Integral trees with diameters 4, 6 and 8*

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Abstract

In this paper, some new families of integral trees with diameters 4, 6 and 8 are given. All these classes are infinite. They are different from those in the existing literature. We also prove that the problem of finding integral trees of diameters 4, 6 and 8 is equivalent to the problem of solving Pell's diophantine equations. The discovery of these integral trees is a new contribution to the search for such trees.

I. Introduction

All graphs considered here are simple. For a graph G, let $V(G)$ denote the vertex set of G and $E(G)$ the edge set. All other notation and terminology can be found in [1–3].

The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974. A graph G is called *integral* if all the zeros of the characteristic polynomial $P(G,x)$ are integers. The 23rd open problem of reference [4] is about trees with purely integral eigenvalues. All integral trees with diameters less than 4 are given in [4, 7]. Results on integral trees with diameters 4, 5, 6 and 8 can be found in [4–14].

Various families of integral balanced trees were studied in [4–7,13,14]. A tree T is called *balanced* if the vertices at the same distance from the center of T have the same degree. According to the parity of the diameter of a tree, balanced trees split into two families. We shall code a balanced tree of diameter $2k$ by the sequence $(n_k, n_{k-1}, \cdots, n_1)$ or the tree $T(n_k, n_{k-1}, \cdots, n_1)$, where n_j $(j = 1, 2, \cdots, k)$ denotes the number of successors of a vertex at distance $k - j$ from the center. Let the tree $K_{1,s} \bullet T(m,t)$ of diameter 4 be obtained by identifying the center w of $K_{1,s}$

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and the center v of $T(m,t)$. Let the tree $K_{1,s} \bullet T(r,m,t)$ of diameter 6 be obtained by identifying the center w of $K_{1,s}$ and the center u of $T(r,m,t)$. Integral trees $T(r,m,t)$, $K_{1,s} \bullet T(m,t)$ and $K_{1,s} \bullet T(r,m,t)$ were investigated in [5–7,12–14].

In this paper, some new families of integral trees with diameters 4, 6 and 8 are given. All these classes are infinite. They are different from those of [4–14]. We also prove that the problem of finding integral trees of diameters 4, 6 and 8 is equivalent to the problem of solving Pell's diophantine equations. This is a new contribution to the search for integral trees. We believe that it is useful for constructing other integral trees.

The following Lemmas 1 and 2 can be found in [1].

Lemma 1 Let $G_1 \cup G_2$ denote the union of two disjoint graphs G_1 and G_2. If $u \in V(G_1)$, $v \in V(G_2)$ and $G = G_1 \cup G_2 + uv$, then

$$P(G,x) = P(G_1,x)P(G_2,x) - P(G_1 - u, x)P(G_2 - v, x).$$

Lemma 2 Let G be a graph. If $u \in V(G), v \notin V(G)$ and $G^* = G + uv$, then

$$P(G^*,x) = xP(G,x) - P(G - u, x).$$

The following Lemmas 3 and 4 can be found in [5].

Lemma 3

1) $P(T(m,t),x) = x^{m(t-1)+1}(x^2 - t)^{m-1}[x^2 - (m+t)]$.

2) $P(T(r,m,t),x) = x^{rm(t-1)+r-1}(x^2 - t)^{r(m-1)}[x^2 - (m+t)]^{r-1} \times[x^4 - (m+t+r)x^2 + rt]$.

Lemma 4 The tree $T(r,m,t)$ of diameter 6 is integral if and only if both t and $m+t$ are perfect squares, and $x^4 - (m+t+r)x^2 + rt$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$.

Clearly the following result in [13] and [14] is a corollary of Lemma 4.

Corollary 1 The tree $T(r,m,t)$ is integral if and only if $t = k^2$, $m = n^2 + 2nk$, $r = \frac{a^2b^2}{k^2}$, where a, b, k, n are positive integers satisfying

$$(k^2 - b^2)(a^2 - k^2) = k^2(n^2 + 2nk), \quad b < k < a.$$

The following Lemmas 5, 6 and 7 can be found in [12].

Lemma 5

1) $P[K_{1,s} \bullet T(m,t),x] = x^{m(t-1)+(s-1)}(x^2 - t)^{m-1}[x^4 - (m+t+s)x^2 + st]$.

2) $P[K_{1,s} \bullet T(r,m,t),x] = x^{rm(t-1)+r+(s-1)}(x^2 - t)^{r(m-1)}[x^2 - (m+t)]^{r-1} \times[x^4 - (m+t+r+s)x^2 + rt + s(m+t)]$.

Lemma 6 The tree $K_{1,s} \bullet T(m,t)$ of diameter 4 is integral if and only if t is a perfect square, and $x^4 - (m+t+s)x^2 + st$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$.

Lemma 7 The tree $K_{1,s} \bullet T(r, m, t)$ of diameter 6 is integral if and only if t and $m+t$ are perfect squares, and $x^4 - (m+t+r+s)x^2 + rt + s(m+t)$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$.

Theorem 1 The tree $T(q, r, m, t)$ of diameter 8 is integral if and only if both t and $m+t$ are perfect squares, $x^4 - (m+t+r)x^2 + rt$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, and $x^4 - (q+m+t+r)x^2 + rt + q(m+t)$ can be factorized as $(x^2 - c^2)(x^2 - d^2)$.

