_1 = 0$, then $p(0) = p(a_2) = 0$ and p is not injective for $a_2 \neq 0$. If $a_2 = a_1 = 0$, then $p(1) = p(w) = 1$ and again p is not injective. We now assume that $a_1 \neq 0$. Since $p(X)$ defines a permutation of $\mathbb{F}_4$ if and only if $p(\frac{1}{a_1}X)$ defines a permutation of $\mathbb{F}_4$, we may assume that $a_1 = 1$ so that $p(X) = X^3 + a_2X^2 + X$. But then $p(1) = p(a_2) = a_2$ and p is not injective. This proves that $p(X)$ has degree at most 2, i.e., $a_3 = 0$.

Suppose that $p(X)$ has degree 2. Then $p(X) = X^2 + a_1X$ and $p(0) = p(a_1) = 0$ so that p is not injective for $a_1 \neq 0$. Clearly, $x \mapsto x^2$ is a permutation of $\mathbb{F}_4$ so that a quadratic polynomial defines a permutation of $\mathbb{F}_4$ if and only if the linear term equals 0.

Since $u^3 = 0$ or 1 for $u = 0$ or $u \neq 0$, respectively, it readily follows that $u^3 + v^3 = 1$ for $u, v \in \mathbb{F}_4$ if and only if either $u = 0$, $v \neq 0$ or $u \neq 0$, $v = 0$. □

Note that maps of the form $x \mapsto a_2x^2 + a_1x$ for $a_2, a_1 \in \mathbb{F}_4$ are additive, that is, they are linear over $\mathbb{F}_2$, the Galois field of order 2. Hence each such permutation of $\mathbb{F}_4$ represents an element of $\mathrm{GL}(2,2)$, the group of all invertible 2×2 matrices over the field $\mathbb{F}_2$. This group has order 6 and obviously the maps for $a_2 \neq 0$, $a_1 = 0$ and $a_2 = 0$, $a_1 \neq 0$ belong to it. Therefore we must have already covered all permutations of this form.

From this point of view one further obtains the inverse of $x \mapsto a_2x^2 + a_1x$ for $a_2, a_1 \in \mathbb{F}_4$, $a_2^3 + a_1^3 = 1$, in closed form. Let

$$a_1x + a_2x^2 = u;$$

then

$$a_2^2x + a_1^2x^2 = u^2.$$

In matrix notation we have

$$\begin{pmatrix} a_1 & a_2 \\ a_2^2 & a_1^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ x^2 \end{pmatrix} = \begin{pmatrix} u \\ u^2 \end{pmatrix}.$$

Since the coefficient matrix of this system of linear equations has determinant $a_1^3 + a_2^3 = 1$, one finds

$$x = a_1^2 u + a_2 u^2.$$

We now consider the partial map $z \mapsto f(x, y, z)$ which is described by a polynomial in Z. By the preceding lemma the coefficient of Z^3 must be 0. Therefore

$$\sum_{i,j=0}^{3} a_{ij3} x^i y^j = 0$$

for all $x, y \in \mathbb{F}_4$. Hence the polynomial $p(X, Y) = \sum_{i,j=0}^{3} a_{ij3} X^i Y^j$ vanishes identically. This implies $a_{ij3} = 0$ for all i and j. Considering the other partial maps, we similarly find that $a_{i,3,k} = a_{3,j,k} = 0$ for all i, j and k from 0 to 3. Therefore $f(X, Y, Z)$ reduces to $f(X, Y, Z) = \sum_{ijk=0}^{2} a_{ijk} X^i Y^j Z^k$ for some $a_{ijk} \in \mathbb{F}_4$.

Furthermore,

$$(\sum_{i,j=0}^{2} a_{ij2} x^i y^j)^3 + (\sum_{i,j=0}^{2} a_{ij1} x^i y^j)^3 = 1$$

for all $x, y \in \mathbb{F}_4$. In particular, for $x = y = 0$ we obtain $a_{002}^3 + a_{001}^3 = 1$ so that either $a_{002} = 0$ or $a_{001} = 0$ and the respective other term being non-zero.

By looking at the other partial maps we obtain the following characterisation.

Proposition 2.5. *$f(X, Y, Z) = \sum_{ijk=0}^{2} a_{ijk} X^i Y^j Z^k$ describes a Laguerre near-plane if and only if*

$$(\sum_{i,j=0}^{2} a_{ij2} x^i y^j)^3 + (\sum_{i,j=0}^{2} a_{ij1} x^i y^j)^3 = 1$$
$$(\sum_{i,k=0}^{2} a_{i2k} x^i z^k)^3 + (\sum_{i,k=0}^{2} a_{i1k} x^i z^k)^3 = 1$$
$$(\sum_{j,k=0}^{2} a_{2jk} y^j z^k)^3 + (\sum_{j,k=0}^{2} a_{1jk} y^j z^k)^3 = 1$$

for all $x, y, z \in \mathbb{F}_4$. In particular, $a_{002}^3 + a_{001}^3 = a_{020}^3 + a_{010}^3 = a_{200}^3 + a_{100}^3 = 1$.

It seems that the corresponding polynomial identities cannot be algebraically used in general to simplify the form of f a great deal although we shall come back to them later on. Note, however, that the above conditions do not involve the coefficient a_{000}. In fact, up to isomorphism, we can always assume that $a_{000} = 0$, see 5.1.

Corollary 2.6. *The inverses of the partial maps with respect to x, y and z are given by*

$$\begin{aligned}
f_{y,z}^{-1}(x) =& f_2^x(y,z)x^2 + f_1^x(y,z)^2 x + f_0^x(y,z)^2 f_2^x(y,z) + f_0^x(y,z) f_1^x(y,z)^2,\\
f_{x,z}^{-1}(y) =& f_2^y(x,z)y^2 + f_1^y(x,z)^2 y + f_0^y(x,z)^2 f_2^y(x,z) + f_0^y(x,z) f_1^y(x,z)^2,\\
f_{x,y}^{-1}(z) =& f_2^z(x,y)z^2 + f_1^z(x,y)^2 z + f_0^z(x,y)^2 f_2^z(x,y) + f_0^z(x,y) f_1^z(x,y)^2,
\end{aligned}$$

respectively, where $f_2^x(y,z)$, $f_1^x(y,z)$, $f_0^x(y,z)$, $f_2^y(x,z)$, $f_1^y(x,z)$, $f_0^y(x,z)$, $f_2^z(x,y)$, $f_1^z(x,y)$ and $f_0^z(x,y)$ are the respective coefficient functions, i.e.,

$$\begin{aligned}
f(x,y,z) =& f_2^x(y,z)x^2 + f_1^x(y,z)x + f_0^x(y,z)\\
=& f_2^y(x,z)y^2 + f_1^y(x,z)y + f_0^y(x,z)\\
=& f_2^z(x,y)z^2 + f_1^z(x,y)z + f_0^z(x,y).
\end{aligned}$$

