On undirected Cayley graphs

A.V. Kelarev

Department of Mathematics
University of Tasmania
G.P.O. Box 252-37, Hobart, Tasmania 7001
Australia
Andrei.Kelarev@utas.edu.au

Abstract

We determine all periodic (and, therefore, all finite) semigroups G for which there exists a non-empty subset S of G such that the Cayley graph of G relative to S is an undirected Cayley graph.

Let G be a semigroup, and let S be a nonempty subset of G. The *Cayley graph* $\mathrm{Cay}(G, S)$ of G relative to S is defined as the graph with vertex set G and edge set $E(S)$ consisting of those ordered pairs (x, y) such that $sx = y$ for some $s \in S$. Cayley graphs of groups are significant both in group theory and in constructions of interesting graphs with nice properties. They have received serious attention in the literature (see, in particular, [1], [2], [5]). The Cayley graph of a semigroup has been introduced by Bohdan Zelinka [9].

In the investigation of the Cayley graphs of semigroups it is first of all interesting to find the analogues of natural conditions which have been used in the group case. For example, it is well known that the Cayley graph $\mathrm{Cay}(G, S)$ of a group G is symmetric or undirected if and only if $S = S^{-1}$. A graph $D = (V, E)$ is said to be *undirected* if and only if, for every $(u, v) \in E$, the edge (v, u) belongs to E, too. It is a common practice even to include the condition $S = S^{-1}$ in the definition of a Cayley graph if the undirected case is being considered. In [8] the authors characterise all vertex-transitive directed Cayley graphs produced by periodic semigroups. (A semigroup G is periodic if, for each $g \in G$, there exist positive integers m, n such that $g^m = g^{m+n}$.)

The aim of this paper is to determine all periodic (and, therefore, all finite) semigroups G for which there exists a non-empty subset S of G such that $\mathrm{Cay}(G, S)$ is an undirected Cayley graph (see Theorem 1). In order to investigate when there is some kind of a substitute for the group-theoretic inversion map, it is convenient to think of every element s of S as 'colour' of all edges (g, sg). This is in fact a multi-colouring of the graph, i.e., each edge may have several colours. We find conditions necessary and sufficient for each edge of colour s always to have a reverse edge of a fixed colour s' depending only on s (see Theorem 2).

We use standard concepts and notation of semigroup theory following [3] and [6]. If $S \subseteq G$, then the subsemigroup generated by S in G is denoted by $\langle S \rangle$. A semigroup is said to be *completely simple* if it has no proper ideals and has an idempotent minimal with respect to the natural partial order defined on the set of all idempotents by $e \leq f \Leftrightarrow ef = fe = e$.

THEOREM 1 *For a periodic semigroup G, the following conditions are equivalent:*

(i) *there exists a subset S of G such that the Cayley graph* $\mathrm{Cay}(G, S)$ *is undirected;*

(ii) *G has a completely simple subsemigroup C such that $CG = G$.*

A *right zero band* is a semigroup satisfying the identity $xy = y$.

THEOREM 2 *Let G be a finite semigroup, and let S be a subset of G. Then the following conditions are equivalent:*

(i) *there exists a one-to-one mapping $s \mapsto s'$ from S to S such that, for every edge (g, sg) of the Cayley graph, there is a reversed edge $(sg, s'sg)$;*