Proof We assume that the vertex u is the center of the tree $T(r, m, t)$, and the vertex w is the center of the tree $T(q, r, m, t)$. Suppose that

$$G_1 = T(r, m, t), \quad G_2 = T(q-1, r, m, t).$$

By Lemma 1, we know that

$$\begin{aligned}
&P(T(q, r, m, t), x) \\
&= P(T(r, m, t), x)P(T(q-1, r, m, t), x) - P^r(T(m, t), x) \times P^{q-1}(T(r, m, t), x) \\
&= P(T(r, m, t), x)[P(T(q-1, r, m, t), x) - P^r(T(m, t), x) \times P^{q-2}(T(r, m, t), x)]
\end{aligned}$$

By induction on q, we have

$$P(T(q, r, m, t), x) = P^{q-1}(T(r, m, t), x)[P(T(1, r, m, t), x) - (q-1)P^r(T(m, t), x)],$$

where the graph $T(1, r, m, t)$ denotes a tree by joining the center of the tree $T(r, m, t)$ to a new vertex w. By Lemma 2, we have

$$P(T(q, r, m, t), x) = P^{q-1}(T(r, m, t), x)[xP(T(r, m, t), x) - qP^r(T(m, t), x)]$$

By Lemma 3, we have

$$\begin{aligned}
P(T(q, r, m, t), x) = \; & x^{qrm(t-1)+q(r-1)+1}(x^2 - t)^{qr(m-1)}[x^2 - (m+t)]^{q(r-1)} \\
& \times[x^4 - (m+t+r)x^2 + rt]^{q-1} \\
& \times[x^4 - (q+m+t+r)x^2 + rt + q(m+t)].
\end{aligned}$$

Thus, the theorem is proved.

Corollary 2 If $q = t$, then the tree $T(q, r, m, t)$ is integral if and only if all $t, m+t$ and $m+t+r$ are perfect squares, and $x^4 - (m+t+r)x^2 + rt$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$.

The following result in [13] is a corollary of Theorem 1.

Corollary 3 The tree $T(q, r, m, t)$ is integral if and only if $t = k^2$, $m = n^2 + 2nk$, $r = \frac{a^2b^2}{k^2}$,$q = \frac{c^2d^2 - a^2b^2}{(n+k)^2}$, where a, b, c, d, k, n are positive integers satisfying

$$(k^2 - b^2)(a^2 - k^2) = k^2(n^2 + 2nk),$$

$$c^2 + d^2 = (n+k)^2 + \frac{a^2b^2}{k^2} + \frac{c^2d^2 - a^2b^2}{(n+k)^2},$$

and

$$b < k < a.$$

II. Some facts in number theory

Lemma 8 If a, b, c and d are real numbers, then

$$(a^2+b^2)(c^2+d^2) = (ad+bc)^2 + (ac-bd)^2.$$

Proof It is easy to check.

The following Lemmas 9, 10, 11 and 12 can be found in [15].

Lemma 9 An odd prime p can be expressed as the sums of two squares if and only if $p \equiv 1 (\bmod 4)$. Moreover, the unordered sum of such two squares is unique.

Lemma 10 Every positive integer m can be uniquely expressed as $m = k^2 l$, where k^2 is a square, and l is 1 or can be written as the product of primes. Then, m can be written as the sums of two squares if and only if $l = 1$ or each prime divisor p of l is the form $p \equiv 1 (\bmod 4)$.

Lemma 11 If a positive integer m can be written as $m = 2^l p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$, then m can be expressed as the sums of two mutually prime squares if and only if $l = 0$ or 1, and $p_i \equiv 1 (\bmod 4)$, for $i = 1, 2, \cdots, s$. There are 2^{s-1} ways for the expression $m = a^2 + b^2$.

Lemma 12 There are infinitely many primes of the form $p \equiv 1 (\bmod 4)$.

The following Lemma 13 can be found in [2].

Lemma 13 If $x > 0$, $y > 0$, $z > 0$, $(x, y) = 1$ and $2|y$, then all solutions of the diophantine equation $x^2 + y^2 = z^2$ are given by

$$x = r^2 - s^2, \;\; y = 2rs, \;\; z = r^2 + s^2,$$

where $(r, s) = 1$, $r > s > 0$ and $2 \nmid r + s$.

Theorem 2 There exist positive integers $N = 2^l p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$, where $l = 0$ or 1, $s \geq 2$, and p_i are primes of the form $p_i \equiv 1 (\bmod 4)$, for $i = 1, 2, \cdots, s$, such that N can be expressed as

$$a^2 + b^2 = c^2 + d^2 \tag{1}$$

satisfying $a|cd$ or $b|cd$, where a, b, c and d are positive integers with $c > a, b > d$, $(a, b) = 1$ and $(c, d) = 1$. In particular, there are such N's with $N = (p_1 p_2 \cdots p_s)^2$.

Proof By Lemma 11, N can be expressed in the form (1). By Lemmas 8, 11 and 13, if $N = (p_1 p_2 \cdots p_s)^2$, then N can also be expressed in the form (1). For the proof of the existence of N satisfying all other conditions of the theorem, we simply list the following examples.

