Concerning maximal arcs and inversive planes

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Abstract

In this article we show that a Thas 1974 maximal arc has an associated inversive plane which is isomorphic in a natural way to the inversive plane obtained from the generalised quadrangle $T_3(\mathcal{O})$ by the method of Payne and Thas, *Finite Generalized quadrangles*, (Pitman, London, 1984), 1.3.3, Proof of Theorem 5.3.1. Moreover we obtain a characterisation of the Thas 1974 maximal arcs in $\mathrm{PG}(2, q^2)$ based on two configurational properties. We show that a maximal arc of degree q satisfying two certain configurational properties in $\mathrm{PG}(2, q^2)$, where $q-1$ is a Mersenne prime, is a Thas 1974 maximal arc. This work is motivated by a paper of Barwick and O'Keefe in 1997, in which the configurational properties of Buekenhout-Metz unitals were examined.

1 Introduction

Denote by π_q a finite projective plane of order q.

In a finite projective plane π_q, a $\{k, n\}-$*arc* is a non-void proper subset of k points of π_q such that some n and no $n+1$ points of the set are collinear [4]. The number k of points in such a set is at most $qn - q + n$. A $\{qn - q + n, n\}-$arc in π_q is called a *maximal arc of degree n*. Alternatively, a maximal arc of degree n in π_q is a non-void proper subset $\mathcal{K}$ of points of π_q such that each line meets $\mathcal{K}$ in 0 or exactly n points; a line of π_q is called an *external* or *secant* line of the maximal arc $\mathcal{K}$ respectively. In π_q, a maximal arc of degree 1 is a point and a maximal arc of degree q is the complement of a line. These examples are known as the *trivial* maximal arcs; a *non-trivial* maximal arc in π_q is a maximal arc of degree n with $1 < n < q$.

An *ovoid* $\mathcal{O}$ of $\mathrm{PG}(3, q)$, $q > 2$, is a set of $q^2 + 1$ points of $\mathrm{PG}(3, q)$, no three collinear; in $\mathrm{PG}(3, 2)$, and *ovoid* is a set of 5 points, no 4 coplanar. Each plane in $\mathrm{PG}(3, q)$ intersects an ovoid in exactly 1 or $q+1$ points and is called respectively a *tangent* or *secant* plane of the ovoid. Note that the $q+1$ points of an ovoid in a secant plane form an oval (or $(q+1)-$arc). All known ovoids in $\mathrm{PG}(3,q)$ are either elliptic quadrics or Tits ovoids, however the classification of ovoids in $\mathrm{PG}(3,q)$ is complete only for all q odd and for q even with $q \leq 32$ (see [13] for a survey of known results).

A finite *inversive plane of order* q is a $3-(q^2+1, q+1, 1)$ design with the blocks of the design called *circles*. In this paper we will discuss only *finite* inversive planes. See [9] for an introduction to inversive planes. The finite *egglike* inversive planes are defined as follows. Let $\mathcal{O}$ be an ovoid in $\mathrm{PG}(3,q)$. The points of $\mathcal{O}$ together with the secant plane sections of $\mathcal{O}$ form the points and circles of a finite inversive plane $I(\mathcal{O})$. An inversive plane of order q is called *egglike* if it is isomorphic to an $I(\mathcal{O})$, for some ovoid $\mathcal{O}$ in $\mathrm{PG}(3,q)$. Corresponding to the two known infinite families of ovoids in $\mathrm{PG}(3,q)$, the elliptic quadrics and the Tits ovoids, are the two known infinite families of finite egglike inversive planes denoted M(q) and S(q) respectively.

One construction of a finite inversive plane arises from the finite generalised quadrangle $T_3(\mathcal{O})$ (see [14] for definitions and results concerning generalised quadrangles), namely if X is a point of type (i) and Y is the point (∞) in $T_3(\mathcal{O})$, then it can be shown using [14, Result 1.3.3] and [14, Proof of Theorem 5.3.1] that the incidence structure with pointset $\{X,Y\}^\perp$, circleset the set of elements $\{Z_1, Z_2, Z_3\}^{\perp\perp}$, where Z_1, Z_2, Z_3 are distinct points in $\{X,Y\}^\perp$, and with natural incidence, is an inversive plane $I_{X3}(\mathcal{O})$ of order q. In this paper, see Theorem 3.6, we obtain inversive planes isomorphic in structure to these inversive planes.

We recall the following representation of π_{q^2}, a translation plane of order q^2 with kernel containing $\mathrm{GF}(q)$, in $\mathrm{PG}(4,q)$ due to André [2] and Bruck and Bose [7, 8]. The construction is discussed in [12, Section 17.7]. We shall refer to this representation as the *André/Bruck and Bose representation of* π_{q^2} *in* $\mathrm{PG}(4,q)$.