Proof. Let $f(x,y,z) = f_2^z(x,y)z^2 + f_1^z(x,y)z + f_0^z(x,y)$. We can write the inverse $f_{x,y}^{-1}$ of the partial map with respect to z in the form $f_{x,y}^{-1}(z) = g_2(x,y)z^2 + g_1(x,y)z + g_0(x,y)$. Expanding the identity $f_{x,y}^{-1}(f(x,y,z)) = z$ one finds

$$\begin{aligned}
g_2(x,y)f_1^z(x,y)^2 + g_1(x,y)f_2^z(x,y) &= 0,\\
g_2(x,y)f_2^z(x,y)^2 + g_1(x,y)f_1^z(x,y) &= 1,\\
g_2(x,y)f_0^z(x,y)^2 + g_1(x,y)f_0^z(x,y) + g_0(x,y) &= 0.
\end{aligned}$$

This is a system of linear equations for $g_2(x,y)$, $g_1(x,y)$ and $g_0(x,y)$. Since its determinant is $f_2^z(x,y)^3 + f_1^z(x,y)^3 = 1$ by Proposition 3.5, this system has a unique solution and one readily finds

$$\begin{aligned}
g_2(x,y) &= f_2^z(x,y),\\
g_1(x,y) &= f_1^z(x,y)^2,\\
g_0(x,y) &= f_0^z(x,y)^2 f_2^z(x,y) + f_0^z(x,y) f_1^z(x,y)^2.
\end{aligned}$$

The inverses of the other partial maps are found likewise. □

Examples 2.7.

(1) Let $f(x,y,z) = (x^2+x)(z^2+z)+x+y+z$. Then f is a Laguerre near-plane describing map. The inverses of the partial maps with respect to x, y and z are $w \mapsto (w^2+w+y^2+y)(z^2+z)+w+y+z^2$, $w \mapsto (x^2+x)(z^2+z)+w+x+z$, $w \mapsto (w^2+w+y^2+y)(x^2+x)+w+x^2+y$, respectively, where $w = f(x,y,z)$.

(2) Let $f(x,y,z) = (x^2+x)(y^2+y)+(y^2+y)(z^2+z)+(x^2+x)(z^2+z)+x+y+z$. Then f is a Laguerre near-plane describing map. The inverses of the partial maps with respect to x, y and z are $w \mapsto f(w,y^2,z^2)$, $w \mapsto f(x^2,w,z^2)$, $w \mapsto f(x^2,y^2,w)$, respectively, where $w = f(x,y,z)$.

(3) Let $f(x,y,z) = (x^2+x)(y^2+y)(z^2+z)+x+y+z$. Then f is a Laguerre near-plane describing map. The inverses of the partial maps with respect to x, y and z are $w \mapsto f(w,y,z)$, $w \mapsto f(x,w,z)$, $w \mapsto f(x,y,w)$, respectively, where $w = f(x,y,z)$.

3. Isomorphisms and Linear Laguerre near-planes of order 4

3.1. Isomorphisms of Laguerre near-planes of order 4. The obvious definition of an isomorphism between Laguerre near-planes of the same order is that we have a bijection between the point sets that takes generators to generators and circles to circles. Using the representation 2.1 for Laguerre near-planes of order 4, every isomorphism is of the form

$$\mathbb{F}_4^4 \to \mathbb{F}_4^4 : (u,v) \mapsto (\alpha(u), \beta_u(v))$$

where α and β_u are permutations of $\mathbb{F}_4$ for each $u \in \mathbb{F}_4$.

Clearly, the group of permutations of $\mathbb{F}_4$ is the symmetric group S_4. Every even permutation can be written as $u \mapsto au + b$ for some $a, b \in \mathbb{F}_4$, $a \neq 0$. The automorphism $u \mapsto u^2$ of $\mathbb{F}_4$ is an odd permutation of $\mathbb{F}_4$ – in fact, a transposition – and every odd permutation of $\mathbb{F}_4$ is of the form $u \mapsto au^2 + b$ for some $a, b \in \mathbb{F}_4$.

The collection of all permutations of $\mathbb{F}_4 \times \mathbb{F}_4$ as above forms a group Γ of order $(4!)^4 = (24)^5 = 2^{15} \cdot 3^5$. We give a set of generators for Γ as permutations of $\mathbb{F}_4 \times \mathbb{F}_4$ and determine how circles and Laguerre near-planes of order 4 are transformed.

(1) $(u,v) \mapsto (u, \beta_u(v))$ where β_u are permutations of $\mathbb{F}_4$ for each $u \in \mathbb{F}_4$. These permutations take C_{c_0,c_1,c_2,c_∞} to $C_{\beta_1(c_0),\beta_\omega(c_1),\beta_{\omega^2}(c_2),\beta_0(c_\infty)}$. A Laguerre near-plane $\mathcal{L}(f)$ is taken to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \beta_0(f(\beta_1^{-1}(x), \beta_\omega^{-1}(y), \beta_{\omega^2}^{-1}(z))$$

for $x, y, z \in \mathbb{F}_4$.

(2) $(u,v) \mapsto (u+t, v)$ for $t \in \mathbb{F}_4$. These permutations take C_{c_0,c_1,c_2,c_∞} to C_{d_0,d_1,d_2,d_∞} where

$$(d_0,d_1,d_2,d_\infty) = \begin{cases} (c_0,c_1,c_2,c_\infty), & \text{if } t=0,\\ (c_\infty,c_2,c_1,c_0), & \text{if } t=1,\\ (c_2,c_\infty,c_0,c_1), & \text{if } t=\omega,\\ (c_1,c_0,c_\infty,c_2), & \text{if } t=\omega^2. \end{cases}$$

A Laguerre near-plane $\mathcal{L}(f)$ is taken to $\mathcal{L}(f')$ where $f' = f$ for $t = 0$ and f' is an inverse of a partial map of f with the other two variables exchanged given by $f'(f(x,y,z),z,y) = x$, $f'(z,f(x,y,z),x) = y$ and $f'(y,x,f(x,y,z)) = z$ for $t = 1$, ω and ω^2, respectively, that is, the maps $(x,y,z) \mapsto f_{z,y}^{-1}(x)$, $(x,y,z) \mapsto f_{z,x}^{-1}(y)$ and $(x,y,z) \mapsto f_{y,x}^{-1}(z)$, respectively.

(3) $(u,v) \mapsto (ru, v)$ for $r \in \mathbb{F}_4$, $r \neq 0$. These permutations take C_{c_0,c_1,c_2,c_∞} to $C_{c_{3-k},c_{1-k},c_{2-k},c_\infty}$ where $r = \omega^k$, $k = 0,1,2$, and the indices $3-k$, $1-k$ and $2-k$ are taken modulo 3. A Laguerre near-plane $\mathcal{L}(f)$ is taken to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \begin{cases} f(x,y,z), & \text{if } r=1,\\ f(y,z,x), & \text{if } r=\omega,\\ f(z,x,y), & \text{if } r=\omega^2. \end{cases}$$

(4) $(u,v) \mapsto (u^2, v)$. This permutation takes C_{c_0,c_1,c_2,c_∞} to C_{c_0,c_2,c_1,c_∞}. A Laguerre near-plane $\mathcal{L}(f)$ is taken to $\mathcal{L}(f')$ where $f'(x,y,z) = f(x,z,y)$.