(i) For $N = 2^l p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$, we have

1) $5 \times 13 = 7^2 + 4^2 = 8^2 + 1^2$,
2) $5 \times 17 = 7^2 + 6^2 = 9^2 + 2^2$,
3) $5 \times 41 = 13^2 + 6^2 = 14^2 + 3^2$,
4) $5 \times 53 = 12^2 + 11^2 = 16^2 + 3^2$,
5) $5 \times 101 = 19^2 + 12^2 = 21^2 + 8^2$,
6) $13 \times 17 = 11^2 + 10^2 = 14^2 + 5^2$,
7) $13 \times 37 = 16^2 + 15^2 = 20^2 + 9^2$,
8) $13 \times 53 = 20^2 + 17^2 = 25^2 + 8^2$,
9) $13 \times 97 = 30^2 + 19^2 = 35^2 + 6^2$,
10) $13 \times 113 = 37^2 + 10^2 = 38^2 + 5^2$,
11) $13 \times 181 = 47^2 + 12^2 = 48^2 + 7^2$,
12) $13 \times 313 = 62^2 + 15^2 = 63^2 + 10^2$,
13) $13 \times 317 = 61^2 + 20^2 = 64^2 + 5^2$,
14) $13 \times 337 = 59^2 + 30^2 = 66^2 + 5^2$,
15) $13 \times 613 = 87^2 + 20^2 = 88^2 + 15^2$,
16) $13 \times 733 = 77^2 + 60^2 = 85^2 + 48^2$,

17) $13 \times 757 = 79^2 + 60^2 = 96^2 + 25^2$,
18) $17 \times 37 = 23^2 + 10^2 = 25^2 + 2^2$,
19) $17 \times 53 = 26^2 + 15^2 = 30^2 + 1^2$,
20) $17 \times 257 = 63^2 + 20^2 = 65^2 + 12^2$,
21) $17 \times 73 = 29^2 + 20^2 = 35^2 + 4^2$,
22) $17 \times 137 = 40^2 + 27^2 = 48^2 + 5^2$,
23) $17 \times 193 = 41^2 + 40^2 = 55^2 + 16^2$,
24) $29 \times 37 = 28^2 + 17^2 = 32^2 + 7^2$,
25) $29 \times 41 = 30^2 + 17^2 = 33^2 + 10^2$,
26) $29 \times 61 = 37^2 + 20^2 = 40^2 + 13^2$,
27) $29 \times 89 = 41^2 + 30^2 = 50^2 + 9^2$,
28) $29 \times 281 = 57^2 + 70^2 = 90^2 + 7^2$,
29) $29 \times 389 = 84^2 + 65^2 = 105^2 + 16^2$,
30) $41 \times 61 = 49^2 + 10^2 = 50^2 + 1^2$,
31) $5 \times 13 \times 17 = 24^2 + 23^2 = 32^2 + 9^2$,
32) $5 \times 13 \times 17 = 31^2 + 12^2 = 32^2 + 9^2$,
33) $5 \times 13 \times 17 = 31^2 + 12^2 = 33^2 + 4^2$,

34) $5 \times 13 \times 17 \times 37 = 167^2 + 114^2 = 194^2 + 57^2$,
35) $257 \times 65537 = 4095^2 + 272^2 = 4097^2 + 240^2$.

(ii) For $N = (p_1 p_2 \cdots p_s)^2$, we have

1) $(5 \times 13)^2 = 56^2 + 33^2 = 63^2 + 16^2$,
2) $(5 \times 29)^2 = 143^2 + 24^2 = 144^2 + 17^2$,
3) $(13 \times 17)^2 = 171^2 + 140^2 = 220^2 + 21^2$,
4) $(17 \times 37)^2 = 460^2 + 429^2 = 621^2 + 100^2$,
5) $(41 \times 61)^2 = 2301^2 + 980^2 = 2499^2 + 100^2$.

Remark 1 We found the above positive integers by checking $5p_1$, $13p_2$, $17p_3$, $29p_4$, where each prime $p_i \equiv 1 (\text{mod } 4)$, for $i = 1, 2, 3, 4$ such that $13 \leq p_1 \leq 1009$, $17 \leq p_2 \leq 1009$, $29 \leq p_3 \leq 229$ and $37 \leq p_4 \leq 557$; while other positive integers are obtained from one by one checking. In addition, we note that some of them are Fermat primes $F_n = 2^{2^n} + 1$, for $n = 1, 2, 3, 4$.

For Theorem 2, we raised the following problems, which are not only useful for the finding of integral trees but also interesting purely as problems in number theory.

Problem 1 Find all the solutions of positive integers for the diophantine equation

$$N = x^2 + y^2 = z^2 + w^2 \tag{2}$$

such that $x|zw$ or $y|zw$, where $z > x, y > w$, $(x, y) = 1$ and $(z, w) = 1$.

Problem 2 For Problem 1, we conjecture that there are infinitely many solutions for diophantine equation (2). In particular, if N is a perfect square, is it true?

Problem 3 Find all solutions for Problem 1, for special $N = (5p_1)^\alpha, (5 \times 13 \times p_1)^\alpha, (p_1 \times p_2)^\alpha, (p_1 \times p_2 \times p_3)^\alpha, \cdots$, where $\alpha = 1, 2, 3, 4, \cdots$, p_i are primes with $p_i \equiv 1 (\bmod 4)$, for $i = 1, 2, 3, \cdots$.

The motivation to raise the above problems is as follows. (i) Construct the integral trees $T(r, m, t)$ and $K_{1,s} \bullet T(m, t)$ from any positive integer solution of diophantine equation (2). (ii) Construct the integral trees $K_{1,s} \bullet T(r, m, t)$ and $T(s, r, m, t)$ from any positive integer solution of diophantine equation (2) of the second part of Problem 2 above. So, it is very important to find all solutions of diophantine equation (2).

For Problem 1 and the first part of Problem 2, we shall give an affirmative answer from the following two different ways.

On the one hand, we find infinitely many solutions of the diophantine equation (2) for Problem 1.

Theorem 3 Let k be any positive integer, and let $u > 1$ be even. Construct x, y, z and w by one of the following formulas.

(1) $x = 3u^2 - 1$, $y = 2u(u^2 - 1)$, $z = 2u^3$ and $w = u^2 - 1$.