Let Σ_∞ be a hyperplane of $\mathrm{PG}(4,q)$ and let $\mathcal{S}$ be a spread of Σ_∞ (that is, a partition of the pointset into lines). Consider the incidence structure whose *points* are the points of $\mathrm{PG}(4,q) \setminus \Sigma_\infty$, *lines* are the planes of $\mathrm{PG}(4,q)$ which do not lie in Σ_∞ but which meet Σ_∞ in a unique line of $\mathcal{S}$ and *incidence* is natural. This incidence structure is an affine translation plane and can be completed to a projective translation plane π_{q^2} of order q^2 and kernel containing $\mathrm{GF}(q)$ by adjoining the line at infinity ℓ_∞ whose points are the elements of the spread $\mathcal{S}$. The line ℓ_∞ is a translation line of π_{q^2} and we shall refer to it as the *line at infinity*; the points of ℓ_∞ will be called *points at infinity* of π_{q^2}. Note that the resulting translation plane is Desarguesian if and only if the spread $\mathcal{S}$ is regular ([8]).

In this representation, planes of $\mathrm{PG}(4,q)$ that are not contained in Σ_∞ and do not contain a line of the spread $\mathcal{S}$ (call such planes *transversal planes*) correspond to Baer subplanes (that is, subplanes of order q) of π_{q^2} secant to ℓ_∞ (that is, meeting ℓ_∞ in $q+1$ points). Consequently, any line of $\mathrm{PG}(4,q)$ that meets Σ_∞ in a unique point corresponds to a Baer subline of π_{q^2} that meets ℓ_∞ in a point. In the case π_{q^2} is the Desarguesian plane $\mathrm{PG}(2,q^2)$ it can be shown using a counting argument that the converse of these results hold.

We shall use the phrase *a subspace of* $PG(4,q)\backslash\Sigma_\infty$ to mean a subspace of $\mathrm{PG}(4,q)$ which is not contained in the hyperplane Σ_∞.

In [15] Thas proves the following construction of a maximal arcs of order q in certain translation planes π_{q^2} of order q^2 with kernel containing $\mathrm{GF}(q)$. We continue with the above notation. Let $\mathcal{O}$ be an ovoid in Σ_∞ and suppose $\mathcal{S}$ is a spread of Σ_∞ such that each line in $\mathcal{S}$ contains a unique point of $\mathcal{O}$. Let π_{q^2} be the translation plane of order q^2 with André/Bruck and Bose representation defined by the spread $\mathcal{S}$ in Σ_∞ as above. Let $\mathcal{K}$ be the set of points of $\mathrm{PG}(4,q)\backslash\Sigma_\infty$ contained in an ovoidal cone with base the ovoid $\mathcal{O}$ and vertex a fixed point X in $\mathrm{PG}(4,q)\backslash\Sigma_\infty$, that is, let $\mathcal{K}$ be the set of points on lines joining X to the ovoid, but not including the ovoid. Then $\mathcal{K}$ is a $\{q^3-q^2+q;q\}-$arc in π_{q^2}; that is, $\mathcal{K}$ is a maximal arc of degree q in π_{q^2}. Note that by definition ℓ_∞ is an external line of the maximal arc $\mathcal{K}$. By counting, for each point P not in $\mathcal{K}$ in π_{q^2}, P is incident with q^2-q+1 secant lines and q external lines of K. We shall call maximal arcs with the above construction *Thas 1974 maximal arcs.* In [15] examples of Thas 1974 maximal arcs are constructed in the Desarguesian planes of even order q^2 and in Lüneburg planes of even order. In [6] it is shown that the constructions of maximal arcs given by Thas in [15] and [16] do not exist for q odd. Note also that in [3] it is proved that for q odd non-trivial maximal arcs do not exist in the Desarguesian plane $\mathrm{PG}(2,q)$. Maximal arcs have been characterised in a number of ways, see for example Hamilton and Penttila [10] and Abatangelo and Larato [1].

2 Thas maximal arcs

Let $\mathcal{K}$ be a Thas 1974 maximal arc in a translation plane π_{q^2} of order q^2 with associated André/Bruck and Bose construction as given in Section 1 and the notation introduced there. Then $\mathcal{K}$ is defined by an ovoid $\mathcal{O}$ in Σ_∞ with the property that each element of the spread $\mathcal{S}$ of Σ_∞ contains exactly one point of $\mathcal{O}$.

Denote by $o_1,\ldots,o_{q^2+1}$ the points of the ovoid $\mathcal{O}$ in Σ_∞ which defines the maximal arc $\mathcal{K}$. Call the lines Xo_i, $i=1,\ldots,q^2+1$, in $\mathrm{PG}(4,q)$, *generator lines of* $\mathcal{K}$. Let π_{o_i} denote the unique tangent plane to $\mathcal{O}$ in Σ_∞ at the point o_i, $i=1,\ldots,q^2+1$. Recall that the unique spread line through a point o_i of $\mathcal{O}$ is contained in the tangent plane π_{o_i} at o_i, since the plane π_{o_i} contains a spread line and each spread line contains a unique (and therefore at least one) point of $\mathcal{O}$. Denote by s_i the spread line incident with the point o_i of $\mathcal{O}$.

There exist $q+1$ hyperplanes of $\mathrm{PG}(4,q)$ which contain the plane $\langle X,s_i\rangle$, for a fixed point o_i of the ovoid $\mathcal{O}$. The hyperplane $\langle X,\pi_{o_i}\rangle$ contains the unique generator line Xo_i of $\mathcal{K}$ and therefore the $q-1$ planes in $\langle X,\pi_{o_i}\rangle$ about the spread line s_i, besides π_{o_i} and the plane $\langle X,s_i\rangle$, represent $q-1$ external lines of $\mathcal{K}$ on the point at infinity of π_{q^2} represented by s_i. These $q-1$ external lines together with ℓ_∞ are all the external lines to $\mathcal{K}$ on the point at infinity of π_{q^2} represented by s_i.