Note that inverses of partial maps can be obtained as a composition of permutations of types (2), (3) and (4). More precisely, the inverse of the partial map with respect to x, y and z can be found after the permutation $(u,v) \mapsto (u^2+1, v)$, $(u,v) \mapsto (\omega(u^2+1), v)$ and $(u,v) \mapsto (\omega^2(u^2+1), v)$, respectively.

3.2. Linear Laguerre near-planes of order 4. A nice class of examples are the *linear Laguerre near-planes of order 4*. In this case, $\mathcal{C}$ is an affine subspace of $\mathbb{F}_4^4$, that is, $\mathcal{C}$ is of the form

$$\mathcal{C} = \{(c_0, c_1, c_2, c_\infty) \in \mathbb{F}_4^4 \mid a_0c_0 + a_1c_1 + a_2c_2 + a_\infty c_\infty = b\}$$

for some $a_0, a_1, a_2, a_\infty, b \in \mathbb{F}$, $(a_0, a_1, a_2, a_\infty) \neq (0,0,0,0)$. For $\mathcal{C}$ to be a hypersurface as described in 2.1 we have to require that each a_i, $i \in I$, is non-zero. Furthermore, because $(a_0, a_1, a_2, a_\infty, b)$ and $\lambda(a_0, a_1, a_2, a_\infty, b)$ for $\lambda \in \mathbb{F}$, $\lambda \neq 0$, describe the same linear Laguerre near-plane, we can assume that $a_\infty = 1$. The associated map f then is

$$f(x,y,z) = a_0x + a_1y + a_2z + b.$$

We denote this Laguerre near-plane by $\mathcal{L}(a_0, a_1, a_2, b)$. Clearly, the parabola model of a Laguerre near-plane of order 4 is of this form. More precisely, it can be obtained for $(a_0, a_1, a_2, b) = (1,1,1,0)$. In fact, all affine subspaces essentially yield the same model. From 3.1 we see that the permutation of type (1)

$$(u,v) \mapsto \begin{cases} (u, a_0v), & \text{if } u = 1, \\ (u, a_1v), & \text{if } u = \omega, \\ (u, a_2v), & \text{if } u = \omega^2, \\ (u, v+b), & \text{if } u = 0, \end{cases}$$

yields an isomorphism from $\mathcal{L}(a_0, a_1, a_2, b)$ to $\mathcal{L}(1,1,1,0)$. Hence, we have the following result.

Proposition 3.3. *Every linear Laguerre near-plane of order 4 is isomorphic to the Laguerre near-plane $\mathcal{L}(1,1,1,0)$ obtained from the Miquelian Laguerre plane of order 4 by deleting one generator.*

As for models of Laguerre near-planes of order 4 that are not isomorphic to $\mathcal{L}(1,1,1,0)$ we begin with a closer description of circles of $\mathcal{L}(1,1,1,0)$. They are of the form

$$\{(u, au^2 + bu + c) \mid u \in \mathbb{F}_4\}$$

for $a, b, c \in \mathbb{F}_4$, and they fall into three classes. First, there are the graphs of the four constant polynomials obtained for $a = b = 0$. Then there are the graphs of the 24 permutation polynomials obtained for $a = 0$, $b \neq 0$ and $a \neq 0$, $b = 0$. Third, there are the graphs of the remaining 36 polynomials obtained for $a, b \neq 0$; these polynomials take on exactly two values and each of these values occurs exactly twice.

Note that the same picture emerges if we delete a different generator from the Miquelian Laguerre plane of order 4 because the automorphism group of this plane is transitive on the point set.

Example 3.4. We now modify the above model to obtain a new Laguerre near-plane of order 4. To this end, we consider the circles that are entirely contained in

$$S = \mathbb{F}_4 \times \{\omega, \omega^2\}.$$

There are 8 such circles, two of the first kind and six of the third kind.

These 8 circles cover 32 *admissible* triples of points, that is, triples of points such that no two of the points are on the same generator. We now replace these circles by 8 new circles covering the same 32 admissible triples of points. From this property it will be clear that we again obtain a Laguerre near-plane of order 4. The new circles are obtained as the images of the 8 old circles under the map

$$\phi : (u, v) \mapsto \begin{cases} (u, v), & \text{if } u \neq 0, \\ (u, v^2), & \text{if } u = 0; \end{cases}$$

that is, the points $(0, \omega)$ and $(0, \omega^2)$ are swapped and all other points remain unchanged.

However, since we still have circles of all three types and the new circles where one of the values ω or ω^2 occurs thrice and the other once, this Laguerre near-plane cannot be obtained from a Laguerre plane of order 4 by deleting one generator.

In order to represent this Laguerre near-plane by a function f as in 2.1 one at the eight circles that are replaced. In the parabola model these circles are described as

$$C_{x,y,z,x+y+z}$$

for all $x, y, z \in \{\omega, \omega^2\}$. These circles are replaced by

$$C_{x,y,z,x+y+z+1}.$$

Hence $f(x, y, z) = x + y + z + g(x, y, z)$ where $g(x, y, z)$ is a function that has value 1 if $x, y, z \in \{\omega, \omega^2\}$ and value 0 else. Now $g(x, y, z)$ can be found as

$$\begin{aligned} g(x, y, z) =& p_\omega(x)p_\omega(y)p_\omega(z) + p_\omega(x)p_\omega(y)p_{\omega^2}(z) \\ &+ p_\omega(x)p_{\omega^2}(y)p_\omega(z) + p_\omega(x)p_{\omega^2}(y)p_{\omega^2}(z) \\ &+ p_{\omega^2}(x)p_\omega(y)p_\omega(z) + p_{\omega^2}(x)p_\omega(y)p_{\omega^2}(z) \\ &+ p_{\omega^2}(x)p_{\omega^2}(y)p_\omega(z) + p_{\omega^2}(x)p_{\omega^2}(y)p_{\omega^2}(z) \\ =&(p_\omega(x) + p_{\omega^2}(x))(p_\omega(y) + p_{\omega^2}(y))(p_\omega(z) + p_{\omega^2}(z)) \\ =&(x^2 + x)(y^2 + y)(z^2 + z) \end{aligned}$$

Therefore

$$f(x, y, z) = (x^2 + x)(y^2 + y)(z^2 + z) + x + y + z;$$

see Example 2.7.3 for f being a Laguerre near-plane describing map.

4. The Classification

We return to the permutations of type (1) listed in 3.1 and and have a closer look at how they transform a Laguerre near-plane $\mathcal{L}(f)$.

4.1 Normal form. Each permutation of $\mathbb{F}_4$ is generated by permutations of the form $u \mapsto u+t$, $u \mapsto ru$ and $u \mapsto u^2$ for some $r,t \in \mathbb{F}_4$, $r \neq 0$. Correspondingly, permutations of type (1) listed in 3.1 are generated by the following transformations.

