Cycles through a given arc in certain almost regular multipartite tournaments

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is regular and if $i_g(D) \leq 1$, then D is almost regular.

A c-partite tournament is an orientation of a complete c-partite graph. In 1998, Y. Guo showed, if every arc of a regular c-partite tournament is contained in a directed cycle of length three, then every arc belongs to a directed cycle of length n for each $n \in \{4, 5, \ldots, c\}$. In this paper we present the following generalization of Guo's result for $n \geq 6$.

Let $V_1, V_2, \ldots, V_c$ be the partite sets of an almost regular c-partite tournament. If $c \geq 6$ and $|V_1| = |V_2| = \ldots = |V_c| \geq 2$, then every arc of D is contained in a directed cycle of length n for each $n \in \{4, 5, \ldots, c\}$.

1. Terminology and introduction

In this paper all digraphs are finite without loops or multiple arcs. The vertex set and arc set of a digraph D is denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X, Y)$. If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length m is called an *m-cycle*. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

There are several measures of how much a digraph differs from being regular. In [7], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$$

over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leq 1$, then D is called *almost regular*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$.

It is very easy to see that every arc of a regular tournament belongs to a 3-cycle. The next example shows that this is not valid for regular multipartite tournaments in general.

Example 1.1 Let C, C' and C'' be three induced cycles of length 4 such that $C \to C' \to C'' \to C$. The resulting 6-partite tournament D_1 is 5-regular, but no arc of the three cycles C, C', and C'' is contained in a 3-cycle.

Let H, H_1, and H_2 be three copies of D_1 such that that $H \to H_1 \to H_2 \to H$. The resulting 18-partite partite tournament is 17-regular, but no arc of the cycles corresponding to the cycles C, C', and C'' is contained in a 3-cycle.

If we continue this process, we arrive at regular c-partite tournaments with arbitrary large c which contain arcs that do not belong to any 3-cycle.

However, recently the author [5] showed that every arc of a regular c-partite tournament belongs to a 4-cycle, when $c \geq 6$. We even proved the following more general result.

Theorem 1.2 (Volkmann [5]) Let $V_1, V_2, \ldots, V_c$ be the partite sets of an almost regular c-partite tournament D. If $|V_1| = |V_2| = \ldots = |V_c| = r$ and $c \geq 6$, then every arc of D is contained in a 4-cycle.

The condition $c \geq 6$ in Theorem 1.2 is in the following sense best possible. There exist 4- and 5-partite regular tournaments with $r \geq 2$ which contain arcs that do not belong to any 4-cycle.

In 1998, Y. Guo [2] proved the following generalization of Alspach's classical result [1] that every regular tournament is arc pancyclic.

Theorem 1.3 (Guo [2]) Let D be a regular c-partite tournament with $c \geq 3$. If every arc of D is contained in a 3-cycle, then every arc of D is contained in an

n-cycle for each $n \in \{4, 5, \ldots, c\}$.

Using Theorem 1.2 as the basis of induction, we present in this paper the following generalization of Theorem 1.3 for $c \geq 6$. If D is an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c| \geq 2$ and $c \geq 6$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$. This result is also a supplement to a theorem of Jacobson [3], which states that in an almost regular tournament with $c \geq 7$ vertices, every arc is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.

2. Main results

If D is a regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$, then $|V_1| = |V_2| = \ldots = |V_c| = |V(D)|/c = r$ and $d^+(x) = d^-(x) = r(c-1)/2$ for every vertex x of D. The next lemma is immediate.

Lemma 2.1 If D is an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c| = r$, then

$$\frac{(c-1)r-1}{2} \leq d^+(x), d^-(x) \leq \frac{(c-1)r+1}{2}$$

for every vertex x of D.

It may be noted that an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c| = r$ is regular if and only if c is odd or c and r are even.

Theorem 2.2 Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. If $c \geq 6$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.

Proof. We prove the theorem by induction on n. For $n = 4$ the result follows from Theorems 1.2. Now let e be an arbitrary arc of D and assume that e is contained in an n-cycle $C = a_n a_1 a_2 \ldots a_{n-1} a_n$ with $e = a_n a_1$ and $4 \leq n < c$. Suppose that $e = a_n a_1$ is not contained in any $(n+1)$-cycle.

Firstly, we observe that $N^+(v) - V(C) \neq \emptyset$ for each $v \in V(C) = \{a_1, a_2, \ldots, a_n\}$, because otherwise Lemma 2.1 yields the contradiction

$$n = |V(C)| \geq d^+(v) + 2 \geq \frac{(c-1)r-1}{2} + 2 > c.$$

Analogously, one can show that $N^-(v) - V(C) \neq \emptyset$ for each $v \in V(C)$.