The remaining q hyperplanes on $\langle X, s_i \rangle$ each intersect the ovoidal cone in an oval cone with vertex X. Let Σ be such a hyperplane, so that Σ contains q generator lines of $\mathcal{K}$ besides Xo_i. Planes about s_i in Σ, besides $\Sigma \cap \Sigma_\infty$, intersect the oval cone in q points of $\mathcal{K}$ and of these planes all, except $\langle X, s_i \rangle$, intersect the same q generator lines of $\mathcal{K}$. We have the following well known result:

Result 2.1 *Let $\mathcal{K}$ be a Thas 1974 maximal arc (of degree q) with base point X and axis line ℓ_∞ in a translation plane π_{q^2} of order q^2, where ℓ_∞ is the translation line of π_{q^2}. Let P be a point of ℓ_∞, then the secant lines of $\mathcal{K}$ incident with P besides XP are partitioned into q classes of $q-1$ lines such that the lines in a class intersect the same generator lines of $\mathcal{K}$.* □

3 Thas maximal arcs and Inversive planes

Motivated by [17] we have the following definition.

Definition 3.1 *An* **O'Nan** *configuration is a set of six distinct points with the following properties. The set contains four distinct points A, B, C, D of which no three are collinear and the remaining two points E, F are such that $\{E\} = AC \cap BD$ and $\{F\} = AB \cap CD$. The six points A, B, C, D, E, F are called the* **vertices** *of the configuration.*

Let $\mathcal{K}$ be a maximal arc in a projective plane π_q of order q. Let X be a point of $\mathcal{K}$.
We say $\mathcal{K}$ satisfies property
I_X: If $\mathcal{K}$ contains no O'Nan configurations with X a vertex.
II_X: If l is a secant line of $\mathcal{K}$ not through X, m a secant line of $\mathcal{K}$ through X meeting l in a point of $\mathcal{K}$ and $Y (Y \neq X, Y \notin l)$ a point of $\mathcal{K}$ on m, then there exists a line $l' \neq m$ incident with Y and meeting every line through X that meets l in a point of $\mathcal{K}$ and such that l' intersects each such line in a point of $\mathcal{K}$.

We now show that a Thas 1974 maximal arc $\mathcal{K}$ with base point X satisfies I_X and II_X and these properties lead to defining an inversive plane associated to the Thas maximal arc.

Let $\mathcal{K}$ be a Thas 1974 maximal arc with base point X in a translation plane π_{q^2} of order q^2 with translation line ℓ_∞. Note that π_{q^2} has an André/Bruck and Bose representation in PG$(4, q)$ with the usual notation.

Lemma 3.1 *$\mathcal{K}$ satisfies I_X.*

Proof: Suppose there exists an O'Nan configuration in $\mathcal{K}$ with X a vertex. Let m_1 and m_2 be the two secant lines of $\mathcal{K}$ not incident with X in the configuration. Let P_i be the point of intersection of m_i and ℓ_∞, $i = 1, 2$. The three points of $\mathcal{K}$ on m_1 in the O'Nan configuration correspond to three generator lines l_1, l_2, l_3 of $\mathcal{K}$ and m_2 intersects these same generator lines of $\mathcal{K}$ in the O'Nan configuration. By Result 2.1 and the comments preceding it, in the André/Bruck and Bose representation of π_{q^2}, l_1, l_2, l_3 generate a hyperplane of $\mathrm{PG}(4,q)\backslash\Sigma_\infty$ which contains the spread lines corresponding to $P_1, P_2 \in \ell_\infty$, a contradiction since Σ_∞ is the only hyperplane of $\mathrm{PG}(4,q)$ which contains two distinct elements of the spread $\mathcal{S}$. Therefore there exist no O'Nan configurations in $\mathcal{K}$ with X a vertex. □

Lemma 3.2 *$\mathcal{K}$ satisfies II_X*

Proof: Let l be a secant line of $\mathcal{K}$ not on X and let $l \cap \ell_\infty = \{P\}$. The result now follows from Result 2.1. □.

Consider the incidence structure $I'_{\mathcal{K}}$ defined by:

Points: generator lines of $\mathcal{K}$, $\mathcal{K}$ a Thas 1974 maximal arc;
Blocks: secant lines of $\mathcal{K}$ not incident with X; identifying blocks with their points and using the property II_X to eliminate repeated blocks;
Incidence: is inherited from the translation plane.

Lemma 3.3 *$I'_{\mathcal{K}}$ is a 2-$(q^2+1, q, q-1)$ design.*

Proof: There are q^2+1 generator lines of $\mathcal{K}$, corresponding to the points of the ovoid in the construction of $\mathcal{K}$, therefore the number v' of points of $I'_{\mathcal{K}}$ is q^2+1. A secant line of $\mathcal{K}$ which is not incident with X intersects q generator lines of $\mathcal{K}$, hence the number k' of points in a block is q.