(1a) $(u,v) \mapsto (u, v+tp_w(u)) = \begin{cases} (u,v), & \text{if } u \neq w, \\ (u,v+t), & \text{if } u = w, \end{cases}$ for $t,w \in \mathbb{F}_4$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x,y,z) = f(x+tp_w(1), y+tp_w(\omega), z+tp_w(\omega^2)) + tp_w(0)$$
$$= \begin{cases} f(x,y,z)+t, & \text{if } w = 0, \\ f(x+t,y,z), & \text{if } w = 1, \\ f(x,y+t,z), & \text{if } w = \omega, \\ f(x,y,z+t), & \text{if } w = \omega^2. \end{cases}$$

(1b) $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \neq w, \\ (u,rv), & \text{if } u = w, \end{cases}$ for $r,w \in \mathbb{F}_4$, $r \neq 0$, takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \begin{cases} rf(x,y,z), & \text{if } w = 0, \\ f(r^2x,y,z), & \text{if } w = 1, \\ f(x,r^2y,z), & \text{if } w = \omega, \\ f(x,y,r^2z), & \text{if } w = \omega^2. \end{cases}$$

(1c) $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \neq w, \\ (u,v^2), & \text{if } u = w, \end{cases}$ for $w \in \mathbb{F}_4$ takes $\mathcal{L}(f)$ to $\mathcal{L}(f')$ where

$$f'(x,y,z) = \begin{cases} f(x,y,z)^2, & \text{if } w = 0, \\ f(x^2,y,z), & \text{if } w = 1, \\ f(x,y^2,z), & \text{if } w = \omega, \\ f(x,y,z^2), & \text{if } w = \omega^2. \end{cases}$$

Note that permutations of type (1a) can be used to yield a map that takes (0,0,0) to 0 whereas permutations of type (1b) and (1c) allow us to replace any of the coordinates by a fixed multiple or by its square, respectively. Recall that permutations of type (3) and (4) allow us to obtain any permutation of the coordinates x, y and z. This can be applied to obtain some normalizations for some of the coefficients of f. In particular, applying an isomorphism of type (1a), we can achieve that $a_{000} = 0$. This means that $C_{0,0,0,0}$ is then a circle in our Laguerre near-plane. Furthermore, since $a_{002}^3 + a_{001}^3 = a_{020}^3 + a_{010}^3 = a_{200}^3 + a_{100}^3 = 1$, see 2.5, we can use isomorphisms of type (1c), if necessary, to achieve $a_{002} = a_{020} = a_{200} = 0$.

Finally, using isomorphisms of type (1b), if necessary, we can further assume that $a_{001} = a_{010} = a_{100} = 1$. Then $f(x,0,0) = x$, $f(0,y,0) = y$ and $f(0,0,z) = z$ for all $x, y, z \in \mathbb{F}_4$. We say that f is in *normal form* if the above identities are satisfied. All examples 2.7 are in normal form. With this notation we have proved the following.

Proposition 4.2. *A Laguerre near-plane $\mathcal{L}(f)$ is isomorphic to a Laguerre near-plane $\mathcal{L}(f')$ where f' is in normal form.*

Note that normal form is preserved under isomorphisms of type (3) and (4) and the transformations $(u,v) \mapsto (u, rv)$, $r \neq 0$, (type (1b)), and $(u,v) \mapsto (u, v^2)$ (type (1c)).

A map f in normal form can obviously be written in the form $f(x,y,z) = g(x,y,z) + x + y + z$ where the polynomial $g(X,Y,Z)$ corresponding to g has no pure terms X^i, Y^j or Z^k.

Proposition 4.3. *A Laguerre near-plane $\mathcal{L}(f)$ is isomorphic to the Laguerre near-plane obtained from the Miquelian Laguerre plane of order 4 by deleting one generator if and only if $f + f(0,0,0)$ is additive, that is,*

$$f(x,y,z) = a_2x^2 + a_1x + b_2y^2 + b_1y + c_2z^2 + c_1z + d$$

for some $a_2, a_1, b_2, b_1, c_2, c_1, d \in \mathbb{F}_4$, $a_2^3 + a_1^3 = b_2^3 + b_1^3 = c_2^3 + c_1^3 = 1$.

A Laguerre near-plane $\mathcal{L}(f)$ with f in normal form is isomorphic to the parabola model of a Laguerre near-plane of order 4 if and only if $f(x,y,z) = x + y + z$.

Proof. Each permutation of the form (1) takes an affine subspace of the affine space $\mathbb{F}_4^4$ over the prime field $\mathbb{F}_2$ to such a subspace. Furthermore, each such subspace that meets each parallel of the coordinates axes in exactly one point can be described by a function f as in the proposition. Hence every Laguerre near-plane isomorphic to $\mathcal{L}(1,1,1,0)$ can be represented in this form.

Conversely, $\mathcal{L}(f)$ with f as above is isomorphic to $\mathcal{L}(f')$ for some f' in normal form. Furthermore, f' still has the same overall form. Hence $f'(x,y,z) = x+y+z$, that is, $\mathcal{L}(f')$ is the parabola model. □

In the proof of Proposition 4.3 we found another characterization of the parabola model.

Corollary 4.4. *A Laguerre near-plane $\mathcal{L}(f)$ is isomorphic to the parabola model of a Laguerre near-plane of order 4 if and only if the graph of f is an affine subspace of $\mathbb{F}_4^4$ over $\mathbb{F}_2$.*

Let $f(x,x,y) = \sum_{ijk=0}^{2} a_{ijk}x^iy^jz^k$ for some $a_{ijk} \in \mathbb{F}_4$. We say that f *has degree* n if $n = \max\{i+j+k \mid a_{ijk} \neq 0\}$. Note that isomorphisms of types (1a), (1b), (3) and (4) do not change the degree. In the following we discuss the degrees from 1 to 4 separately and do a computer search for degrees 5 and 6. Using an isomorphism of type (1a), if necessary, we may always assume that $a_{000} = 0$.

Degree 1. In this case, we clearly have $f(x,y,z) = ax+by+cz$ for some $a, b, c \in \mathbb{F}_4$, $a, b, c \neq 0$, and $\mathcal{L}(f)$ is isomorphic to $\mathcal{L}(1,1,1,0)$ by Proposition 4.3.

Degree 2. In this case, $f(x,y,z) = a_{200}x^2 + a_{020}y^2 + a_{002}z^2 + a_{110}xy + a_{101}xz + a_{011}yz + a_{100}x + a_{010}y + a_{001}z$ for some $a_{ijk} \in \mathbb{F}_4$ where at least one coefficient a_{ijk} is non-zero for $i + j + k = 2$. Rewriting f as a partial map in x we have

$$\begin{aligned} f(x,y,z) =& a_{200}x^2 \\ &+ (a_{110}y + a_{101}z + a_{100})x \\ &+ a_{020}y^2 + a_{002}z^2 + a_{011}yz + a_{010}y + a_{001}z. \end{aligned}$$

By Lemma 2.4 we have either $a_{200} = 0$ or $a_{110}y + a_{101}z + a_{100} = 0$ for all $y, z \in \mathbb{F}_4$. If $a_{200} \neq 0$, then we obtain $a_{110} = a_{101} = a_{100} = 0$; if $a_{200} = 0$, then $a_{110}y + a_{101}z + a_{100} \neq 0$ for any $y, z \in \mathbb{F}_4$ and one finds $a_{110} = a_{101} = 0$. Therefore $a_{110} = a_{101} = 0$ in any case. Considering the partial map in y one likewise obtains $a_{011} = 0$. Hence $a_{110} = a_{101} = a_{011} = 0$ and $\mathcal{L}(f)$ is isomorphic to $\mathcal{L}(1,1,1,0)$ by Proposition 4.3.