Next let S be the set of vertices that belong to partite sets not represented on C and define

$$X = \{x \in S | C \to x\}, \quad Y = \{y \in S | y \to C\}.$$

Assume that $X \neq \emptyset$ and let $x \in X$. If there is a vertex $w \in N^-(a_n) - V(C)$ such that $x \to w$, then $a_n a_1 a_2 \ldots a_{n-2} x w a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. If $(N^-(a_n) - V(C)) \to x$, then $|N^-(x)| \geq |N^-(a_n) - V(C)| + |V(C)| \geq |N^-(a_n)| + 2$, a contradiction to the hypothesis that $i_g(D) \leq 1$. If there exists a vertex $b \in (N^-(a_n) - V(C))$ such that $V(b) = V(x)$, then b is adjacent with all vertices of C. In the case that $N^-(b) \cap V(C) \neq \emptyset$, let $k = \max_{1 \leq i \leq n-1}\{i | a_i \to b\}$. Then $a_n a_1 \ldots a_k b a_{k+1} \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. It remains the case that $N^-(b) \cap V(C) = \emptyset$. If there is a vertex $u \in (N^-(b) - V(C)) = N^-(b)$ such that $x \to u$, then $a_n a_1 a_2 \ldots a_{n-3} x u b a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Otherwise, $N^-(b) \to x$, and we arrive at the contradiction $d^-(x) \geq d^-(b) + |V(C)|$. Altogether, we have seen that $X \neq \emptyset$ is not possible, and analogously we find that $Y \neq \emptyset$ is impossible. Consequently, from now on we shall assume that $X = Y = \emptyset$.

By the definition of S, every vertex of $V(C)$ is adjacent to every vertex of S, and from our assumption $n < c$, we deduce that $S \neq \emptyset$. Now we distinguish different cases.

Case 1. There exists a vertex $v \in S$ with $v \to a_n$. Since $Y = \emptyset$, there is a vertex $a_i \in V(C)$ such that $a_i \to v$. If $k = \max_{1 \leq i \leq n-1}\{i | a_i \to v\}$, then $a_n a_1 \ldots a_k v a_{k+1} \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $a_n \to S$.

Case 2. There exists a vertex $v \in S$ with $a_1 \to v$. Since $X = \emptyset$, there is a vertex $a_i \in V(C)$ such that $v \to a_i$. If $k = \min_{2 \leq i \leq n-1}\{i | v \to a_i\}$, then $a_n a_1 \ldots a_{k-1} v a_k \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $S \to a_1$.

Case 3. There exists a vertex $v \in S$ such that $v \to a_{n-1}$. If there is a vertex $a_i \in V(C)$ with $2 \leq i \leq n-2$ such that $a_i \to v$, then we obtain as above an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-1}\}$. Because of $S \to a_1$, we note that every vertex of $N^+(a_1)$ is adjacent to v. If there is a vertex $x \in (N^+(a_1) - V(C))$ such that $x \to v$, then $a_n a_1 x v a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Therefore we assume now that $v \to (N^+(a_1) - V(C))$. This leads to $d^+(v) \geq d^+(a_1) + 1$, and thus, because of $i_g(D) \leq 1$, it follows that $N^+(v) = N^+(a_1) \cup \{a_1\}$ and $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$. This is a contradiction, when D is regular.

It remains the case that D is not regular, and thus c even and $r \geq 3$ odd. Now let $H = N^+(a_1) - V(C)$, $Q = N^-(v) - \{a_n\}$, and $R = V(D) - (H \cup Q \cup V(v) \cup V(C))$. With respect to Lemma 2.1, we see that

$$|R| \leq cr - \left\{\frac{(c-1)r-1}{2} - (n-2) + \frac{(c-1)r-1}{2} - 1 + r + n\right\} = 0.$$

If there is an arc $x a_2$ with $x \in H$, then $a_n a_1 x a_2 a_3 \ldots a_n$ is an $(n+1)$-cycle through the arc $a_n a_1$, a contradiction.

Subcase 3.1. Let $n \geq 5$. If there is an arc xy with $x \in H$ and $y \in Q$, then $a_n a_1 x y v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, it remains the case that $(Q \cup \{a_1, a_2, v\}) \rightsquigarrow H$. Hence, since $|R| = 0$, for every $x \in H$, we conclude that $d(x, V(D) - H) \le r + n - 3$ and thus, it follows from Lemma 2.1

$$d^+_{D[H]}(x) = d^+(x) - d(x, V(D) - H) \ge \frac{(c-1)r - 1}{2} - r - n + 3.$$

This implies

$$\begin{aligned} \frac{|H|(|H|-1)}{2} &\ge |E(D[H])| = \sum_{x \in H} d^+_{D[H]}(x) \\ &\ge |H| \Big\{ \frac{(c-1)r-1}{2} - r - n + 3 \Big\}. \end{aligned} \tag{1}$$

The conditions $d^+(v) \ge d^+(a_1) + 1$, $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$, and Lemma 2.1 yield $|H| = d^+(a_1) - (n-2) = \frac{(c-1)r-1}{2} - n + 2$. Combining this with inequality (1), we obtain

$$|H| - 1 = \frac{(c-1)r-1}{2} - n + 1 \ge 2 \Big\{ \frac{(c-1)r-1}{2} - r - n + 3 \Big\}.$$

It is straightforward to verify that this inequality is equivalent with $2n \ge (c-5)r + 9$. Because of $c - 1 \ge n$ and $r \ge 3$, this leads to the contradiction $c \le 4$.