By Result 2.1, each point of ℓ_∞ corresponds q distinct blocks of $I'_{\mathcal{K}}$ and since each secant line of $\mathcal{K}$ intersects ℓ_∞ in a unique point, blocks corresponding to distinct points of ℓ_∞ are distinct. Therefore the number b' of blocks of $I'_{\mathcal{K}}$ is therefore $q(q^2+1) = q^3+q$.

By Result 2.1 there exist $q-1$ secants on a point $P \in \ell_\infty$ which define the same block of $I'_{\mathcal{K}}$. A generator line of $\mathcal{K}$ has $q-1$ points of $\mathcal{K}$ besides X and there exist q^2 secant lines not containing X through each such point. Therefore in $I'_{\mathcal{K}}$, the number r' of blocks containing a point is $q^2(q-1)/(q-1) = q^2$.

Consider two generator lines of $\mathcal{K}$; they each have $q-1$ points besides X. From above a block is defined by $q-1$ distinct secant lines of $\mathcal{K}$ and therefore the number λ'_2 of blocks containing two fixed points is $(q-1)^2/(q-1) = q-1$.

It follows that $I'_{\mathcal{K}}$ is a 2-$(q^2+1, q, q-1)$ design. □

We have that a block, B_P say, of $I'_{\mathcal{K}}$ is determined by $q-1$ distinct secant lines of $\mathcal{K}$ each incident with a common point $P \in \ell_\infty$. Thus to each block B_P in $I'_{\mathcal{K}}$ is associated a unique point not incident with the block, namely, the generator line of $\mathcal{K}$ on the line XP. We use this fact to define a new incidence structure as follows.

Definition 3.4 *Let $I_{\mathcal{K}}$ be the incidence structure defined by:*

Points:	*generator lines of $\mathcal{K}$, $\mathcal{K}$ a Thas 1974 maximal arc;*
Circles:	*$\{\{$Block B_P of $I'_{\mathcal{K}}\} \cup \{$the generator line of $\mathcal{K}$ in $XP\}$; for all blocks B_P in $I'_{\mathcal{K}}\}$;*
Incidence:	*containment.*

Lemma 3.5 *The incidence structure $I_{\mathcal{K}}$ is a 3-$(q^2+1, q+1, 1)$ design, namely a finite inversive plane of order q.*

Proof: $I_{\mathcal{K}}$ has the same number of points and blocks as $I'_{\mathcal{K}}$ therefore $v = v' = q^2+1$ and $b = b' = q^3+q$. The number k of points in a block of $I_{\mathcal{K}}$ is $b = b'+1 = q+1$.

The number r of blocks on a fixed point of $I_{\mathcal{K}}$ is given by

$$r = r' + \{\text{the number of blocks of } I'_{\mathcal{K}} \text{ determined by secant lines on a fixed point of } \ell_\infty\}.$$

Using the definition of blocks of $I'_{\mathcal{K}}$ and Result 2.1 we have $r = r'+q = q^2+q$.

It remains to show that for any three distinct points of $I_{\mathcal{K}}$ there exists a unique block containing them.

Let l_1, l_2, l_3 be three distinct points of $I_{\mathcal{K}}$, that is, l_1, l_2, l_3 are three generator lines of $\mathcal{K}$ in the André/Bruck and Bose representation of the translation plane. The three lines span a unique hyperplane Σ in PG$(4,q)$ which intersects Σ_∞ in a plane containing a unique spread element; denote this spread element by P. Since the hyperplane Σ intersects the ovoidal cone of the Thas maximal arc in three generator lines, Σ contains an oval cone of generator lines. Thus the planes in Σ about P represent secant lines of $\mathcal{K}$ and define a unique block of $I_{\mathcal{K}}$ containing the points l_1, l_2, l_3.

We have shown therefore that $I_{\mathcal{K}}$ is an inversive plane. □

Theorem 3.6 *The inversive plane $I_{\mathcal{K}}$ associated to a Thas maximal arc $\mathcal{K}$ with vertex X and base ovoid $\mathcal{O}$ in a translation plane π_{q^2} is isomorphic to the inversive plane $I_{X3}(\mathcal{O})$ obtained from the generalized quadrangle $T_3(\mathcal{O})$ (defined in the PG$(4,q)$ with ovoid $\mathcal{O}$ of the construction of $\mathcal{K}$.)*
The inversive planes are egglike.

Proof: The result follows from the above discussion of the construction in PG$(4,q)$ of $I_{\mathcal{K}}$ and the known results of $T_3(\mathcal{O})$ discussed in Section 1. □

Remark: The inversive plane associated to a Buekenhout-Metz unital (see Barwick and O'Keefe [5]) is isomorphic in a natural way to the inversive planes of Theorem 3.6 defined with the same ovoid $\mathcal{O}$ of PG$(3,q)$, since both Thas maximal arcs and Buekenhout-Metz unitals are defined using an ovoidal cone in PG$(4,q)$ with base an ovoid $\mathcal{O}$ in a hyperplane of PG$(4,q)$.

4 A characterisation of Thas maximal arcs

In this section we endeavour to find a converse to the main result of Section 3. We attempt to characterise Thas 1974 Maximal Arcs with the configurational properties I_X and II_X. We weaken our hypothesis and obtain a partial converse.