Degree 3. Writing f as a partial map in z we have

$$\begin{aligned} f(x,y,z) =& (a_{102}x + a_{012}y + a_{002})z^2 \\ &+ (a_{201}x^2 + a_{111}xy + a_{021}y^2 + a_{101}x + a_{011}y + a_{001})z \\ &+ a_{210}x^2y + a_{120}xy^2 + a_{200}x^2 + a_{110}xy + a_{020}y^2 \\ &\quad + a_{100}x + a_{010}y \end{aligned}$$

for some $a_{ijk} \in \mathbb{F}_4$ where at least one coefficient a_{ijk} is non-zero for $i + j + k = 3$.

Equating the coefficient of z^2 to 0 describes a line unless $a_{102} = a_{012} = 0$. Equating the coefficient of z to 0 describes a conic or a line unless $a_{201} = a_{111} = a_{021} = a_{101} = a_{011} = 0$. Since a line has four points and a conic has at most eight points (in case of a degenerate conic representing two parallel lines), a line and a conic or line cannot cover all 16 points of $\mathbb{F}_4^2$. Therefore, both coefficients of z^2 and z must be constant so that $f(x,y,z) = a_{002}z^2 + a_{001}z + a_{210}x^2y + a_{120}xy^2 + a_{200}x^2 + a_{020}y^2 + a_{110}xy + a_{100}x + a_{010}y$. In particular, $a_{102} = a_{012} = a_{201} = a_{111} = a_{021} = 0$.

A simimlar argument for the partial map in x shows that $a_{210} = a_{120} = 0$ – a contradiction to f being of degree 3. Hence degree 3 cannot occur.

Since example 2.7.1 gives us a map f of degree 4, and by Proposition 4.3 and the above we have the following characterization.

Proposition 4.5. *A Laguerre near-plane describing map f cannot have degree 3. Furthermore, $\mathcal{L}(f)$ is isomorphic to the parabola model $\mathcal{L}(1,1,1,0)$ if and only if f has degree at most 3.*

Degree 4. Let

$$\begin{aligned} f(x,y,z) =& a_{220}x^2y^2 + a_{022}y^2z^2 + a_{202}x^2z^2 \\ &+ a_{211}x^2yz + a_{121}xy^2z + a_{112}xyz^2 \\ &+ a_{210}x^2y + a_{201}x^2z + a_{120}xy^2 + a_{021}y^2z + a_{102}xz^2 + a_{012}yz^2 \\ &+ a_{111}xyz \\ &+ a_{200}x^2 + a_{020}y^2 + a_{002}z^2 + a_{110}xy + a_{011}yz + a_{101}xz \\ &+ a_{100}x + a_{010}y + a_{001}z \end{aligned}$$

for some $a_{ijk} \in \mathbb{F}_4$ where at least one coefficient a_{ijk} is non-zero for $i+j+k=4$. There are two types of terms of degree 4, one involving all three variables (like in xyz^2) and the other involving only two variables (like in x^2y^2).

We now suppose that terms of the first type occur. Using isomorphisms of types (3) and (4), we may assume that $a_{112} \neq 0$. Rewriting f as a partial map in z we have

$$\begin{aligned} f(x,y,z) =& (a_{202}x^2 + a_{112}xy + a_{022}y^2 + a_{102}x + a_{012}y + a_{002})z^2 \\ &+ (a_{211}x^2y + a_{121}xy^2 + a_{201}x^2 + a_{111}xy + a_{021}y^2 \\ &\qquad + a_{101}x + a_{011}y + a_{001})z \\ &+ a_{220}x^2y^2 + a_{210}x^2y + a_{120}xy^2 + a_{200}x^2 + a_{110}xy + a_{020}y^2 \\ &\qquad + a_{100}x + a_{010}y. \end{aligned}$$

Equating the coefficient $c_2(x,y)$ of z^2 to 0 describes a nondegenerate quadric which has at least one point and at most five points or a pair of intersecting lines which have seven points. Equating the coefficient $c_1(x,y)$ of z to 0 yields at most ten points unless this coefficient is identically 0. (The equation $(a_{121}x + a_{021})y^2 + (a_{211}x^2 + a_{111}x + a_{011})y + a_{201}x^2 + a_{101}x + a_{001} = 0$ has at most two solutions y for each x unless $a_{121}X + a_{021}$ is a common factor of $a_{211}X^2 + a_{111}X + a_{011}$ and $a_{201}X^2 + a_{101}X + a_{001}$ in which case one may have $4 + 3 \cdot 2 = 10$ solutions; if the coefficient of y^2 is identically 0, then a similar consideration shows that at most 10 solutions can occur.) By Lemma 2.4 the sets of zeros $Z_2 = \{(x,y) \in \mathbb{F}_4 \times \mathbb{F}_4 \mid c_2(x,y) = 0\}$ and $Z_1 = \{(x,y) \in \mathbb{F}_4 \times \mathbb{F}_4 \mid c_1(x,y) = 0\}$ parition $\mathbb{F}_4 \times \mathbb{F}_4$; furthermore the first set is nonempty. Hence, in order to cover 16 points, $c_2(x,y) = 0$ must describe a pair of intersecting lines and Z_1 must contain nine points. To get this number of points we must have $a_{121} \neq 0$ (otherwise $c_1(x,y) = 0$ has only at most eight points) and $a_{121}X + a_{021}$ must be a common factor of $a_{211}X^2 + a_{111}X + a_{011}$ and $a_{201}X^2 + a_{101}X + a_{001}$. Hence all points on the vertical line $\{a_{211}^{-1}a_{101}\} \times \mathbb{F}_4$ are solutions. However, such a line intersects at least one of the two non-parallel lines determined by $c_2(x,y) = 0$ – a contradiction to the fact that the two sets Z_2 and Z_1 must be disjoint.

This shows that terms of the first type cannot occur. We thus have $a_{112} = a_{121} = a_{211} = 0$.

The isomorphism $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \neq \omega^2, \\ (u,v^2), & \text{if } u = \omega^2, \end{cases}$ replaces f by the map f' given by $f'(x,y,z) = f(x,y,z^2)$, see 3.1. Moreover, f' has the same degree as f. Applying the same argument as before to f' we then see that $a_{111} = 0$.

Using isomorphisms of types (3) and (4) we may now assume that $a_{202} \neq 0$. Considering f as a partial map in z as before we see that the equations $c_2(x,y) = 0$ and $c_1(x,y) = 0$ describe a quadric or a quadric/line, respectively. Such a configuration can cover 16 points if and only if we have degenerate quadrics describing two parallel lines in both cases; together, one has a full bundle of parallel lines. In particular $a_{201} \neq 0$.