Subcase 3.2. Let $n = 4$. Because of $a_4 \to S$, it holds $S \cup \{a_1\} \subseteq N^+(a_4)$. This implies together with Lemma 2.1 that $\frac{(c-1)r+1}{2} \ge d^+(a_4) \ge |S| + 1 \ge (c-4)r + 1$, a contradiction, when $c \ge 7$. Therefore, it remains the case that $c = 6$ and $r \ge 3$. Now let $F = N^-(a_4) - V(C)$ and $L = N^+(a_3) - V(C)$. If there is a vertex $w \in F \cap L$, then $a_4a_1a_2a_3wa_4$ is a 5-cycle through a_4a_1, a contradiction. If there is an arc xy with $x \in L$ and $y \in F$, then $a_4a_1a_3xya_4$ is a 5-cycle, a contradiction. Consequently, it remains the case that $F \cap L = \emptyset$ and $F \leadsto (L \cup \{a_3, a_4\})$. According to Lemma 2.1, we obtain

$$|L| = |N^+(a_3)| - 1 \ge \frac{(c-1)r-1}{2} - 1 = \frac{5r-3}{2},$$

and thus it follows for every $x \in F$ that

$$d(V(D) - F, x) \le 6r - |F| - |L| - 2 \le \frac{7r}{2} - |F| - \frac{1}{2}.$$

This leads to

$$d^-_{D[F]}(x) = d^-(x) - d(V(D) - F, x) \ge \frac{5r-1}{2} - \frac{7r}{2} + |F| + \frac{1}{2} = |F| - r$$

for every $x \in F$. Hence, we conclude on the one hand that

$$|E(D[F])| = \sum_{x \in F} d^-_{D[F]}(x) \ge |F|(|F| - r).$$

On the other hand, since $F \cap S = \emptyset$, the subdigraph $D[F]$ is 3-partite, and the well known Theorem of Turán [4] yields

$$|E(D[F])| \le \frac{1}{3}|F|^2.$$

The last two inequalities imply $r \geq 2|F|/3$. Since $|F| = |N^-(a_4) - V(C)| \geq d^-(a_4) - 2$, we deduce from Lemma 2.1 that

$$r \geq \frac{2|F|}{3} \geq \frac{2}{3}\Big(\frac{5r-1}{2} - 2\Big) = \frac{5r}{3} - \frac{5}{3}.$$

Therefore, $2r \leq 5$, a contradiction to $r \geq 3$.

Summarizing the investigations of Case 3, we see that there remains the case that $a_{n-1} \rightarrow S$.

Case 4. There exists a vertex $v \in S$ such that $a_2 \rightarrow v$. If we consider the converse of D, then analogously to Case 3, it remains the case that $S \rightarrow a_2$.

If $C = a_n a_1 a_2 \dots a_n$ and $v \in S$, then the following three sets play an important role in our investigations

$$H = N^+(a_1) - V(C), \quad F = N^-(a_n) - V(C), \quad Q = N^-(v) - V(C).$$

Summarizing the investigations in the Cases 1 - 4, we can assume in the following, usually without saying so, that

$$\{a_{n-1}, a_n\} \rightarrow S \rightarrow \{a_1, a_2\} \leadsto H \tag{2}$$

Case 5. Let $n = 4$. Because of (2), we have $a_4 \rightarrow S$, and thus $S \cup \{a_1\} \subseteq N^+(a_4)$. This implies together with Lemma 2.1 $\frac{(c-1)r+1}{2} \geq d^+(a_4) \geq |S| + 1$, a contradiction, when $c \geq 7$ or $|S| \geq 3r$, when $c = 6$. Therefore, it remains the case that $c = 6$, $|S| = 2r$, $D[V(C)]$ is a tournament, and $D[S]$ is a bipartite tournament.

Subcase 5.1. Assume that $a_2 \rightarrow a_4$. If $a_1 \rightarrow a_3$ and $v \in S$, then $a_4 a_1 a_3 v a_2 a_4$ is a 5-cycle through $a_4 a_1$, a contradiction. Let now $a_3 \rightarrow a_1$. If there are vertices $v \in S$ and $x \in H$ such that $x \rightarrow v$, then $a_4 a_1 x v a_2 a_4$ is a 5-cycle, a contradiction. Otherwise, we have $S \rightarrow H$. If we choose $v, w \in S$ such that $v \rightarrow w$, then $N^+(a_1) = H \cup \{a_2\}$ and $N^+(v) \supseteq H \cup \{a_1, a_2, w\}$, a contradiction to $i_g(D) \leq 1$.