(2) $x = 36k^2 + 6k + 1$, $y = 18k^2 + 6k$, $z = 36k^2 + 12k$ and $w = 18k^2 - 6k - 1$.

(3) $x = 36k^2 + 30k + 7$, $y = 18k^2 + 18k + 4$, $z = 36k^2 + 36k + 8$ and $w = 18k^2 + 6k - 1$.

Then $x^2 + y^2 = z^2 + w^2$ such that y divides zw, with $z > x, y > w$, $(x, y) = 1$ and $(z, w) = 1$.

Proof (1) Because u ($u > 1$) is even, $x = 3u^2 - 1$, $y = 2u(u^2 - 1)$, $z = 2u^3$ and $w = u^2 - 1$, we have that

$x^2 + y^2 = z^2 + w^2, z - x = u^2(2u - 3) + 1 > 0, z - y = 2u > 0, z - w = u^2(2u - 1) + 1 > 0, x - w = 2u^2 > 0, y - w = u[u(2u - 1) - 2] + 1 > 0$, and $y | zw$, where $(z, w) \neq 2$, $(x, y) \neq 2$.

We assume that $(z, w) = d \neq 1, 2$. Then $d | z = 2u^3$, $d | w = u^2 - 1$. We get that $d | u$. Let $u = kd$, k a positive integer. Then $w = u^2 - 1 = k^2 d^2 - 1$. Hence, $d \not| w$, This is a contradiction.

We assume that $(x, y) = m \neq 1, 2$. Then $m | x$, $d | y = 2u(u + 1)(u - 1)$. We know that there exists $m_1 \neq 1, 2$ such that $m_1 | m$. We discuss the following three cases.

Case 1 If $m_1 | u$, then $u = km_1$, where k is a positive integer. We get that $x = 3u^2 - 1 = 3k^2 m_1^2 - 1$. Hence, $m_1 \not| x$. This is a contradiction.

Case 2 If $m_1 | (u + 1)$, then $u + 1 = km_1$, where k is a positive integer. We get that $x = 3u^2 - 1 = 3k^2 m_1^2 - 6km_1 + 2$. Hence, $m_1 \not| x$. This is a contradiction.

Case 3 If $m_1 | (u - 1)$, then $u - 1 = km_1$, where k is a positive integer. We get that $x = 3u^2 - 1 = 3k^2 m_1^2 + 6km_1 + 2$. Hence, $m_1 \not| x$. This is a contradiction.

Therefore, we have that $(z, w) = 1$ and $(x, y) = 1$.

For (2) and (3), it is easy to check by this similar method.

Remark 2 In fact, we note that it is not easy to find all the solutions of positive integers of the diophantine equation (2) of Problem 1. But we also can construct different integral trees if we omit the conditions $(x, y) = 1$ and $(z, w) = 1$ for the diophantine equation (2) of Problem 1.

Theorem 4 Let k, u and q be any positive integers, and $u > 1$ be odd. Construct x, y, z and w by one of the following formulas.

(1) $x = 3u^2 - 1$, $y = 2u(u^2 - 1)$, $z = 2u^3$ and $w = u^2 - 1$.

(2) $x = 2k^2 + 2k + 2$, $y = k(k + 2)$, $z = 2k(k + 2)$ and $w = k^2 - 2k - 2$.

(3) $x = |k^2 - 3q^2|$, $y = 4kq$, $z = k^2 + 3q^2$ and $w = 2kq$, where $0 < k < q$ or $k > 3q$, and $2|(k + q)$.

(4) $x = k^2 + 3q^2$, $y = 2kq$, $z = 4kq$ and $w = |k^2 - 3q^2|$, where $q < k < 3q$.

Then $x^2 + y^2 = z^2 + w^2$ such that y divides zw, with $z > x, y > w$,

Proof It is easy to check.

On the other hand, we only discuss the case $y|zw$ for diophantine equation (2) of Problem 1. The following cases are distinguished.

(i) If $(y, w) = 1$, let $z = cy$; then diophantine equation (2) is changed into

$$x^2 - (c^2 - 1)y^2 = w^2. \tag{3}$$

Now, we assume that $w = 1$; then diophantine equation (3) is changed into

$$x^2 - (c^2 - 1)y^2 = 1. \tag{4}$$

(ii) If $(y, w) = h$, let $y = hY$, $z = cY$ and $w = hW$; then diophantine equation (2) is changed into

$$x^2 - (c^2 - h^2)Y^2 = h^2W^2, \tag{5}$$

where $c > h$, c and h are positive integers, and $c^2 - h^2$ is not a perfect square.

Remark 3 If $z = w$, then diophantine equation (2) is changed into

$$x^2 + y^2 = 2z^2. \tag{6}$$

From [5, 7], one can find all solutions of diophantine equation (6). Also, from any positive integer solution of equation (6) one can construct integral trees $T(r, m, t)$ of diameter 6 and $K_{1,s} \bullet T(m, t)$ of diameter 4 in [5, 7, 12].

Now, by the following results for diophantine equations in Number Theory, we shall study whether there exist solutions for diophantine equations (3), (4) and (5).

The following Lemmas 14 and 15 can be found in [2].

Lemma 14 Let x_1, y_1 be the least positive solution of the diophantine equation

$$x^2 - dy^2 = 1, \tag{7}$$

where d ($d > 1$) is a positive integer that is not a perfect square. Then all the positive solutions x_k, y_k are given by

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k, \tag{8}$$

for $k = 1, 2, 3, \cdots$.