4.1 A sequence of lemmata

Let $\mathcal{K}$ be a (maximal) $\{q^3 - q^2 + q; q\}$-arc in a translation plane π_{q^2} of order q^2 with kernel containing $GF(q)$. Then π_{q^2} has an André/Bruck and Bose representation in $\mathrm{PG}(4, q)$ defined by a spread in the hyperplane Σ_∞ of $\mathrm{PG}(4, q)$. Denote by ℓ_∞ the translation line of π_{q^2} corresponding to Σ_∞ and suppose ℓ_∞ is an external line of $\mathcal{K}$. Note that if $q = 2$, then $\mathcal{K}$ is a Thas 1974 maximal arc in $\mathrm{PG}(2, 4)$; hence we consider the case $q > 2$.

Let X be a fixed point of $\mathcal{K}$.
We say $\mathcal{K}$ satisfies:

- I_X: (As in Section 3.)
- II_X^*: If l is a secant line of $\mathcal{K}$ not through X and P is the point of intersection of lines l and ℓ_∞, then there exist $q-2$ further secant lines of $\mathcal{K}$ incident with P and which intersect every line through X that meets l (these intersections are all in $\mathcal{K}$).

Suppose $\mathcal{K}$ satisfies properties I_X and II_X^*.

We proceed with a sequence of lemmata and determine some properties of $\mathcal{K}$, but first we introduce some terminology.

Each line on X contains $q - 1$ points of $\mathcal{K}$ besides X; call such a set of $q - 1$ points of $\mathcal{K}$ on a line through X a **variety**. For a variety V (on a line l through X), label the point at infinity of l, namely $l \cap \ell_\infty$, by P_V. We shall sometimes refer to P_V as the **point at infinity of the variety** V.

Let l be a secant line of $\mathcal{K}$ not on X. Then l is incident with q varieties and by II_X^* there exist $q - 2$ further secants of $\mathcal{K}$ incident with these same q varieties and concurrent with l in a point P on ℓ_∞. Call such a collection of q varieties a **block** b and call the associated point P on ℓ_∞ the **point at infinity of the block** b and say b **is a block of** P.

Lemma 4.1.1 *For a point $P \in \ell_\infty$,*
(i) Distinct blocks of P are disjoint (they have no varieties in common).
(ii) P is the point at infinity of exactly q blocks.

Proof: Let P be a point on ℓ_∞.
(i) Let b_1 and b_2 be two blocks of P. Suppose b_1 and b_2 intersect in a variety V_1. Let l_1 be a secant line of $\mathcal{K}$ on P incident with b_1 (and therefore incident with every variety in b_1). Since l_1 is incident with the variety V_1 of block b_2 and l_1 passes through P, then l_1 must be one of the $q-1$ secant lines of $\mathcal{K}$ on P incident with every variety in b_2 by II_X^*. Since l_1 intersects $\mathcal{K}$ in exactly q points, blocks b_1 and b_2 must coincide. We have shown therefore that distinct blocks of P are disjoint.

(ii) There exist q^2-q secant lines of $\mathcal{K}$ on P besides the line XP. For each block of P there exist $q-1$ secant lines of $\mathcal{K}$ on P which determine that block and since by (i) distinct blocks of P are disjoint, there are exactly q blocks of P. □

Lemma 4.1.2 *Let P and Q be two points on ℓ_∞ and let b_P, b_Q be a block of P, Q respectively. Then the blocks b_P and b_Q intersect in exactly $0, 1, 2$ or q varieties.*

Proof: If $P = Q$ then by Lemma 4.1.1 b_P intersects b_Q in 0 or q varieties.

If $P \neq Q$, suppose b_P and b_Q have three varieties V_1, V_2, V_3 in common; V_i contained in line l_i, $i = 1, 2, 3$, incident with X. Let R be a point of $\mathcal{K}$ in V_1. By II_X^*, the line RP is a secant line of $\mathcal{K}$ incident with P and incident with the varieties in b_P; also the line RQ is a secant line of $\mathcal{K}$ on Q incident with b_Q. The lines RP, RQ, l_2 and l_3 are four lines of an O'Nan configuration in $\mathcal{K}$ with X as a vertex; a contradiction, as $\mathcal{K}$ satisfies I_X, thus in this case b_P and b_Q have at most 2 varieties in common. □

Lemma 4.1.3 *There are exactly q^3+q blocks in $\mathcal{K}$.*

Proof: By Lemma 4.1.1 there are q blocks corresponding to each of the q^2+1 points of ℓ_∞ and by definition (or the proof of Lemma 4.1.2) a block corresponds to a unique point at infinity. The result follows. □

Lemma 4.1.4 *Let V_1 and V_2 be two distinct varieties. There exist exactly $q-1$ blocks containing both V_1 and V_2.*

Proof: Let V_1 be on line l_1 through X and let V_2 be on line l_2 through X. Let R be a point (of $\mathcal{K}$) in V_1. The join of R to each point of V_2 defines $q-1$ secant lines m_i $(i = 1, \ldots, q-1)$ of $\mathcal{K}$, not on X and with distinct points $P_1, \ldots P_{q-1}$ (say) on the line at infinity. The line m_i defines block B_i, containing both varieties V_1 and V_2, and with point at infinity P_i (for $i = 1, \ldots, q-1$). Thus there exist at least $q-1$ blocks containing both V_1 and V_2.