Subcase 5.2. Assume that $a_4 \rightarrow a_2$. Firstly, let $a_1 \rightarrow a_3$. If there are vertices $v \in S$ and $x \in F = N^-(a_4) - V(C)$ such that $v \rightarrow x$, then $a_4 a_1 a_3 v x a_4$ is a 5-cycle, a contradiction. Otherwise, we have $F \rightarrow S$. If we choose $v, w \in S$ such that $v \rightarrow w$, then we see that $N^-(a_4) = F \cup \{a_3\}$ and $N^-(w) \supseteq F \cup \{a_3, a_4, v\}$, a contradiction to $i_g(D) \leq 1$. In the remaining case that $a_3 \rightarrow a_1$, it follows from Lemma 2.1

$$\begin{aligned} 6r &= |V(D)| \geq |H| + |F| + |S| + |V(C)| - |H \cap F| \\ &\geq \frac{5r-1}{2} - 1 + \frac{5r-1}{2} - 1 + 2r + 4 - |H \cap F| \\ &= 7r + 1 - |H \cap F|. \end{aligned}$$

Consequently, $|H \cap F| \geq r + 1$ and thus, $H \cap F$ consists of at least two partite sets. If we choose $u_2, u_3 \in H \cap F$ such that $u_2 \rightarrow u_3$, then $C' = a_4 a_1 u_2 u_3 a_4$ is also a 4-cycle through $a_4 a_1$. Since $u_2 \rightarrow a_4$, the cycle C' fulfills the conditions of Subcase 5.1, and we obtain similarly a contradiction.

Altogether, we have shown in the meantime that every arc of D belongs to a 5-cycle.

Case 6. Let $n \geq 5$ and assume that there exists a vertex $v \in S$ such that $v \rightarrow a_{n-2}$. If there is a vertex $a_i \in V(C)$ with $3 \leq i \leq n-3$ such that $a_i \rightarrow v$, then we obtain, as in Case 1, an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \rightarrow \{a_1, a_2, \ldots, a_{n-2}\}$. If there is a vertex $x \in H$ such that $x \rightarrow v$, then $a_n a_1 x v a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Therefore we assume now that $v \rightarrow H$. This leads to $d^+(v) \geq d^+(a_1)$, and thus, because of $i_g(D) \leq 1$, it follows that $a_1 \rightarrow \{a_2, a_3, \ldots, a_{n-1}\}$ or $a_1 \rightarrow \{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\}$ for some $a_j \in \{a_3, a_4, \ldots, a_{n-1}\}$ and $a_j \rightarrow a_1$ or $V(a_1) = V(a_j)$.

Subcase 6.1. Assume that $a_1 \rightarrow \{a_2, a_3, \ldots, a_{n-1}\}$. If there is a vertex $x \in H$ such that $x \rightarrow a_n$, then $a_n a_1 a_3 a_4 \ldots a_{n-1} v x a_n$ is an $(n+1)$-cycle, a contradiction. Therefore, we may assume now that $a_n \rightarrow (H - V(a_n))$. If $a_{i-1} \rightarrow a_n$ for $3 \leq i \leq n-1$, then $a_n a_1 a_i a_{i+1} \ldots a_{n-1} v a_2 a_3 \ldots a_{i-1} a_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains the case that $a_n \rightarrow a_{i-1}$ or $a_{i-1} \in V(a_n)$ for $2 \leq i \leq n-1$. Let $\{a_1, a_2, \ldots, a_{n-2}\} = A \cup B$ such that $a_n \rightarrow A$ and $B \subseteq V(a_n)$. Then $N^+(a_1) = H \cup \{a_2, a_3, \ldots, a_{n-1}\}$ and $N^+(a_n) \supseteq A \cup S \cup (H - (V(a_n) - (B \cup \{a_n\})))$. This leads to

$$d^+(a_n) \geq |A| + |S| + |H| - (r - (|B| + 1)) = d^+(a_1) + |S| - r + 1. \tag{3}$$

This yields a contradiction, when D is regular or $|S| \geq 2r$. It remains the case that D is not regular and $|S| = r$, and thus $n = c - 1$, c even and $r \geq 3$ odd. Furthermore, we see that $B = \emptyset$ and so $a_n \rightarrow \{a_1, a_2, \ldots, a_{n-2}\}$. If we define $R = V(D) - (H \cup F \cup S \cup V(C))$, then by Lemma 2.1, we find that

$$|R| \leq cr - \left\{\frac{(c-1)r-1}{2} - (n-2) + \frac{(c-1)r-1}{2} - 1 + r + n\right\} = 0.$$

If there is an arc xy with $x \in H$ and $y \in F$, then $a_n a_1 a_4 \ldots a_{n-1} v x y a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, it remains the case that $(F \cup \{a_1, a_2, a_n, v\}) \rightsquigarrow H$. Hence, since $|R| = 0$, for every $x \in H$, we conclude that $d(x, V(D) - H) \leq r + c - 5$ and thus, it follows from Lemma 2.1

$$d^+_{D[H]}(x) = d^+(x) - d(x, V(D) - H) \geq \frac{(c-1)r-1}{2} - r - c + 5.$$

This implies

$$\begin{aligned} \frac{|H|(|H|-1)}{2} &\geq |E(D[H])| = \sum_{x \in H} d^+_{D[H]}(x) \\ &\geq |H| \left\{\frac{(c-1)r-1}{2} - r - c + 5\right\}. \end{aligned} \tag{4}$$

According to (3), we have $d^+(a_n) \geq d^+(a_1) + 1$ and thus, it follows from Lemma 2.1 that $|H| = d^+(a_1) - (n-2) = \frac{(c-1)r-1}{2} - c + 3$. Combining this with inequality (4), we obtain

$$|H| - 1 = \frac{(c-1)r-1}{2} - c + 2 \geq 2\left\{\frac{(c-1)r-1}{2} - r - c + 5\right\}.$$

The last inequality is equivalent with $2c \geq (c-5)r+15$. Because of $r \geq 3$, this leads to the contradiction $2c \geq 3c$.