Lemma 15 Let u, v be the least positive solution of diophantine equation (7), where d ($d > 1$) is a positive integer that is not a perfect square. Then the diophantine equation

$$x^2 - dy^2 = -1 \tag{9}$$

has solutions if and only if there exist positive integer solutions s and t for the equations

$$s^2 + dt^2 = u, \quad 2st = v,$$

and moreover s and t are the least positive solution of diophantine equation (9).

The following Lemma 16 can be found in [3].

Lemma 16 Let x_1, y_1 be the least positive solution of diophantine equation (9), where d ($d > 1$) is a positive integer that is not a perfect square. Then

(1) All the positive integer solutions x_k, y_k of equation (9) are given by

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k, \tag{10}$$

for $k = 1, 3, 5, \cdots$.

(2) All the positive integer solutions x_k, y_k of equation (7) are given by

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k, \tag{11}$$

for $k = 2, 4, 6, \cdots$.

The following Lemma 17 can be found in [2].

Lemma 17

(1) If there is a solution for the diophantine equation

$$x^2 - dy^2 = m, \tag{12}$$

where m is an integer and d ($d > 1$) is a positive integer that is not a perfect square, then diophantine equation (12) has infinitely many solutions.

(2) Let x_1, y_1 be the least positive solution of diophantine equation (7). Let u_1, v_1 be the least positive solution of diophantine equation (12). Then all the positive integer solutions u_k, v_k of equation (12) are given by

$$u_k + v_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k(u_1 + v_1\sqrt{d}), \tag{13}$$

for $k = 1, 2, 3, \cdots$.

(3) Let x_1, y_1 be the least positive solution of the diophantine equation

$$x^2 - dy^2 = 4, \tag{14}$$

where d ($d > 1$) is a positive integer that is not a perfect square. Then all the positive integer solutions x_k, y_k of equation (14) are given by

$$\frac{x_k + y_k\sqrt{d}}{2} = (\frac{x_1 + y_1\sqrt{d}}{2})^k, \tag{15}$$

for $k = 1, 2, 3, \cdots$.

Theorem 5 Let x_1, y_1 be the least positive solution of diophantine equation (4), where c ($c > 1$) is a positive integer. Then all the positive integer solutions x_k, y_k of equation (4) are given by

$$x_k + y_k\sqrt{c^2 - 1} = (x_1 + y_1\sqrt{c^2 - 1})^k, \tag{16}$$

for $k = 1, 2, 3, \cdots$.

Proof This follows directly from Lemma 14.

Theorem 6 If there is a solution for the diophantine equation

$$x^2 - (c^2 - h^2)y^2 = h^2w^2, \tag{17}$$

where $c > h$, c, h and w are positive integers, and $c^2 - h^2$ is not a perfect square, let x_1, y_1 be the least positive solution of diophantine equation

$$x^2 - (c^2 - h^2)y^2 = 1. \tag{18}$$

Let u_1, v_1 be the least positive solution of diophantine equation (17). Then all the positive integer solutions x_k, y_k of equation (17) are given by

$$x_k + y_k\sqrt{c^2 - h^2} = (x_1 + y_1\sqrt{c^2 - h^2})^k(u_1 + v_1\sqrt{c^2 - h^2}), \tag{19}$$

for $k = 1, 2, 3, \cdots$.

Proof This follows directly from Lemma 17.

From Theorems 5 and 6, we shall construct infintely many Pell's diophantine equations from every identity in the list of Theorem 2.

Example 1 Note that $5 \times 13 = 7^2 + 4^2 = 8^2 + 1^2$. Then we get the Pell's diophantine equation (4), where c ($c > 1$) is a positive integer.

Choosing $c = 2, 3, 4, \cdots$, successively in the above diophantine equation (4), we get the following Pell's diophantine equations

$$x^2 - 3y^2 = 1, \quad x^2 - 8y^2 = 1, \quad x^2 - 15y^2 = 1, \quad \cdots.$$

Example 2 Note that $5 \times 17 = 7^2 + 6^2 = 9^2 + 2^2$. Then we get the Pell's diophantine equation

$$x^2 - (c^2 - 4)y^2 = 4, \tag{20}$$

where c ($c > 2$) is a positive integer and $c^2 - 4$ is not a perfect square.

Choosing $c = 3, 4, 5, \cdots$, successively in the above diophantine equation (20), we get the following Pell's diophantine equations

$$x^2 - 5y^2 = 4, \quad x^2 - 12y^2 = 4, \quad x^2 - 21y^2 = 4, \quad \cdots.$$

Example 3 Note that $13 \times 37 = 16^2 + 15^2 = 20^2 + 9^2$. Then we get the Pell's diophantine equation

$$x^2 - (c^2 - 9)y^2 = 81, \tag{21}$$

where c ($c > 3$) is a positive integer and $c^2 - 9$ is not a perfect square.

Choosing $c = 4, 6, 7, \cdots$, successively in the above diophantine equation (21), we get the following Pell's diophantine equations

$$x^2 - 7y^2 = 81, \quad x^2 - 27y^2 = 81, \quad x^2 - 40y^2 = 81, \quad \cdots.$$

The following Lemma 18 can be found in [2].

Lemma 18 Let m be a positive integer; if $2 \nmid m$ or $4|m$, then there exist positive integer solutions for the diophantine equation

$$x^2 - y^2 = m. \tag{22}$$

Remark 4 We shall give a method by the following case for finding the solutions of the diophantine equation (22). We discuss m. Suppose that $m = m_1m_2$. Let $x - y = m_1$, $x + y = m_2$ and $2|(m_1 + m_2)$. Then the solutions of the diophantine equation (22) can be found.

III. Integral trees with diameters 4, 6 and 8

In this section, we shall construct infinitely many integral trees with diameters 4, 6 and 8. These classes are different from those in [4–14].