(ii) *$SG = G$, $\langle S \rangle$ is isomorphic to a direct product $H \times R$ of a group H and a right zero band R, and for each $g \in H$*

$$|S \cap (\{g\} \times R)| \quad = \quad |S \cap (\{g^{-1}\} \times R)|.$$

Suppose that H is a group, I and Λ are nonempty sets, and $P = [p_{\lambda i}]$ is a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in H$ for all $\lambda \in \Lambda$, $i \in I$. The *Rees matrix semigroup* $M(H; I, \Lambda; P)$ over H with *sandwich-matrix* P consists of all triples $(h; i, \lambda)$, where $i \in I$, $\lambda \in \Lambda$, and $h \in H$, with multiplication defined by the rule

$$(h_1; i_1, \lambda_1)(h_2; i_2, \lambda_2) = (h_1 p_{\lambda_1 i_2} h_2; i_1, \lambda_2),$$

for all $h_1, h_2 \in H$, $i_1, i_2 \in I$, $\lambda_1, \lambda_2 \in \Lambda$. A semigroup is *simple* if it has no proper ideals. All information on Rees matrix semigroups and completely simple semigroups required for the proofs is collected in the following Rees theorem (see [6], Theorems 3.3.1, 3.2.3 and 3.2.11).

THEOREM 3 (Rees Theorem) *Every completely simple semigroup is isomorphic to a Rees matrix semigroup $M(H; I, \Lambda; P)$ over a group H. Conversely, every semigroup $M(H; I, \Lambda; P)$ is completely simple if and only if each row and column of P contains at least one nonzero entry. All periodic simple semigroups are completely simple.*

Our proof of Theorem 1 uses the following more general technical lemma of independent interest.

LEMMA 4 *Let G be a semigroup, and let S be a subset of G, which generates a periodic subsemigroup $\langle S\rangle$. Then the following conditions are equivalent:*

(i) *the Cayley graph* $\mathrm{Cay}(G, S)$ *is undirected;*

(ii) $SG = G$*, the semigroup* $\langle S\rangle = M(H; I, \Lambda; P)$ *is completely simple and, for each* $(g; i, \lambda) \in S$ *and every* $j \in I$*, there exists* $\mu \in \Lambda$ *such that* $(p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}; j, \mu) \in S$.

PROOF. (i)⇒(ii): Suppose that the Cayley graph $\mathrm{Cay}(G, S)$ is undirected. First, we claim that the following auxiliary condition holds:

$$t \in Sst \text{ for all } s \in S, t \in G. \qquad (1)$$

Indeed, take any elements $s \in S, t \in G$. By the definition of a Cayley graph (t, st) is an edge of $\mathrm{Cay}(G, S)$. Since the graph is undirected, (st, t) is also an edge. Hence there exists $u \in S$ such that $t = ust$, and so $t \in Sst$, i.e., (1) holds. It immediately follows from (1) that $SG = G$.

The Cayley graph $\mathrm{Cay}(\langle S\rangle, S)$ is a subgraph of $\mathrm{Cay}(G, S)$ induced by the set $\langle S\rangle$ of vertices. Therefore $\mathrm{Cay}(\langle S\rangle, S)$ is undirected, too.

Take any elements x, y in $\langle S\rangle$. There exist $x_1, \ldots, x_m, y_1, \ldots, y_n \in S$ such that $x = x_1 x_2 \cdots x_m$ and $y = y_1 y_2 \ldots y_n$. Condition (1) shows that $t_i x_i x_{i+1} = x_{i+1}$ for some $t_i \in S$, where $i = 1, \ldots, m-1$. Similarly, by (1) there exists $t_m \in S$ such that $t_n x_m y_1 = y_1$. Hence

$$\begin{aligned}(t_m t_{m-1} \ldots t_2 t_1) xy &= t_m x_1 y_1 \ldots y_n \\ &= y_1 \ldots y_n = y,\end{aligned}$$

and so y belongs to the ideal generated by x in $\langle S\rangle$. This means that $\langle S\rangle$ is simple. It follows from Theorem 3 that $\langle S\rangle$ is completely simple, and so $\langle S\rangle = M(H; I, \Lambda; P)$ is a Rees matrix semigroup over a group H.