Subcase 6.2. Assume that there exists exactly one $j \in \{a_3, a_4, \ldots, a_{n-1}\}$ such that $a_1 \to (\{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_j) = V(a_1)$. This condition implies $d^+(v) \geq d^+(a_1) + 1$. Therefore, it remains the case that D is not regular, c even and $r \geq 3$. If we define $R = V(D) - (H \cup Q \cup V(v) \cup V(C))$, then it follows from $Q = N^-(v) - \{a_{n-1}, a_n\}$ and Lemma 2.1

$$|R| \leq cr - \left\{\frac{(c-1)r-1}{2} - (n-3) + \frac{(c-1)r-1}{2} - 2 + r + n\right\} = 0.$$

Subcase 6.2.1. Let $n \geq 6$. If there is an arc xy with $x \in H$ and $y \in Q$, then $a_n a_1 x y v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains the case that $(Q \cup \{a_1, a_2, v\}) \rightsquigarrow H$. However, in this situation we obtain, analogously to Case 3, the contradiction $c \leq 6$.

Subcase 6.2.2. Let $n = 5$ and assume that $a_1 \to \{a_2, a_3\}$ and $a_4 \to a_1$ or $V(a_4) = V(a_1)$. If there is a vertex $x \in H$ such that $x \to a_5$, then $a_5 a_1 a_3 a_4 v x a_5$ is a 6-cycle, a contradiction. Therefore, we may assume that $a_5 \to (H - V(a_5))$. If $a_2 \to a_5$, then $a_5 a_1 a_3 a_4 v a_2 a_5$ is a 6-cycle, a contradiction. Hence, it remains the case that $a_5 \to a_2$ or $V(a_2) = V(a_5)$. Let $\{a_1, a_2\} = A \cup B$ such that $a_5 \to A$ and $B \subseteq V(a_5)$. Then $N^+(a_1) = H \cup \{a_2, a_3\}$ and $N^+(a_5) \supseteq A \cup S \cup (H - (V(a_5) - (B \cup \{a_5\})))$. This leads to

$$d^+(a_5) \geq |A| + |S| + |H| - (r - (|B| + 1)) = d^+(a_1) + |S| - r + 1. \tag{5}$$

This yields a contradiction, when $|S| \geq 2r$. It remains the case that D is not regular and $|S| = r$, and thus $c = 6$ and $r \geq 3$ odd. Furthermore, $D[V(C)]$ is a tournament and so $a_5 \to \{a_1, a_2\}$ and $a_4 \to a_1$. In the case that $a_5 \to a_3$, we deduce analogously to (5) the contradiction $d^+(a_5) \geq d^+(a_1) + 2$. Hence, we assume that $a_3 \to a_5$. In addition, we find that $d^+(v) \geq d^+(a_1)+1$. If we define $R = V(D)-(H \cup Q \cup S \cup V(C))$, then it follows from $Q = N^-(v) - \{a_4, a_5\}$ and Lemma 2.1

$$|R| \leq 6r - \left\{\frac{5r-1}{2} - 2 + \frac{5r-1}{2} - 2 + r + 5\right\} = 0.$$

If there is an arc xy with $x \in H$ and $y \in Q$, then $a_5 a_1 x y v a_3 a_5$ is a 6-cycle, a contradiction. Hence, it remains the case that $(Q \cup \{a_1, a_2, a_n, v\}) \rightsquigarrow H$. However, in this situation we obtain analogously to Case 3 a contradiction.

Subcase 6.2.3. Let $n = 5$ and assume that $a_1 \to \{a_2, a_4\}$ and $a_3 \to a_1$ or $V(a_3) = V(a_1)$. If there exist vertices $x, y \in H$ such that $x \to y$ and $y \to a_5$, then $a_5 a_1 a_4 v x y a_5$ is a 6-cycle, a contradiction. Let now $W = H - V(a_5)$ and $U = \{x \in W | d^-_{D[H]}(x) = 0\}$. It follows that U is a subset of one partite set and $a_5 \to (W - U)$. Since $|U| \leq r-1$, we note that $|W - U| \geq \frac{5r-1}{2} - 2 - 2(r-1) = \frac{r-1}{2} > 0$. If $a_3 \to a_5$, then $a_5 a_1 a_4 v a_2 a_3 a_5$ is a 6-cycle, a contradiction. Hence, it remains the case that $a_5 \to a_3$ or $V(a_3) = V(a_5)$. Let $\{a_1, a_3\} = A \cup B$ such that $a_5 \to A$ and $B \subseteq V(a_5)$. Then $N^+(a_1) = H \cup \{a_2, a_4\}$ and $N^+(a_5) \supseteq A \cup S \cup (H - ((V(a_5) - (B \cup \{a_5\})) \cup U))$ and therefore

$$d^+(a_5) \geq |A| + |S| + |H| - (r - (|B| + 1)) - |U| \geq d^+(a_1) + |S| - 2r + 2. \tag{6}$$