By II_X^*, for each block B_i there exist $q-2$ further lines through P_i incident with both V_1 and V_2, thus giving all the possible (secant) lines joining a point of V_1 and a point of V_2. Thus there exist exactly $q-1$ blocks containing both V_1 and V_2. □

Lemma 4.1.5 *There are exactly q^2 blocks containing a given variety V.*

Proof: Let P_V be the point at infinity of a fixed variety V. For each point P on the line at infinity besides P_V, V lies in a block of P, since there exist secant lines of $\mathcal{K}$ on P incident with points in V. Therefore by Lemmata 4.1.1 and 4.1.2, V lies in exactly one block of P ($P \in \ell_\infty \backslash \{P_V\}$), with no two distinct points at infinity determining the same block containing V. Since there are q^2 points on ℓ_∞ besides P_V, there exist exactly q^2 blocks containing the variety V. □

Let $\mathcal{V}$ be the set of varieties and $\mathcal{B}$ be the set of blocks and with incidence $\mathbf{I}$ the natural containment relation. We define an incidence structure $\mathcal{I}' = (\mathcal{V}, \mathcal{B}, \mathbf{I})$.

Lemma 4.1.6 *The incidence structure $\mathcal{I}' = (\mathcal{V}, \mathcal{B}, \mathbf{I})$ is a 2-$(q^2+1, q, q-1)$ design with parameters $v' = q^2+1$, $k' = q$, $b' = q^3+q$, $r' = q^2$ and $\lambda_2' = q-1$.*

Proof: Lemmata 4.1.1, 4.1.2, 4.1.3, 4.1.3, 4.1.4 and 4.1.5 determine the parameters of $\mathcal{I}'$. □

Next we define a new incidence structure $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathbf{I})$ based on $\mathcal{I}'$. Let the set of varieties $\mathcal{V}$ of $\mathcal{I}'$ be the points $\mathcal{P}$ of $\mathcal{I}$ and let

$$\mathcal{C} = \{\{\text{varieties in a block } B_P \text{ of a point } P\} \cup \{\text{the variety contained in the line } XP\} : \text{ for all blocks } B_P \text{ of a point } P, \text{ for all points } P \text{ on } \ell_\infty\}.$$

Call the elements of $\mathcal{C}$ **circles** and call $\mathcal{C}$ the **set of circles** in $\mathcal{I}$.

There is a natural one-to-one correspondence between blocks of $\mathcal{I}'$ and circles of $\mathcal{I}$ since each block of $\mathcal{I}'$ is contained in a unique circle and conversely each circle of $\mathcal{I}$ contains a unique block of $\mathcal{I}'$.

Lemma 4.1.7 *The incidence structure $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathbf{I})$ is a 2-$(q^2+1, q+1, q+1)$ design with parameters $v = q^2+1$, $k = q+1$, $b = q^3+q$, $r = q^2+q$ and $\lambda_2 = q+1$.*

Proof: Now $v = v' = q^2+1$ and $b = b' = q^3+q$ using the definition of $\mathcal{I}$ and the natural one-to-one correspondence between circles and blocks. The number k of varieties in a circle is one more than the number k' of varieties in a block, therefore $k = k'+1 = q+1$.

For a variety V with point at infinity P, the number of circles containing V equals the number of blocks containing V plus the number of blocks of P, therefore $r = r'+q = q^2+q$.

Lastly, consider two varieties V_1 and V_2 with points at infinity P_1 and P_2 respectively. Variety V_1 lies in a unique block of P_2 and similarly variety V_2 lies in a unique block of P_1 and there are $q-1$ blocks containing both V_1 and V_2. Therefore the number λ_2 of circles containing both V_1 and V_2 is $\lambda_2 = \lambda_2' + 2 = q+1$. □

Corollary 4.1.8 *The following four statements are equivalent for the incidence structure $\mathcal{I}$:*

(i) *three distinct varieties are contained in at least one circle;*
(ii) *three distinct varieties are contained in at most one circle;*
(iii) *the design $\mathcal{I}$ has parameter $\lambda_3 = 1$;*
(iv) *the design $\mathcal{I}$ is a finite inversive plane.*

Proof: If three distinct varieties are contained in a unique circle, for any choice of three distinct varieties, then $\mathcal{I}$ is a 3-$(q^2+1, q+1, 1)$ design with the parameters given in Lemma 4.1.7 together with $\lambda_3 = 1$, that is, $\mathcal{I}$ is a finite inversive plane.

Let λ_{3_i}, $i = 1, \ldots, \binom{v}{3}$, be the number of circles containing three given (distinct) varieties V_1, V_2, V_3, for all $\binom{v}{3}$ possible choices of V_1, V_2, V_3. We now count in two ways the number of 3-flags of $\mathcal{I}$

$$\sum_{i=1}^{\binom{v}{3}} \lambda_{3_i} = b\binom{k}{3}.$$

Thus the average number $\lambda_{3_i ave}$ of circles on three varieties is given by

$$\begin{aligned}\lambda_{3_i ave} &= b\tbinom{k}{3}/\tbinom{v}{3}\\ &= 1.\end{aligned}$$

Therefore if $\lambda_{3_i} \geq 1$ for all i then $\lambda_{3_i} = 1$ for all i. Similarly if $\lambda_{3_i} \leq 1$ for all i then $\lambda_{3_i} = 1$ for all i. □

Lemma 4.1.9 *Let V_1 and V_2 be two distinct varieties in a block b_P of a point P ($P \in \ell_\infty$). Let l_i be the lines on X containing V_i, with the point at infinity of l_i denoted by Q_i, $i = 1, 2$.*

If a Baer subplane B of π_{q^2} contains P, Q_1, Q_2 and X then
either B contains no points of V_1 or V_2
or B contains the same number of points of V_1 as of V_2.