Theorem 7 Let m_1, t_1, r_1, a, b, c and d be positive integers satisfying the following conditions:

$$m_1 + t_1 + r_1 = a^2 + b^2 = c^2 + d^2,$$

where $c > a, b > d$, $(a, b) = 1$, $(c, d) = 1$ and $a|cd$ or $b|cd$. For the tree $T(r, m, t)$ of Lemma 3, we have

(1) If $a|cd$, for any positive integer n, let $m = m_1n^2$, $m_1 = b^2 - (\frac{cd}{a})^2$, $t = t_1n^2$, $t_1 = (\frac{cd}{a})^2$, $r = r_1n^2$ and $r_1 = a^2$. Then $T(r, m, t)$ is an integral tree with diameter 6.

(2) If $b|cd$, for any positive integer n, let $m = m_1n^2$, $m_1 = a^2 - (\frac{cd}{b})^2$, $t = t_1n^2$, $t_1 = (\frac{cd}{b})^2$, $r = r_1n^2$ and $r_1 = b^2$. Then $T(r, m, t)$ is an integral tree with diameter 6.

Proof By Lemma 4 and Theorem 2, this is easy to check.

Example 4 Note that $5 \times 13 = 7^2 + 4^2 = 8^2 + 1^2$. From Theorem 7, if we let $t = 4n^2, r = 16n^2$ and $m = 45n^2$ for any positive integer n, then the tree $T(r, m, t)$ is an integral one with diameter 6. Its sprectrum is

$$Spec(T(16n^2, 45n^2, 4n^2) =$$

$$\begin{pmatrix} 0 & \pm n & \pm 2n & \pm 7n & \pm 8n \\ 2880n^6 - 720n^4 + 16n^2 - 1 & 1 & 720n^4 - 16n^2 & 16n^2 - 1 & 1 \end{pmatrix}.$$

If $n = 1$, we know that the tree $T(16, 45, 4)$ is an integral one with diameter 6, the order of which is 3617, which is much smaller than those given in [5–9].

In fact, by the same methods as in Example 4, we can construct a family of integral trees with diameter 6 from every identity in the list of Theorem 2. The family of integral trees given in Example 4 is obtained exactly from the first identity in the list of Theorem 2.

We also note that the integral tree $T(36n^2, 120n^2, 49n^2)$ can be constructed from the identity $5 \times 41 = 13^2 + 6^2 = 14^2 + 3^2$, which is Theorem 3 of [4].

Theorem 8 For the tree $T(q, r, m, t)$ of diameter 8, let the numbers m, t, r, m_1, t_1, r_1, a, b, c and d be the same as those in Theorem 7, and let $q = t$ and $m_1 + t_1 + r_1$ be a perfect square. Then $T(q, r, m, t)$ is an integral tree with diameter 8, and $T(r, m, t)$ is an integral tree with diameter 6.

Proof By Corollary 2, Lemma 4 and Theorem 2, this is easy to check.

Remark 5 From Theorem 8 and Lemma 3, it is interesting to observe that the trees $T(t, r, m, t)$, $T(r, m, t)$, $T(m, t)$ and $K_{1,t}$ are all integral ones. We call this kind of tree $T(t, r, m, t)$ *serially integral.*

Example 5 Note that $(5 \times 13)^2 = 56^2 + 33^2 = 63^2 + 16^2$. From Theorem 8, if we let $t = q = (18n)^2, m = 765n^2$ and $r = (56n)^2$ for any positive integer n, then the tree $T(q, r, m, t)$ is an integral one with diameter 8. Its sprectrum is

$$Spec(T(324n^2, 3136n^2, 765n^2, 324n^2) =$$

$$\begin{pmatrix} 0 & \pm 16n & \pm 18n & \pm 33n & \pm 63n & \pm 65n \\ a & b & c & d & b & 1 \end{pmatrix},$$

where $a = 251841623040n^8 - 777288960n^6 + 1016064n^4 - 324n^2 + 1$, $b = 324n^2 - 1$, $c = 777288960n^6 - 1016064n^4 + 1$ and $d = 1016064n^4 - 324n^2$. To our knowledge, this is the first time infinitely many integral trees with diameter 8 have been found. By setting $n = 1$, we get a minimal integral tree $T(324, 3136, 765, 324)$ with diameter 8 in this class, the order of which is 252,619,928,389.

In fact, by the same methods as in Example 5, we can construct a family of integral trees with diameter 8 from every identity in the second half of the list in Theorem 2. All these classes are different from those of [13].

Theorem 9 Let u and n be positive integers, and $u > 1$, $s = 4u^2(u^2 - 1)^2, m = 8u^4 - 6u^2 + 1$ and $t = u^4$. Then $K_{1,sn^2} \bullet T(mn^2, tn^2)$ and $K_{1,tn^2} \bullet T(mn^2, sn^2)$ are integral trees with diameter 4.

Proof By Lemma 4 and Theorem 3, this is easy to check.

Remark 6 We can also find that the tree $T(sn^2, mn^2, tn^2)$ is an integral one with diameter 6 from Theorem 3 of [14] and Theorem 3.5 of [13].