Using the isomorphism $(u,v) \mapsto \begin{cases} (u,v), & \text{if } u \neq \omega^2, \\ (u,v^2), & \text{if } u = \omega^2, \end{cases}$, if necessary, we may now assume that $a_{002} = 0$. Note that the degree has not changed. Using isomorphisms of type (1b), we may further assume that $a_{202} = 1$, $a_{001} = 1$ and $a_{022} = 0$ or 1. Let $a = a_{022}$. Since $x^2 + ay^2 + a_{102}x + a_{012}y = 0$ describes a pair of parallel lines, we must have $a_{102} \neq 0$ and we can achieve that $a_{102} = 1$ by an isomorphism of type (1b). But then $a_{012} = a$ and the coefficient $c_2(x, y)$ of z^2 now has the form

$$x^2 + ay^2 + x + ay = (x + ay)(x + ay + 1).$$

(Note that $a = 0$ or 1 so that $a^2 = a$.) Since $c_1(x, y) = 0$ must represent the two remaining parallel lines in the bundle, one finds that

$$\begin{aligned} a_{201}x^2 + a_{021}y^2 + a_{101}x + a_{011}y + 1 &= a_{201}(x + ay + \omega)(x + ay + \omega^2) \\ &= a_{201}(x^2 + ay^2 + x + ay + 1); \end{aligned}$$

In particular, $a_{201} = 1$.

We now rewrite f as a partial map in x. We find

$$\begin{aligned} f(x, y, z) =&(z^2 + z + a_{220}y^2 + a_{210}y + a_{200})x^2 \\ &+ (z^2 + z + a_{120}y^2 + a_{110}y + a_{100})x \\ &+ ay^2z^2 + ayz^2 + ay^2z + ayz + a_{020}y^2 + a_{010}y + z. \end{aligned}$$

After applying the isomorphism $(u, v) \mapsto \begin{cases} (u, v), & \text{if } u \neq 1, \\ (u, v^2), & \text{if } u = 1, \end{cases}$, if necessary, we may assume that $a_{200} = 0$ and $a_{100} \neq 0$. Note that the substitution of x by x^2 in f does not change the degree. Equating the coefficients of x^2 and x to 0 gives us again two quadrics and these must describe four parallel lines between them. Let $b = a_{210}$. Then $a_{220} = b^2$ and $z^2 + z + a_{120}y^2 + a_{110}y + a_{100} = z^2 + z + b^2y^2 + by + 1$ as before. Hence $a_{100} = 1$, $a_{110} = b$ and $a_{120} = b^2$.

Finally, rewriting f as a partial map in y, we find

$$\begin{aligned} f(x, y, z) =&(az^2 + az + b^2x^2 + b^2x + a_{020})y^2 \\ &+ (az^2 + az + bx^2 + bx + a_{010})y \\ &+ (x^2 + x)(z^2 + z) + x + z. \end{aligned}$$

Using the isomorphism $(u, v) \mapsto \begin{cases} (u, v), & \text{if } u \neq \omega, \\ (u, v^2), & \text{if } u = \omega, \end{cases}$, if necessary, we may assume that $a_{020} = 0$ and $a_{010} \neq 0$. Note that the substitution of y by y^2 in f does not change the degree. The same arguments as before yield the following form for f:

$$f(x, y, z) = (x^2 + x)(z^2 + z) + a(y^2 + y)(z^2 + z) + b(x^2 + x)(y^2 + y) + x + y + z$$

where $a, b \in \{0, 1\}$.

Clearly, $(a, b) = (0, 1)$ and $(a, b) = (1, 0)$ yield isomorphic Laguerre near-planes; we just swap the roles of x and y. $(a, b) = (0, 0)$ yields the plane in Example 2.7.1. Furthermore, in this case, the inverse of the partial map with respect to x essentially is the above map with $(a, b) = (0, 1)$, see 2.7.1. Hence $(a, b) = (0, 1)$, $(a, b) = (1, 0)$ and $(a, b) = (0, 0)$ yield isomorphic Laguerre near-planes of order 4. In summary, we have obtained the following result.

Proposition 4.6. *A Laguerre near-plane $\mathcal{L}(f)$ with f of degree 4 is isomorphic to a Laguerre near-plane $\mathcal{L}(f_1)$ or $\mathcal{L}(f_2)$ where f_1 and f_2 are the maps defined by*

$$f_1(x, y, z) = (x^2 + x)(y^2 + y) + (y^2 + y)(z^2 + z) + (x^2 + x)(z^2 + z) + x + y + z$$

and

$$f_2(x, y, z) = (x^2 + x)(z^2 + z) + x + y + z,$$

respectively.

As a consequence of Propositions 4.5 and 4.6 we obtain that if $\mathcal{L}(f)$ is a Laguerre near-plane where f has degree at least 5, then for any isomorphic model $\mathcal{L}(f')$ the describing function f' also has degree at least 5. So we may assume that the function f is in normal form in the following.

Degrees 5 and 6. For these last two remaining cases we did a computer search for functions f. In fact, we searched for all functions f in normal form, not necessarily of degree 5 or 6. Let

$$\begin{aligned} f(x, y, z) = & a_{222}x^2y^2z^2 + a_{221}x^2y^2z + a_{212}x^2yz^2 + a_{122}xy^2z^2 + \\ & a_{220}x^2y^2 + a_{022}y^2z^2 + a_{202}x^2z^2 \\ & + a_{211}x^2yz + a_{121}xy^2z + a_{112}xyz^2 \\ & + a_{210}x^2y + a_{201}x^2z + a_{120}xy^2 + a_{021}y^2z + a_{102}xz^2 + a_{012}yz^2 \\ & + a_{111}xyz \\ & + a_{110}xy + a_{011}yz + a_{101}xz + x + y + z \end{aligned}$$

be in normal form where $a_{ijk} \in \mathbb{F}_4$. Furthermore, under an isomorphism $(x, y) \mapsto (x, ry)$, $r \in \{\omega, \omega^2\}$, we can replace f by $(x, y, z) \mapsto rf(r^2x, r^2y, r^2z)$. Then the coefficients a_{222} of $X^2Y^2Z^2$ and a_{111} of XYZ are replaced by ra_{222} and ra_{111}, respectively. Hence we may further assume that

$$a_{222} = 0 \text{ or } 1 \text{ and } a_{111} = 0 \text{ or } 1 \text{ if } a_{222} = 0$$

We proceed in three steps.
Step 1: We determine all coefficients $b_{ij} \in \mathbb{F}_4$, $i = 0, 1, 2$, $j = 1, 2$, such that $(b_{22}x^2 + b_{12}x + b_{02})^3 + (b_{21}x^2 + b_{11}x + b_{01})^3 = 1$ for all $x \in \mathbb{F}_4$. We found 96 solution vectors $b = (b_{22}, b_{12}, b_{02}, b_{21}, b_{11}, b_{01})$. Note that one can restrict the search to $b_{0,2} = 0$ and $b_{01} = 1$; this yields 16 solution vectors. These vectors are

b_{22}	0	0	0	0	0	0	0	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
b_{12}	0	0	0	0	0	0	0	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2
b_{02}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
b_{21}	0	1	1	ω	ω	ω^2	ω^2	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1
b_{11}	0	ω	ω^2	1	ω	1	ω^2	1	ω^2	ω	ω	1	ω^2	ω^2	ω	1
b_{01}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

.

