A note on critical sets

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Abstract

First, a counterexample to a published theorem on critical sets in Latin squares is given. Second, an example is given showing that a published theorem on critical sets in F-squares cannot be strengthened.

1 Counterexample

A *Latin square*, L, of order n, is an $n \times n$ array with symbols chosen from a set N of size n, such that each element of N occurs precisely once in each row and column. So L may be thought of as a set of n^2 triples $(i, j; k)$, where the cell (i, j) of L contains the symbol k. Latin squares and their properties have been much studied. A *partial Latin square*, P, of order n, is an $n \times n$ array with entries from a set N of size n, such that each element of N occurs at most once in each row and each column. Again P may be thought of as a set of p triples which correspond to the p filled cells in P where $0 \leq p \leq n^2$. One property of interest is the size of the smallest partial Latin square that uniquely defines a Latin square of a specified order. More precisely, a subset of the entries of a Latin square, L, of order n, is called a *critical set*, C, of L if L is the only Latin square of order n containing all the entries from C and any proper subset of C is contained in at least two distinct Latin squares. A long list of papers studying critical sets can be found in Gower [5]. Cooper, Donovan and Seberry [2] define a *strongly* critical set of a Latin square, L, with symbols from the set N, to be a critical set with the property that there exists a set $\{P_1, P_2, \ldots P_f\}$ of partial Latin squares of order n with $f = n^2 - |C|$ such that:
1. $C = P_1 \subset P_2 \subset \ldots P_f = L$
2. for any i, $2 \leq i \leq f$, where $P_i = P_{i-1} \cup \{(r_{i-1}, s_{i-1}, t_{i-1})\}$, the set $P_{i-1} \cup \{(r_{i-1}, s_{i-1}, t'_{i-1})\}$ is not a partial Latin square for any $t' \in N - \{t\}$.

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	2		4
3		1	
4			1

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Figure 1: critical set C and Latin square L

1	2	3	4
2		4	
3	4	1	2
4		2	

1	2	3	4
2			3
3			2
4	3	2	1

1	2	3	4
2			
3			
4			

Figure 2: Nest of $\{x\}$ Nest of $\{y\}$ Nest of $\{x, y\}$

Fitina, Seberry and Chaudhry [3] have tried to study these critical sets more deeply. In order to do this they define the nest of a critical set. Let B be a subset of a critical set C in a Latin square, L, of order n. The *nest* of B, $N(B)$, is defined to be the union of $C - B$ and the largest set that can be uniquely filled from $C - B$. They define a set of triples, A, to be *uniquely filled* from a set $X \subseteq L$, if A is a subset of every Latin square of order n which contains X. They claim to have proved the following theorem:

Theorem 1.1 *If x and y are any two triples in C, a critical set of a Latin square L, then $N(\{x, y\}) = N(\{x\}) \cap N(\{y\})$.*

Certainly, if x and y are triples in C, then $N(\{x, y\}) \subseteq N(\{x\}) \cap N(\{y\})$ is true as they proved. Unfortunately $N(\{x, y\}) \supseteq N(\{x\}) \cap N(\{y\})$ is false. Consider the critical set C in Figure 1. L is the only Latin square that contains C. Let x be (4,4;1) and let y be (3,3;1), then Figure 2 shows $N(\{x\})$, $N(\{y\})$, and $N(\{x, y\})$.

Clearly, $N(\{x, y\}) \neq N(\{x\}) \cap N(\{y\})$ and their theorem is not true. This theorem was an important part of their paper. The rest of the paper must be read with care and some scepticism.

2 An example

Another type of square that has been studied is the Frequency square or F-square. An *F-square* of type $F = F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$ is an $n \times n$ array with symbols chosen from the set $N = \{0, 1, \ldots, v-1\}$ such that each element i occurs α_i times in each row and in each column where $n = \alpha_0 + \alpha_1 + \ldots + \alpha_{v-1}$. As in Latin squares, an F-square can be thought of as a set of triples $(i, j; k)$ where cell (i, j) contains element or symbol k. Again, as in Latin squares, researchers are interested in the size of the smallest set of entries that uniquely defines an F-square of a specified type.

				5
	1	4		
3				
4		2		
			2	

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

Figure 3: critical set C and Latin square L

2	4	1	3	**5**	1	2	3	4	**5**
5	**1**	**4**	3	2	2	**1**	**4**	5	3
3	2	5	4	1	**3**	4	5	1	2
4	5	**2**	1	3	**4**	5	**2**	3	1
1	3	5	**2**	4	5	3	1	**2**	4
1	2	3	4	**5**	2	3	1	4	**5**
2	**1**	**4**	5	3	5	**1**	**4**	3	2
3	4	1	5	2	**3**	2	5	1	4
4	3	**2**	1	1	**4**	5	**2**	5	3
5	5	3	**2**	4	1	4	3	**2**	1

Figure 4: The F-square M

Define a *critical set* of an F-square to be a non-empty subset S of an F-square, FF of type $F{=}F(n;\alpha_0,\alpha_1,\ldots,\alpha_{v-1})$ if FF is the only F-square of type F which has element k in position (i,j) for each $(i,j;k)\in S$ and every proper subset of S is contained in at least two F-squares of type F. One way to find critical sets in larger F-squares is by using products of Latin squares and products of critical sets. Let L be a Latin square of order n and let J_2 be the two by two square of all ones. Then $L\times J_2$ is the F-square of order $2n$ consisting of 4 copies of L and $C\times J_2$ is a partial square of $L\times J_2$ as is shown in the following:

L	L
L	L

C	C
C	C

Fitina and Seberry [4] proved the following theorem.

Theorem 2.1 *If C is a strongly critical set of a Latin square L, then $C\times J_2$ is a critical set of F-square-$(n;2,2,\ldots,2)$, $L\times J_2$.*

It would be nice if the word 'strongly' could be deleted from the theorem. But this is impossible as the next example shows. In Figure 3, C is a critical set of a Latin square L. This example is taken from Adams and Khodkar [1]. Now, $C\times J_2$ is embedded in the F-square $L\times J_2$ but also in the F-square M shown in Figure 4.

References

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