Proof: Let R be a point of V_1 in B. Since PR and l_2 are lines of B, the point $PR \cap l_2$ is a point of B. Since l_1 and l_2 lie in the block b_P of P, by II_X^*, the point $PR \cap l_2$ of B is a point on l_2 of the maximal arc $\mathcal{K}$, that is $PR \cap l_2$ is a point of V_2. The same argument holds if we suppose R is a point of V_2 in B.

It follows that either B contains no points of V_1 and V_2 or B contains the same number of points of V_1 as of V_2. □

In the following lemmata, a *linear* Baer subplane of π_{q^2} is a Baer subplane of π_{q^2} which is represented in PG$(4,q)$ by a (transversal) plane of PG$(4,q)\backslash\Sigma_\infty$ which intersects Σ_∞ in a line which is not a line of the spread $\mathcal{S}$ of Σ_∞; a *linear* Baer subline is a Baer subline of a line of π_{q^2} which is represented by a line of PG$(4,q)\backslash\Sigma_\infty$. Note that a linear Baer subplane of π_{q^2} necessarily contains ℓ_∞ as a line.

Lemma 4.1.10 *Each linear Baer subline which contains X and contains further points of $\mathcal{K}$ contains a constant number n points of $\mathcal{K}$ besides X. Moreover, $1 \leq n \leq q-1$ and n divides $q-1$.*

Proof: There exists a linear Baer subline in π_{q^2} containing X and which contains at least one further point of $\mathcal{K}$. Let l_1 be a line on X containing a linear Baer subline l_{B1}, where l_{B1} contains X and say n points of $\mathcal{K}$ besides X. Let $l_2(\neq l_1)$ be any other line containing a linear Baer subline l_{B2}, with $X \in l_{B2}$, and such that l_{B2} contains further points of $\mathcal{K}$. There exists a linear Baer subplane B of π_{q^2} containing l_{B1} and l_{B2}. Note that the line at infinity is a line of B.

Let l be a line not through X and such that l contains a point of $\mathcal{K}$ in l_{B1} and a point of $\mathcal{K}$ in l_{B2}, then l is a line of B and intersects ℓ_∞ in a point P of B. Thus, as l is a secant line of $\mathcal{K}$ on P and hence the varieties in l_1 and l_2 lie together in a block of P. Now by Lemma 4.1.9, Baer sublines l_{B1} and l_{B2} contain the same number (n) of points of $\mathcal{K}$ besides X. It follows that the linear Baer sublines of π_{q^2} which contain X contain either 0 or n further points of $\mathcal{K}$, where $1 \leq n \leq q-1$ is a fixed integer. Moreover, since each secant line of $\mathcal{K}$ incident with X contains exactly $q-1$ points of $\mathcal{K}$ distinct from X, the integer n divides $q-1$. □

Next we show that if π_{q^2} is the Desarguesian plane, then the parameter n found in Lemma 4.1.10 satisfies $n \neq 1$.

Lemma 4.1.11 *If π_{q^2} is the Desarguesian plane $PG(2,q^2)$, then each linear Baer subline of π_{q^2} which contains X contains either 0 or n further points of $\mathcal{K}$, where $1 < n \leq q-1$ is a fixed integer such that n divides $q-1$.*

Proof: If π_{q^2} is the Desarguesian plane $\mathrm{PG}(2,q^2)$, then by [3] and since π_{q^2} contains a maximal arc $\mathcal{K}$ we have that q is even. Moreover in the André/Bruck and Bose representation of π_{q^2} in $\mathrm{PG}(4,q)$ the 1−spread $\mathcal{S}$ of $\Sigma_\infty = \mathrm{PG}(3,q)$ is then a regular spread. By Lemma 4.1.10 we have that each linear Baer subline of π_{q^2} which contains X contains exactly 0 or n further points of $\mathcal{K}$, where $1 \leq n \leq q-1$ is a fixed integer and n divides $q-1$. Since $q > 2$ and q is even, we have $q \geq 4$. Consider two distinct varieties V_1 and V_2 of $\mathcal{I}'$ contained in lines ℓ_1, ℓ_2 of π_{q^2} respectively. By definition ℓ_1 and ℓ_2 intersect in the point X of $\mathcal{K}$. Denote by P_1 and P_2 the points at infinity of ℓ_1 and ℓ_2 respectively. In the André/Bruck and Bose representation, the points P_1, P_2 on ℓ_∞ correspond to distinct elements P_1^*, P_2^* of the regular spread $\mathcal{S}$ of Σ_∞. In $\mathcal{I}'$, there exist $q-1$ distinct blocks which contain the varieties V_1 and V_2; denote the points at infinity of these blocks by $Q_1, Q_2, \ldots, Q_{q-1}$. In the André/Bruck and Bose representation the points Q_i correspond to $q-1$ distinct elements of the spread $\mathcal{S}$; denote these spread elements by Q_i^*, $i = 1, \ldots, q-1$. There exist $q+1$ reguli in $\mathcal{S}$ containing P_1^* and P_2^*, therefore there exists at least one regulus $\mathcal{R}$ of lines of $\mathcal{S}$ which contains P_1^* and P_2^* but which contains no spread element Q_i^*. Let $\mathcal{R}'$ denote the opposite regulus of $\mathcal{R}$ in Σ_∞. In $\mathrm{PG}(4,q)$, the lines ℓ_1 and ℓ_2 correspond to planes ℓ_1^* and ℓ_2^* in $\mathrm{PG}(4,q)$ respectively; both planes contain X and a line P_1^*, P_2^* respectively of $\mathcal{S}$.