All other solution vectors are then obtained by multiplication by ω and ω^2 and by exchanging the roles of b_{i2} and b_{i1}, i.e., multiplying b by $\begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}$ where 0_3 and I_3 denotes the 3×3 zero and identity matrix, respectively.

Step 2: We determine all coefficients $b_{ijk} \in \mathbb{F}_4$, $i = 0,1,2$, $j = 0,1,2$, $k = 1,2$, where $b_{222} = 0,1$ and $b_{111} = 0,1$ if $b_{222} = 0$ such that $(b_{222}x^2y^2 + b_{212}x^2y + b_{202}x^2 + b_{122}xy^2 + b_{112}xy + b_{102}x + b_{022}y^2 + b_{012}y + b_{002})^3 + (b_{221}x^2y^2 + b_{211}x^2y + b_{201}x^2 + b_{121}xy^2 + b_{111}xy + b_{101}x + b_{021}y^2 + b_{011}y + b_{001})^3 = 1$ for all $x, y \in \mathbb{F}_4$.

For each $y \in \mathbb{F}_4$ we obtain an identity in x, see Lemma 2.4 and Proposition 2.5. More precisely, we find that

$$(b_{202}x^2 + b_{102}x + b_{002})^3 + (b_{201}x^2 + b_{101}x + b_{001})^3 = 1$$

$(y = 0)$,

$$\begin{aligned}&[(b_{222} + b_{212} + b_{202})x^2 + (b_{122} + b_{112} + b_{102})x + b_{022} + b_{012} + b_{002}]^3 \\ +&[(b_{221} + b_{211} + b_{201})x^2 + (b_{121} + b_{111} + b_{101})x + b_{021} + b_{011} + b_{001}]^3 = 1\end{aligned}$$

$(y = 1)$,

$$\begin{aligned}&[(b_{222}\omega^2 + b_{212}\omega + b_{202})x^2 + (b_{122}\omega^2 + b_{112}\omega + b_{102})x + b_{022}\omega^2 + b_{012}\omega + b_{002}]^3 \\ +&[(b_{221}\omega^2 + b_{211}\omega + b_{201})x^2 + (b_{121}\omega^2 + b_{111}\omega + b_{101})x + b_{021}\omega^2 + b_{011}\omega + b_{001}]^3 \\ &= 1\end{aligned}$$

$(y = \omega)$ and

$$\begin{aligned}&[(b_{222}\omega + b_{212}\omega^2 + b_{202})x^2 + (b_{122}\omega + b_{112}\omega^2 + b_{102})x + b_{022}\omega + b_{012}\omega^2 + b_{002}]^3 \\ +&[(b_{221}\omega + b_{2,1,1\omega^2} + b_{201})x^2 + (b_{121}\omega + b_{111}\omega^2 + b_{101})x + b_{021}\omega + b_{011}\omega^2 + b_{001}]^3 \\ &= 1\end{aligned}$$

$(y = \omega^2)$ for all $x \in \mathbb{F}_4$. Hence the vectors

$$\begin{aligned}b_0 =&(b_{202}, b_{102}, b_{002}, b_{201}, b_{101}, b_{000}) \\ b_1 =&(b_{222} + b_{212} + b_{202}, b_{122} + b_{112} + b_{102}, b_{022} + b_{012} + b_{002}, \\ &b_{221} + b_{211} + b_{201}, b_{121} + b_{111} + b_{101}, b_{021} + b_{011} + b_{001}) \\ b_\omega =&(b_{222}\omega^2 + b_{212}\omega + b_{202}, b_{122}\omega^2 + b_{112}\omega + b_{102}, b_{022}\omega^2 + b_{012}\omega + b_{002}, \\ &b_{221}\omega^2 + b_{211}\omega + b_{201}, b_{121}\omega^2 + b_{111}\omega + b_{101}, b_{021}\omega^2 + b_{011}\omega + b_{001}) \\ b_{\omega^2} =&(b_{222}\omega + b_{212}\omega^2 + b_{202}, b_{122}\omega + b_{112}\omega^2 + b_{102}, b_{0,2,2\omega} + b_{012}\omega^2 + b_{002}, \\ &b_{221}\omega + b_{2,1,1\omega^2} + b_{201}, b_{121}\omega + b_{111}\omega^2 + b_{101}, b_{021}\omega + b_{011}\omega^2 + b_{001})\end{aligned}$$

must appear in the list found in Step 1. Furthermore,

$$b_0 + b_1 + b_\omega + b_{\omega^2} = 0.$$

Thus, going through the list found in Step 1 one looks for triples (b_0, b_1, b_ω) such that $b_0 + b_1 + b_\omega$ is also in this list. The coefficients b_{ijk} are then determined by

$$\begin{aligned}
(b_{222}, b_{212}, b_{202}) &= (b_0^1, b_1^1, b_\omega^1) \cdot S \\
(b_{221}, b_{211}, b_{201}) &= (b_0^4, b_1^4, b_\omega^4) \cdot S \\
(b_{122}, b_{112}, b_{102}) &= (b_0^2, b_1^2, b_\omega^2) \cdot S \\
(b_{121}, b_{111}, b_{101}) &= (b_0^5, b_1^5, b_\omega^5) \cdot S \\
(b_{022}, b_{012}, b_{002}) &= (b_0^3, b_1^3, b_\omega^3) \cdot S \\
(b_{021}, b_{011}, b_{001}) &= (b_0^6, b_1^6, b_\omega^6) \cdot S
\end{aligned}$$

where b_c^i denotes the ith entry of b_c, $c = 0, 1, \omega$, and

$$S = \begin{pmatrix} \omega^2 & \omega & 1 \\ \omega & \omega^2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

A total of 4056 solution vectors $(b_{ijk})_{ijk}$ were found.

Step 3: We determine all coefficients $a_{ijk} \in \mathbb{F}_4$ of a Laguerre near-plane describing map in normal form where a_{222} equals 0 or 1 and a_{111} equals 0 or 1 if $a_{222} = 0$. Note that a_{222} and a_{111} appear as b_{222} and b_{111} in each of the three identities associated with the three partial maps so that we can directly use the list found in Step 2. Looking at the partial map with respect to z we see that $a_{ijk} = b_{ijk}$ for $i, j = 0, 1, 2$ and $k = 1, 2$. For the partial map with respect to y we now have $a_{ijk} = b_{ikj}$ for $i, k = 0, 1, 2$ and $j = 1, 2$. Finally, for the partial map with respect to x we obtain $a_{ijk} = b_{jki}$ for $j, k = 0, 1, 2$ and $i = 1, 2$. Hence we search through the list found in step 2 for triples of vectors $b^1 = (b^1_{ijk})$, $b^2 = (b^2_{ijk})$ and $b^3 = (b^3_{ijk})$ that show the following identities