Consider any $s = (g; i, \lambda) \in S$ and $j \in I$. For any $h \in H$, put $t = (h; j, \lambda)$. Condition (1) shows that $t = ust$, for some $u \in S$, say $u = (k; \ell, \mu)$, where $\ell \in I$, $\mu \in \Lambda$. Hence we get

$$\begin{aligned}(h; j, \lambda) &= (k; \ell, \mu)(g; i, \lambda)(h; j, \lambda) \\ &= (k p_{\mu i} g p_{\lambda j} h; \ell, \lambda).\end{aligned}$$

Therefore $\ell = j$ and $h = k p_{\mu i} g p_{\lambda j} h$; whence

$$k = p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}.$$

Thus there exists μ (given by u) such that

$$u = (p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}; j, \mu) \in S.$$

This means that (ii) holds.

(ii)⇒(i): Suppose that condition (ii) holds. Consider any edge (w, xw) of the Cayley graph $\mathrm{Cay}(G, S)$, where $x \in S$, $w \in G$. Since $SG = G$, there exist $y \in S$, $v \in G$ such that $yv = w$. Since $\langle S \rangle = M(H; I, \Lambda; P)$ is a completely simple semigroup, we get $x = (g; i, \lambda)$ and $y = (h; j, \xi)$, for some $g, h \in H, i, j \in I, \lambda, \xi \in \Lambda$. By condition (ii), there exists $\mu \in \Lambda$ such that $z = (p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}; j, \mu) \in S$. Hence

$$\begin{aligned} z(xy) &= (p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}; j, \mu)(g; i, \lambda)(h; j, \xi) \\ &= (p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1} p_{\mu i} g p_{\lambda j} h; j, \xi) \\ &= (h; j, \xi) = y, \end{aligned}$$

and so $zxw = w$. Therefore (xw, w) is also an edge of $\mathrm{Cay}(G, S)$. Thus $\mathrm{Cay}(G, S)$ is undirected. □

Proof of Theorem 1. The implication (i)⇒(ii) follows from Lemma 4 immediately.

(ii)⇒(i): Suppose that $C = M(H; I, \Lambda; P)$ is a completely simple semigroup over a group H, such that $CG = G$. Take any $(g; i, \lambda) \in C$ and $j \in I$. Then $(p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}; j, \mu) \in C$, for each $\mu \in \Lambda$, and so condition (ii) of Lemma 4 is satisfied. Thus $\mathrm{Cay}(G, C)$ is undirected. □

A *band* is a semigroup entirely consisting of idempotents. A band is called a *semilattice* (*left zero band, right zero band, rectangular band*) if it satisfies the identity $xy = yx$ (respectively, $xy = x$, $xy = y$, $xyx = x$). Bands play important roles in several structure theorems of semigroup theory providing decompositions of semigroups into 'simpler' subsemigroups (see [3] and [6]). They have also found applications to the investigation of ring constructions (see the survey [7]). Since every completely simple subsemigroup of a band is a rectangular band, we get the following corollary, which shows how our main technical lemma simplifies in this interesting special case.

Corollary 5 *Let G be a band, and let S be a subset of G. Then the following conditions are equivalent:*

(i) *the Cayley graph* $\mathrm{Cay}(G, S)$ *is undirected;*

(ii) $SG = G$ *and* $\langle S \rangle$ *is a rectangular band.*

Proof of Theorem 2. (i)⇒(ii): Suppose that there exists a one-to-one mapping $s \mapsto s'$ from S to S such that, for every edge (x, sx) of the Cayley graph, $(sx, s'sx)$

is the reversed edge (sx, x) of the graph. Then the Cayley graph is undirected, and Lemma 4 shows that $\langle S\rangle = M(H; I, \Lambda; P)$ is a completely simple semigroup over a group H, and for each $(g; i, \lambda) \in S$, $j \in I$, there exists $\mu \in \Lambda$ such that $(p_{\lambda j}^{-1} g^{-1} p_{\mu i}^{-1}; j, \mu) \in S$.