This yields a contradiction, when $|S| \geq 2r$ and thus for $c \geq 7$. It remains the case that $|S| = r$, and thus $c = 6$ and $r \geq 3$ odd. Furthermore, $D[V(C)]$ is a tournament and so $a_5 \to \{a_1, a_3\}$ and $a_3 \to a_1$. If we define $U' = (N^+(a_1) \cap N^-(a_5)) - V(C)$, then $U' \subseteq U$. Let now $J = N^-(a_5) - (U' \cup V(C))$ and $G = N^+(a_1) - (U' \cup \{a_2, a_4\})$. If there is an arc xy with $x \in G$ and $y \in J \cup U'$, then $a_5a_1a_4vxya_5$ is a 6-cycle, a contradiction. Hence, it remains that $(J \cup U' \cup \{a_1, a_2, a_5, v\}) \rightsquigarrow G$.

Suppose next that there exist vertices $b \in G$ and $w \in S$ such that $b \to w$. If $w \to a_3$, then $a_5a_1bwa_3a_4a_5$ is a 6-cycle, a contradiction. So, we can assume that $a_3 \to w$. If there is a vertex $x \in (N^-(a_5) - V(C))$ such that $w \to x$, then $a_5a_1a_2a_3wxa_5$ is a 6-cycle, a contradiction. Thus, we can assume that $(N^-(a_5) - V(C)) \to w$. Altogether, we see that $N^-(a_5) \subseteq (N^-(a_5) - V(C)) \cup \{a_2, a_4\}$ and $N^-(w) \supseteq (N^-(a_5) - V(C)) \cup \{a_3, a_4, a_5, b\}$ and this yields the contradiction $d^-(w) \geq d^-(a_5) + 2$. Consequently, it remains the case that $S \to G$. If we define $R = V(D) - (H \cup J \cup S \cup V(C))$, then, because of $|J| \geq |N^-(a_5)| - |U'| - 2 \geq \frac{5r-1}{2} - r - 1$, we obtain

$$|R| \leq 6r - \left\{\frac{5r-1}{2} - 2 + \frac{5r-1}{2} - r - 1 + r + 5\right\} = r - 1.$$

Hence, for each $x \in G$, we conclude that $d(x, V(D) - G) \leq r + 1$ and thus it follows

$$d^+_{D[G]}(x) = d^+(x) - d(x, V(D) - G) \geq \frac{5r-1}{2} - r - 1 = \frac{3r-3}{2}.$$

This implies

$$\frac{|G|(|G|-1)}{2} \geq |E(D[G])| = \sum_{x \in G} d^+_{D[G]}(x) \geq |G|\frac{3r-3}{2}. \quad (7)$$

In view of Lemma 2.1, we have $|G| = d^+(a_1) - |U'| - 2 \leq d^+(a_1) - 2 \leq \frac{5r+1}{2} - 2 = \frac{5r-3}{2}$. Combining this with inequality (7), we obtain $\frac{5r-3}{2} - 1 \geq |G| - 1 \geq 3r - 3$, and thus the contradiction $r \leq 1$.

Summarizing the investigations of Case 6, we see that there remains the case that $a_{n-2} \to S$.

Case 7. Let $n = 5$. If we consider the cycle $C^{-1} = a_1a_5a_4a_3a_2a_1 = b_5b_1b_2b_3b_4b_5$ in the converse D^{-1} of D, then $\{b_4, b_5\} \to S \to \{b_1, b_2, b_3\}$. Since this is exactly the situation of Case 6, there exists in D^{-1} a 6-cycle, containing the arc $b_5b_1 = a_1a_5$, and hence there exists in D a 6-cycle through a_5a_1.

Case 8. Let $n \geq 6$. Assume that there exists a vertex $v \in S$ such that $a_3 \to v$. If we consider the converse of D, then in view of Case 6, it remains the case that $S \to a_3$.

Case 9. Let $c > n \geq 6$. If there exist vertices $v \in S$ and $x \in H$ such that $x \to v$, then $a_na_1xva_3a_4 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, we assume now that $S \to H$. Let $v \in S$. If there exists a vertex $x \in H$ such that $x \to a_n$, then $a_na_1a_2 \ldots a_{n-2}vxa_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains the case that $(S \cup \{a_1, a_2, a_n\}) \rightsquigarrow H$.

If $a_1 \to a_i$ and $a_{i-1} \to a_n$ for $i \in \{3, 4, \ldots, n-1\}$, then the $(n+1)$-cycle $a_na_1a_i \ldots a_{n-1}va_2 \ldots a_{i-1}a_n$ yields a contradiction. Thus, if $a_1 \to a_i$ for an $i \in$