$$\begin{aligned}
b^1_{222} &= b^2_{222} = b^3_{222} & \quad b^1_{211} &= b^2_{121} = b^3_{112} \\
b^1_{221} &= b^2_{212} = b^3_{122} & \quad b^1_{121} &= b^2_{211} = b^3_{121} \\
b^1_{212} &= b^2_{221} = b^3_{212} & \quad b^1_{112} &= b^2_{112} = b^3_{211} \\
b^1_{122} &= b^2_{212} = b^3_{221} & \quad b^1_{111} &= b^2_{111} = b^3_{111}
\end{aligned}$$

$$\begin{aligned}
b^1_{022} &= b^2_{022} & \quad b^1_{202} &= b^3_{202} & \quad b^2_{202} &= b^3_{022} \\
b^1_{021} &= b^2_{012} & \quad b^1_{201} &= b^3_{102} & \quad b^2_{201} &= b^3_{012} \\
b^1_{012} &= b^2_{021} & \quad b^1_{102} &= b^3_{201} & \quad b^2_{102} &= b^3_{021} \\
b^1_{011} &= b^2_{011} & \quad b^1_{101} &= b^3_{101} & \quad b^2_{101} &= b^3_{011}
\end{aligned}$$

Then

$$\begin{aligned}
a_{ijk} &= b^1_{ikj} \text{ for } i, k = 0, 1, 2 \text{ and } j = 1, 2, \\
a_{ij0} &= b^2_{i0j} \text{ for } i = 0, 1, 2, \text{ and } j = 1, 2.
\end{aligned}$$

Note that $a_{200} = 0$, $a_{100} = 1$ and $a_{000} = 0$ by our assumptions.

A total of 36 maps were found. Of these one is of degree 1, 21 are of degree 4 and 14 are of degree 6. The map of degree 1 is $(x, y, z) \mapsto x + y + z$. The maps f_1 and f_2 are among the 21 maps of degree 4 all of which can be transformed to either f_1 or f_2. This observation agrees with (and confirms) our previous results on maps of degrees at most 4. The 14 maps of degree 6 are listed in the table below where column i shows the coefficients of map i.

Each of these maps can be transformed to f_3 (column 1) or f_4 (column 14), see the proposition below. In fact, colums 2 to 8 transform into column 1 and columns 9 to 13 transform into column 14. For example, one can use an isomorphism of type (1c) to replace f by $(x, y, z) \mapsto f(x^2, y^2, z^2)^2$. This has the effect that each coefficient is squared so that 0 and 1 are fixed and ω and ω^2 are exchanged. Hence the coefficent vectors in columns 10, 11 and 13 can be transformed into those in columns 14, 12 and 9, respectively. The coefficent vectors in columns 9 and 13 are transformed into those in column 14 by swapping x with y and x with z, respectively.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_{222}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a_{221}	1	1	1	1	1	1	1	1	ω	ω^2	ω	ω^2	ω^2	ω
a_{212}	1	1	1	1	1	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
a_{122}	1	1	1	1	1	1	1	1	ω	ω	ω^2	ω	ω^2	ω^2
a_{220}	0	1	0	1	0	1	0	1	1	1	1	1	1	1
a_{211}	1	1	1	1	1	1	1	1	1	ω	1	1	1	ω^2
a_{202}	0	0	0	0	1	1	1	1	1	1	1	1	1	1
a_{121}	1	1	1	1	1	1	1	1	ω^2	1	1	1	ω	1
a_{112}	1	1	1	1	1	1	1	1	1	1	ω	ω^2	1	1
a_{022}	0	0	1	1	0	0	1	1	1	1	1	1	1	1
a_{210}	0	1	0	1	0	1	0	1	ω^2	ω	ω^2	ω	ω	ω^2
a_{201}	0	0	0	0	1	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
a_{120}	0	1	0	1	0	1	0	1	ω^2	ω^2	ω	ω^2	ω	ω^2
a_{111}	1	1	1	1	1	1	1	1	ω	ω^2	ω^2	ω	ω^2	ω
a_{102}	0	0	0	0	1	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
a_{021}	0	0	1	1	0	0	1	1	ω^2	ω^2	ω	ω^2	ω	ω
a_{012}	0	0	1	1	0	0	1	1	ω^2	ω^2	ω	ω^2	ω	ω
a_{200}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_{110}	0	1	0	1	0	1	0	1	ω	ω^2	ω	ω^2	ω^2	ω
a_{101}	0	0	0	0	1	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
a_{020}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_{011}	0	0	1	1	0	0	1	1	ω	ω	ω^2	ω	ω^2	ω^2
a_{002}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_{100}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a_{010}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a_{001}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a_{000}	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Proposition 4.7. *A Laguerre near-plane $\mathcal{L}(f)$ with f of degree > 4 is isomorphic to a Laguerre near-plane $\mathcal{L}(f_3)$ or $\mathcal{L}(f_4)$ where f_3 and f_4 are the maps defined by*

$$f_3(x,y,z) = (x^2+x)(y^2+y)(z^2+z) + x + y + z$$

and

$$\begin{aligned} f_4(x,y,z) =&(x^2+\omega^2x)(y^2+\omega y)(z^2+\omega z) + (x^2+\omega^2x)(y^2+\omega^2y) \\ &+ (x^2+\omega^2x)(z^2+\omega^2z) + (y^2+\omega y)(z^2+\omega z) + x + y + z, \end{aligned}$$

respectively. There are no Laguerre near-planes $\mathcal{L}(f)$ where f has degree 5.

Proof. Examining the 36 solutions one finds that either all coefficients a_{222}, a_{221}, a_{212}, a_{122}, a_{211}, a_{121}, a_{112} and a_{111} are non-zero or they are all zero. It is clear that under an isomorphism of type (1a), (1b) (1c), (3) or (4), this property is preserved. From Corollary 2.6 it follows that under an isomorphism of type (2) these coefficients are permuted among themselves and perhaps squared so that they remain all non-zero or all zero. Since for a map of degree 5 some of these coefficients would have to be zero and some others would have to be non-zero, the above argument shows that there are no Laguerre near-plane describing maps of degree 5. Furthermore, if f has degree 6 it must be tranformed into one of the maps of degree 6 found in Step 3. This proves the proposition. □

So far we have established that a Laguerre near-plane of order 4 is isomorphic to one of the Laguerre near-planes $\mathcal{L}(f_i)$, $i = 0,1,2,3,4$. In [10] we investigated the automorphism groups $\Gamma(f_i)$ of the Laguerre near-planes $\mathcal{L}(f_i)$ and gave characterisations of some of these planes in terms of their automorphism groups. We found that $\Gamma(f_i)$ has order $2^{10}\cdot 3^2$, $2^{10}\cdot 3$, 2^9, $2^7\cdot 3$ and 2^7 for $i = 0,1,2,3,4$, respectively. Hence the Laguerre near-planes $\mathcal{L}(f_i)$, $i = 0,1,2,3,4$, are mutually non-isomorphic. In summary we obtain the following classification result.

Proposition 4.8. *A Laguerre near-plane of order 4 is isomorphic to percisely one of the planes $\mathcal{L}(f_i)$, $i = 0,1,2,3,4$.*

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