Theorem 10 Let k, q and n be positive integers; let $m = m_1n^2$,$t = t_1n^2$, $r = r_1n^2$,

(1) $m_1 = 3(6k+1)(72k^2-6k-1)$, $t_1 = 4(18k^2-6k-1)^2$, $r_1 = (18k^2+6k)^2$,
(2) $m_1 = 9(2k+1)(72k^2+42k+5)$, $t_1 = 4(18k^2+6k-1)^2$, $r_1 = (18k^2+18k+4)^2$,
(3) $m_1 = 12(2k+1)(k-1)(k+1)$, $t_1 = 4(k^2-2k-2)^2$, $r_1 = k^2(k+2)^2$, $k > 2$,
(4) $m_1 = (k^2-3q^2)^2 - (\frac{k^2+3q^2}{2})^2$, $t_1 = \frac{(k^2+3q^2)^2}{4}$, $r_1 = 16k^2q^2$, where $0 < k < q$ or $k > 3q$, and $2|(k+q)$.
(5) $m_1 = 3(k^2-q^2)(9q^2-k^2)$, $t_1 = 4(k^2-3q^2)^2$, $r_1 = 4k^2q^2$, where $q < k < 3q$.

Then $K_{1,r} \bullet T(m,t)$, $K_{1,t} \bullet T(m,r)$ and $T(r,m,t)$ are integral trees with diameter 4, 4, and 6, respectively.

Proof By Lemmas 4 and 6 and Theorems 3, 4 and 7, this is easy to check.

Theorem 11 For any positive integer n, let $t = t_1n^2$, $m = m_1n^2$, $r = r_1n^2$ and $s = s_1n^2$, and t_1, m_1, r_1 and s_1 are given in the following table. Then $K_{1,s} \bullet T(r,m,t)$ is an integral tree with diameter 6.

Proof By Lemma 7, this is easy to check.

t_1	m_1	r_1	s_1	t_1	m_1	r_1	s_1
4	672	225	616	9	9792	1225	6336
9	9792	1225	38784	16	105	144	676
16	560	729	360	16	560	729	2736
36	693	1600	1209	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Remark 7 In [13], the authors used a computer to find 182 "small" solutions t_1, m_1, r_1 and s_1, and constructed integral trees $T(s_1n^2, r_1n^2, m_1n^2, t_1n^2)$ with diameter 8. Here, we construct integral trees $K_{1,s} \bullet T(r,m,t)$ with diameter 6 by these parameters. These integral trees are different from those of [5–9,12–14].

Theorem 12 If there exists a solution for the diophantine equation (17), then we let c, h, w, u_1, v_1, x_k and y_k $(k = 1, 2, \cdots)$ be the same as those of Theorem 6. For any positive integer n, if $t = (cwn)^2$, $m = [x_k^2 - (cw)^2]n^2$ and $r = (hy_kn)^2$, then $T(r,m,t)$ is an integral tree with diameter 6, and $K_{1,r} \bullet T(m,t)$ and $K_{1,t} \bullet T(m,r)$ are integral trees with diameter 4.

Proof By Theorems 2, 5, 6 and Lemmas 4 and 6, it is not difficult to prove the theorem.

Remark 8 We know that integral trees $T(r,m,t)$ and $K_{1,r} \bullet T(m,t)$ were studied in [5–7,12–14]. Here, from Theorems 2, 6 and 11, we get infinitely many such new integral trees, which are different from those of [5–7,12–14] because the Pell's diophantine equations (3), (4) and (17) are not the same as those of [5–7,12–14].

We know that trees of diameter 4 can be formed joining the centers of r stars $K_{1,m_1}, K_{1,m_2}, \cdots, K_{1,m_r}$ to a new vertex v. Let the tree be denoted by $S(r, m_i)$ or $S(r; m_1, m_2, \cdots, m_r)$.

The following Lemma 19 can be found in [5].

Lemma 19 If $m_2 = m_3 = \cdots = m_r$, then $S(r, m_i)$ is integral if and only if m_2 is a perfect square, and $x^4-(m_1+m_2+r)x^2+m_1(r-1)+m_2(m_1+1)$ can be factorized as $(x^2-a^2)(x^2-b^2)$.

Theorem 13 Let $m_1 = a(a+1)-\epsilon$, $r = a(a+1)+\epsilon-c^2+1$, $(\epsilon \in \{-c, c\})$, $m_2 = m_3 = \cdots = m_r = c^2$, $r > 1$, $m_1 > 0$, $c > 0$, and a, c be positive integers. Then $S(r, m_i)$ is an integral tree with diameter 4.

Proof By Lemma 19, we know that

$$\begin{cases} a^2+b^2 & = m_1+m_2+r \\ a^2b^2 & = m_1(r-1)+m_2(m_1+1) \end{cases}$$

Let $m_2 = c^2$; then we get that

$$a^2+b^2 = m_1+c^2+r \tag{23}$$

$$a^2b^2 = m_1(r-1+c^2)+c^2 \tag{24}$$

By (23) and (24) and Lemma 18, we have $(ab+c)(ab-c) = (a^2+b^2-c^2-r)(r-1+c^2)$. We assume that $a^2+b^2-c^2-r = ab-\epsilon$, and $r-1+c^2 = ab+\epsilon$, where $\epsilon \in \{-c, c\}$, then we get $(a-b)^2 = 1$. Thus, let $b = a+1$. Hence, $m_1 = a(a+1)-\epsilon$, $r = a(a+1)+\epsilon-c^2+1$, $(\epsilon \in \{-c, c\})$, $m_2 = m_3 = \cdots = m_r = c^2$, $r > 1$, $m_1 > 0$, $c > 0$.

The proof is complete.

The following result in [7] is a corollary of Theorem 13.

Corollary 4 If $m_1 = a(a+1)-\epsilon$, $r = a(a+1)+\epsilon$, $(\epsilon \in \{-1, 1\})$, $m_2 = m_3 = \cdots = m_r = 1$ and a is a positive integer, then $S(r, m_i)$ is an integral tree with diameter 4.