$\{3, 4, \ldots, n-1\}$, then we may assume that $a_n \to a_{i-1}$ or $V(a_i) = V(a_n)$. Let $N = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ be exactly the subset of $V(C) - \{a_2\}$ with the property that $a_1 \to N$. Then we define $A \cup B = \{a_{i_1-1}, a_{i_2-1}, \ldots, a_{i_k-1}\}$ such that $a_n \to A$ and $B \subseteq V(a_n)$. This definition and the fact that $a_n \to (H - V(a_n))$ lead to $N^+(a_1) = \{a_2\} \cup N \cup H$ and $N^+(a_n) \supseteq \{a_1\} \cup A \cup S \cup (H - (V(a_n) - (B \cup \{a_n\})))$. This implies

$$\begin{aligned} d^+(a_n) &\geq |A| + |S| + 1 + |H| - (r - (|B| + 1)) \\ &= |A| + |B| + 2 + |S| - r + d^+(a_1) - |N| - 1 \\ &= d^+(a_1) + |S| - r + 1. \end{aligned}$$

This yields a contradiction, when D is regular or $c \geq n+2$. It remains the case that D is not regular and $|S| = r$, and thus $c = n + 1 \geq 8$ even and $r \geq 3$ odd. Furthermore, $D[V(C)]$ is a tournament, $B = \emptyset$, and it follows by Lemma 2.1

$$d^+(a_n) = d^+(a_1) + 1 = \frac{(c-1)r+1}{2}. \tag{8}$$

Subcase 9.1. There exists a vertex $v \in S$ such that $v \to a_{n-3}$. If there is a vertex $a_i \in V(C)$ with $4 \leq i \leq n-4$ such that $a_i \to v$, then we obtain, as in Case 1, an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-3}\}$. If $R = V(D) - (H \cup Q \cup S \cup V(C))$, then because of $|H| = |N^+(a_1) - V(C)| \geq d^+(a_1) - (n-2)$ and $|Q| = |N^-(v) - V(C)| \geq d^-(v) - 3$, we see with respect to Lemma 2.1 that

$$|R| \leq cr - \left\{\frac{(c-1)r-1}{2} - (n-2) + \frac{(c-1)r-1}{2} - 3 + r + n\right\} = 2.$$

If there is an arc xy with $x \in H$ and $y \in Q$, then $a_n a_1 x y v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, it remains the case that $(Q \cup S \cup \{a_1, a_2, a_n\}) \rightsquigarrow H$. Hence, since $|R| \leq 2$, for every $x \in H$, we conclude that $d(x, V(D) - H) \leq n - 3 + 2 = c - 2$ and thus, it follows from Lemma 2.1

$$d^+_{D[H]}(x) = d^+(x) - d(x, V(D) - H) \geq \frac{(c-1)r-1}{2} - c + 2 = \frac{(c-1)r+3}{2} - c.$$

This implies

$$\frac{|H|(|H|-1)}{2} \geq |E(D[H])| = \sum_{x \in H} d^+_{D[H]}(x) \geq |H| \left\{\frac{(c-1)r+3}{2} - c\right\}. \tag{9}$$

Since $d^+(v) \geq |H| + (n-3) = |H| + c - 4$ and $d^+(v) \leq d^+(a_1) + 1$, we deduce from Lemma 2.1 and (8)

$$|H| \leq d^+(v) - (n-3) \leq d^+(a_1) - c + 5 = \frac{(c-1)r-1}{2} - c + 5.$$

Combining this with inequality (9), we obtain

$$\frac{(c-1)r-1}{2} - c + 4 \geq |H| - 1 \geq (c-1)r + 3 - 2c.$$

This inequality is equivalent with $2c \geq (c-1)r - 1$. Since $r \geq 3$, this leads to the contradiction $c \leq 4$.

Subcase 9.2. Finally, we assume that $a_{n-3} \to S$. If there is a vertex $w \in H \cap F$, then $a_n a_1 a_2 \ldots a_{n-2} v w a_1$ is a $(n+1)$-cycle, a contradiction. Now let $H \cap F = \emptyset$, and let $R = V(D) - (H \cup F \cup S \cup V(C))$. We have seen above that $|H| = d^+(a_1) - |N| - 1$ and $|N^+(a_n) \cap V(C)| \geq |N| + 1$. Hence $|N^-(a_n) \cap V(C)| \leq n - |N| - 2$, and thus $|F| = |N^-(a_n) - V(C)| \geq d^-(a_n) - (n - 2 - |N|)$. It follows from Lemma 2.1 that

$$|R| \leq cr - \left\{\frac{(c-1)r-1}{2} - |N| - 1 + \frac{(c-1)r-1}{2} - n + 2 + |N| + r + n\right\} = 0.$$

According to (8), we have $|H| = \frac{(c-1)r-3}{2} - |N|$, and therefore, (8) and $|R| = 0$ show that $|F| = d^-(a_n) - (n - 2 - |N|) = \frac{(c-1)r+5}{2} - c + |N|$. If there is an arc xy with $x \in H$ and $y \in F$, then $a_n a_1 a_2 \ldots a_{n-3} v x y a_n$ is an $(n+1)$-cycle, a contradiction. If there is an arc uy with $u \in S$ and $y \in F$, then $a_n a_1 a_2 \ldots a_{n-2} u y a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, it remains the case that $(F \cup S \cup \{a_1, a_2, a_n\}) \rightsquigarrow H$ and $F \rightsquigarrow (\{a_1, a_n\} \cup S \cup H)$.