Since $n = 1$, the q points of $\mathcal{K}$ in ℓ_1 are represented in $\mathrm{PG}(4,q)$ by the point X and $q-1$ further points of $\ell_1^* \backslash \{P_1^*\}$ on distinct lines of ℓ_1^* through X. Similarly for the points of $\mathcal{K}$ incident with ℓ_2. In $\mathrm{PG}(4,q)$, since $q \geq 4$ there exists a line m in the opposite regulus of $\mathcal{R}$ such that the plane $B = \langle m, X \rangle$ contains a point of $\mathcal{K}$ in

ℓ_1^* besides X and a point of $\mathcal{K}$ in ℓ_2^* besides X; denote these two points of $\mathcal{K}$ in B, which are distinct from X, by Y_1^*, Y_2^* respectively. Each point Y_i^* corresponds to a point Y_i in π_{q^2} incident with the variety V_i for $i = 1, 2$. The line Y_1Y_2 is distinct from ℓ_∞ and intersects ℓ_∞ in a point Q which is necessarily the point at infinity of a block containing both varieties V_1 and V_2. In PG$(4, q)$, Q corresponds to a spread element Q^* contained in the regulus $\mathcal{R}$ of $\mathcal{S}$; a contradiction, since the regulus $\mathcal{R}$ contains no element which is the André/Bruck and Bose representation of a point of infinity of a block containing the varieties V_1 and V_2. Hence $n \neq 1$ and therefore $n > 1$ as required. □

Note that a *Mersenne prime* is a prime number which can be written in the form $2^p - 1$ for some positive integer p which is necessarily prime (see [11, Theorem 18]). There are 31 known Mersenne primes and it is conjectured that there exist an infinite number of Mersenne primes.

Corollary 4.1.12 *Suppose $\mathcal{K}$ is a maximal $\{q^3 - q^2 + q; q\}-$arc in the Desarguesian plane $PG(2, q^2)$ satisfying properties I_X and II_X^* for some point X in $\mathcal{K}$. If $q - 1$ is (Mersenne) prime, then $\mathcal{K}$ is a Thas maximal arc with base point X and axis line ℓ_∞.* □

References

[1] V. Abatangelo and B. Larato, A characterisation of Denniston's maximal arcs, *Geom. Dedicata* **30** (1989), no. 2, 197–203.

[2] J. André, Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe, *Math. Z.* **60** (1954), 156–186.

[3] S. Ball, A. Blokuis and F. Mazzocca, Maximal arcs in Desarguesian planes of odd order do not exist, *Combinatorica* **17** (1997), 31–41.

[4] A. Barlotti, Sui $\{k, n\}-$Archi di un Piano Lineare Finito, *Boll. Un. Mat. Ital.* **11** (1956), 553–556.

[5] S.G. Barwick and C.M. O'Keefe, Unitals and inversive planes, *J. Geom.* **58** (1997), 43–52.

[6] A. Blokhuis, N. Hamilton and H. Wilbrink, On the non-existence of Thas maximal arcs in odd order projective planes, *European J. Combin.* **19 (4)** (1998), 413–417.

[7] R.H. Bruck and R.C. Bose, The construction of translation planes from projective spaces, *J. Algebra* **1** (1964), 85–102.

[8] R.H. Bruck and R.C. Bose, Linear representations of projective planes in projective spaces, *J. Algebra* **4** (1966), 117–172.

[9] P. Dembowski, *Finite Geometries*, Springer, New York, 1968.

[10] N. Hamilton and T. Penttila, A characterisation of Thas maximal arcs in translation planes of square order, *J. Geom.* **51** (1994), no.1-2, 60–66.

[11] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Fourth Ed., Clarendon Press, Oxford, 1960.

[12] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.

[13] C.M. O'Keefe, Ovoids in $\mathrm{PG}(3,q)$: a survey, *Discrete Math.* **151** (1996) 175–188.

[14] S.E. Payne and J.A. Thas, *Finite generalized quadrangles*, Pitman, London (1984).

[15] J.A. Thas, Some results concerning $\{(q+1)(n-1);n\}$−arcs and $\{(q+1)(n-1)+1;n\}$−arcs in finite projective planes of order q, *J. Combin. Theory.* **A19** (1974), 228–232.

[16] J.A. Thas, Construction of maximal arcs and dual ovals in translation planes, *European J. Combin.* **1** (1980), 189–192.

[17] H. Wilbrink, A characterization of the classical unitals, *Finite geometries, Lecture Notes in Pure and Applied Math.* **82** (1983), 445–454.

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