Theorem 14 Let $m_1 = b^2+k$, $r = a^2-c^2-k(>1)$, $m_2 = m_3 = \cdots = m_r = c^2$ and k be a positive integer. If there is a positive integer solution a, b and c for the diopantine equation

$$kx^2-(k+1)y^2+z^2 = k^2+k, \tag{25}$$

then $S(r, m_i)$ is an integral tree with diameter 4.

Proof By Lemma 19, m_2 is a perfect square. We have that

$$\begin{aligned} & x^4-(m_1+m_2+r)x^2+m_1(r-1)+m_2(m_1+1) \\ & = x^4-(b^2+k+c^2+a^2-c^2-k)x^2+(b^2+k)(a^2-c^2-k-1)+c^2(b^2+k+1) \\ & = x^4-(a^2+b^2)x^2+a^2b^2+ka^2-(k+1)b^2+c^2-k^2-k \\ & = (x^2-a^2)(x^2-b^2) \end{aligned}$$

Thus, the theorem is proved.

Corollary 5 Let $k = 1$; if $m_1 = b^2+1$, $r = a^2-c^2-1$ $(r > 1)$, $m_2 = m_3 = \cdots = m_r = c^2$ and a, b and c are positive integers solutions for the diopantine equation

$$x^2-2y^2+z^2 = 2, \tag{26}$$

then $S(r, m_i)$ is an integral tree with diameter 4.

Remark 9 (1) Let $a > 2$ be a positive integer; then $x = a$, $y = a-1$ and $z = a-2$ are positive integers solutions of the diophantine equation (26).

(2) Let $z = 2$; the diophantine equation (26) is changed

$$x^2 - 2y^2 = -2. \tag{27}$$

Then all the positive integer solutions x_k, y_k of the diophantine equation (27) are given by

$$x_k + y_k\sqrt{2} = (3 + 2\sqrt{2})^k(4 + 3\sqrt{2}), \quad k = 0, 1, 2, \cdots.$$

The following result in [8] is a corollary of Theorem 14.

Corollary 6 For the diophantine equation (25), let $k = z = 1$. If a and b are positive integer solutions of the diophantine equation $x^2 - 2y^2 = 1$, let $m_1 = b^2 + 1$, $r = a^2 - 2$ (or $m_1 = a^2 - 2$, $r = b^2 + 1$) and $m_2 = m_3 = \cdots = m_r = 1$. Then $S(r, m_i)$ is an integral tree with diameter 4.

We shall give this method by the following cases for finding the solutions of the diophantine equation (25). We discuss k and z. Choosing (1) $k = z = 2$, (2) $k = 2$, $z = 3$, (3) $k = z = 3$, $\cdots$, we get the following corollary.

Corollary 7 (1) If $m_1 = b^2 + 2$, $r = a^2 - 6$ $(r > 1)$, $m_2 = m_3 = \cdots = m_r = 4$ and a, b are positive integers satisfying the diophantine equation

$$2x^2 - 3y^2 = 2, \tag{28}$$

then $S(r, m_i)$ is an integral tree with diameter 4.

(2) If $m_1 = b^2 + 2$, $r = a^2 - 11$ $(r > 1)$, $m_2 = m_3 = \cdots = m_r = 9$ and a, b are positive integers satisfying the diophantine equation

$$2x^2 - 3y^2 = -3, \tag{29}$$

then $S(r, m_i)$ is an integral tree with diameter 4.

(3) If $m_1 = b^2 + 3$, $r = a^2 - 12$ $(r > 1)$, $m_2 = m_3 = \cdots = m_r = 9$ and a, b are positive integers satisfying the diophantine equation

$$3x^2 - 4y^2 = 3, \tag{30}$$

then $S(r, m_i)$ is an integral tree with diameter 4.

For the diophantine equations (28), (29) and (30), they can be changed into the diophantine equations

$$x^2 - 6(\frac{y}{2})^2 = 1, \tag{31}$$

$$y^2 - 6(\frac{x}{3})^2 = 1, \tag{32}$$

$$x^2 - 12(\frac{y}{3})^2 = 1. \tag{33}$$

Using Lemma 16 and Theorem 5, we get the following results.

(1) All the positive solutions x_k, y_k of the diophantine equation (31) are given by

$$x_k + \frac{y_k}{2}\sqrt{6} = (5 + 2\sqrt{6})^k, \quad k = 1, 2, \cdots.$$

(2) All the positive solutions x_k, y_k of the diophantine equation (32) are given by

$$y_k + \frac{x_k}{3}\sqrt{6} = (5 + 2\sqrt{6})^k, \quad k = 1, 2, \cdots.$$

(3) All the positive solutions x_k, y_k of the diophantine equation (33) are given by

$$x_k + \frac{y_k}{3}\sqrt{12} = (7 + 2\sqrt{12})^k, \quad k = 1, 2, \cdots.$$

Theorem 15 Let a, b and c be positive integers, and ab be a perfect square. If $a > b$, $t = abc^2$ and $m = (a - b)^2c^2$, then $K_{1,t} \bullet T(m, t)$ is an integral tree with diameter 4.

Proof It is easy to check by Lemma 6.

The following result in [5] is a corollary of Theorem 15.

Corollary 8 Let a, b and c be positive integers. If $a > b$, $t = a^2b^2c^2$ and $m = (a^2 - b^2)^2c^2$, then $K_{1,t} \bullet T(m, t)$ is an integral tree with diameter 4.

Finally, we point out that the second part of Problem 2 and Problem 3 remain open. We conclude this paper by asking the following:

Problem 4 Are there any integral trees with diameters 7, 9 or 10?

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