Subcase 9.2.1. Assume that $|N| \geq \frac{c}{2}$. Since $|R| = 0$, for every $x \in H$, we conclude that $d(x, V(D) - H) \leq n - 3 = c - 4$ and thus, it follows from Lemma 2.1

$$d^+_{D[H]}(x) = d^+(x) - d(x, V(D) - H) \geq \frac{(c-1)r-1}{2} - c + 4 = \frac{(c-1)r+7}{2} - c.$$

This implies

$$\frac{|H|(|H|-1)}{2} \geq |E(D[H])| = \sum_{x \in H} d^+_{D[H]}(x) \geq |H| \left\{\frac{(c-1)r+7}{2} - c\right\}.$$

Because of $|H| = \frac{(c-1)r-3}{2} - |N| \leq \frac{(c-1)r-3}{2} - \frac{c}{2}$, we obtain

$$\frac{(c-1)r-3}{2} - \frac{c}{2} - 1 \geq |H| - 1 \geq (c-1)r + 7 - 2c.$$

This inequality is equivalent with $3c \geq (c-1)r + 19$. The condition $r \geq 3$ leads to the contradiction $0 \geq 16$.

Subcase 9.2.2. Assume that $|N| \leq \frac{c}{2} - 1$. Since $|R| = 0$, for every $y \in F$, we conclude that $d(V(D) - F, y) \leq n - 2 = c - 3$ and thus, it follows from Lemma 2.1

$$d^-_{D[F]}(x) = d^-(y) - d(V(D) - F, y) \geq \frac{(c-1)r-1}{2} - c + 3 = \frac{(c-1)r+5}{2} - c.$$

This implies

$$\frac{|F|(|F|-1)}{2} \geq |E(D[F])| = \sum_{y \in F} d^-_{D[F]}(y) \geq |F| \left\{\frac{(c-1)r+5}{2} - c\right\}.$$

Because of $|F| = \frac{(c-1)r+5}{2} - c + |N| \leq \frac{(c-1)r+3}{2} - \frac{c}{2}$, we obtain

$$\frac{(c-1)r+3}{2} - \frac{c}{2} - 1 \geq |F| - 1 \geq (c-1)r + 5 - 2c.$$

This inequality is equivalent with $3c \geq (c-1)r + 9$. The condition $r \geq 3$ leads to the contradiction $0 \geq 6$. This completes the proof of the theorem. □

From the theorem of Alspach [1] and Theorem 2.2 we can immediately deduce the following result.

Corollary 2.3 If D is a regular c-partite tournament with $c \geq 6$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.

Example 2.4 Let $A_1 = \{u, u_2, u_3\}$, $A_2 = \{v, v_2, v_3\}$, $A_3 = \{w_1, w_2, w_3\}$, and $A_4 = \{x_1, x_2, x_3\}$ be the partite sets of a 4-partite tournament such that $u \to v \to u_2 \to (A_4 \cup \{v_3\})$, $(A_4 \cup \{u_3\}) \to v_2 \to u \to (A_4 \cup \{v_3\})$, $(A_4 \cup \{u_3\}) \to v \to A_3 \to u$, $v_2 \to A_3 \to u_2$, $v_2 \to u_2$, $v_3 \to A_3 \to u_3 \to A_4 \to v_3 \to u_3$, $w_1 \to x_1 \to w_2 \to x_2 \to w_3 \to x_3 \to w_2$, $x_2 \to w_1$, $x_1 \to w_3$, and $x_3 \to w_1$ (see Figure 1). The resulting 4-partite tournament is almost regular, every arc belongs to a 3-cycle, however, the arc uv is not contained in a 4-cycle.

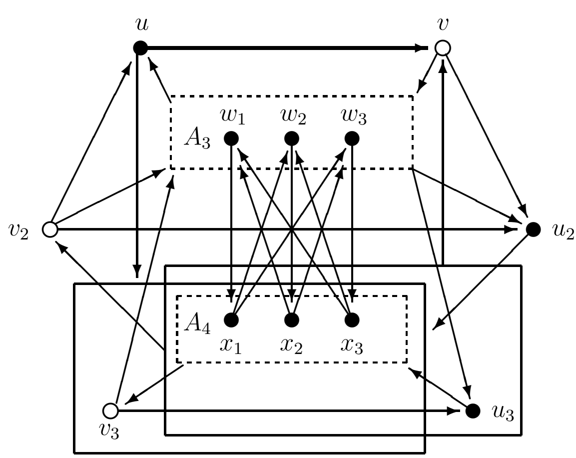

Figure 1: An almost regular 4-partite tournament with the property that every arc belongs to a 3-cycle, but the arc uv is not contained in a 4-cycle

Remark 2.5 Example 2.4 shows that it is not possible to generalize Guo's Theorem 1.3 to almost regular 4-partite tournaments with the partite sets V_1, V_2, V_3, V_4 such that $|V_1| = |V_2| = |V_3| = |V_4| = r$ is odd.

Remark 2.6 In [6], the author has constructed an almost regular 6-partite tournament D with $|V_1| = |V_2| = |V_3| = 2$, $|V_4| = |V_5| = 3$, and $|V_6| = 4$ which contains an arc that does not belong to any 4-cycle. This example shows that Theorem 2.2 is not valid for almost regular c-partite ($c \geq 6$) tournaments in general.

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