Take any element $s \in S$, say $s = (h; i, \lambda)$, where $h \in H$, $i \in I$, $\lambda \in \Lambda$. There exist $h' \in H$, $i' \in I$, $\lambda' \in \Lambda$ such that $s' = (h'; i', \lambda')$. Suppose that $|I| > 1$, and consider two cases.

First, assume that $i' = i$. Then we can choose $j \in I \setminus \{i\}$, pick some $g \in H$, and put $x = (g; j, \lambda)$. By (i), $(sx, s'sx) = (sx, x)$, and so $s'sx = x$. Hence we get

$$\begin{aligned} (g; j, \lambda) &= (h'; i', \lambda')(h; i, \lambda)(g; j, \lambda) \\ &= (h' p_{\lambda' i} h p_{\lambda i'} g; i', \lambda), \end{aligned}$$

and so $j = i'$, a contradiction.

Second, assume that $i' \neq i$. Then we pick some $g \in H$, and put $x = (g; i, \lambda)$. Condition (i) implies $s'sx = x$; and so

$$\begin{aligned} (g; i, \lambda) &= (h'; i', \lambda')(h; i, \lambda)(g; i, \lambda) \\ &= (h' p_{\lambda' i} h p_{\lambda i'} g; i', \lambda). \end{aligned}$$

Therefore $i = i'$, and we get a contradiction again.

Thus both cases give us a contradiction, and therefore $|I| = 1$. We can introduce a multiplication on the set Λ by putting $\lambda\mu = \mu$ for all $\lambda, \mu \in \Lambda$, and obtain a right zero band R. Then it is known in the literature and it is not difficult to verify that the Rees matrix semigroup $\langle S\rangle = M(H; \{i\}, \Lambda; P)$ is isomorphic to the direct product $H \times R$, where the isomorphism is given by

$$(g; i, \lambda) \mapsto (g p_{\lambda i}^{-1}, \lambda),$$

for all $g \in H$, $\lambda \in \Lambda = R$.

Take arbitrary elements $s = (g, r) \in S \cap (H \times R)$ and $x = (h, v) \in H \times R$. The edge reversed to $(x, sx) = ((h, v), (gh, v))$ is $((gh, v), (h, v))$. Since it also belongs to the Cayley graph, there exists $\overline{r} \in R$ such that $(g^{-1}, \overline{r}) \in S$, because $(g^{-1}, \overline{r})(gh, v) = (h, v)$. Since the correspondence $s \mapsto s'$ given in (i) is one-to-one, it follows that the induced correspondence $r \mapsto \overline{r}$ is also a one-to-one mapping from the set $\{r \in R \mid (g, r) \in S\}$ to $\{r \in R \mid (g^{-1}, r) \in S\}$. It follows that

$$|S \cap (\{g\} \times R)| = |S \cap (\{g^{-1}\} \times R)|.$$

(ii)⇒(i): Suppose that $SG = G$, $\langle S\rangle$ is isomorphic to a direct product $H \times R$ of a group H and a right zero band R, and for each $g \in H$

$$|S \cap (\{g\} \times R)| = |S \cap (\{g^{-1}\} \times R)|.$$

Hence there exists a one-to-one correspondence $r \mapsto \overline{r}$ from the set $\{r \in R \mid (g,r) \in S\}$ to $\{r \in R \mid (g^{-1},r) \in S\}$. We define a mapping $s \mapsto s'$ from S to S by putting

$$(g,r)' = (g^{-1},\overline{r}),$$

for each $s = (g,r) \in S$. Consider any edge (x, sx) of the Cayley graph, where $x = (h,v) \in H \times R$ and $s = (g,r) \in S$. We get

$$\begin{aligned} s'sx &= (g^{-1},\overline{r})(g,r)(h,v) \\ &= (g^{-1}gh, v) \\ &= (h,v) = x. \end{aligned}$$

Hence the reversed edge (sx, x) coincides with $(sx, s'sx)$. This means that condition (i) holds. □

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