On lower bounds on the number of perfect matchings in n-extendable bricks

Tomislav Došlić

Dept. of Inform. and Mathematics
Faculty of Agriculture, University of Zagreb
Svetošimunska c. 25
10000 Zagreb
CROATIA
doslic@faust.irb.hr

Abstract

Using elements of the structural theory of matchings and a recently proved conjecture concerning bricks, it is shown that every n-extendable brick (except K_4, $\overline{C_6}$ and the Petersen graph) with p vertices and q edges contains at least $q - p + (n-1)!!$ perfect matchings. If the girth of such an n-extendable brick is at least five, then this graph has at least $q - p + n^{n-1}$ perfect matchings. As a consequence, the best currently known lower bound on the number of perfect matchings in a fullerene graph is obtained.

1 Introduction

All graphs considered in this paper will be finite, simple and connected. For all terms and concepts not defined here, we refer the reader to the book [7].

Let us consider a graph G with p vertices and q edges, and denote its vertex set by $V(G)$, and its edge set by $E(G)$. A **matching** in G is a collection M of edges of G such that no two edges from M have a vertex in common. In other words, every vertex from $V(G)$ is incident with at most one edge from M. If every vertex from $V(G)$ is incident with exactly one edge from M, the matching M is **perfect**. The number of perfect matchings in a given graph G we denote by $\Phi(G)$.

The problem of determining $\Phi(G)$ is, in an algorithmic sense, a difficult one; it is NP-hard even for the bipartite case ([12]). So it makes sense to seek good upper and lower bounds for $\Phi(G)$ in various classes of graphs.

Let n be an integer with $0 \leq n \leq \frac{p}{2} - 1$. A graph G is n**-extendable** if G has a matching of size n, and every such matching extends to (i.e. is contained in) a perfect matching in G. 0-extendable graphs are the graphs with a perfect matching. The greatest $n \in \mathbb{N}$ such that G is n-extendable is called the **extendability number** of G (or simply the extendability of G) and is denoted by $ext(G)$.

There are many results on n-extendable graphs, concerning their various invariants and structural properties ([9], [10], [11], [8]), but we are not aware of any results connecting the numbers $\Phi(G)$ and $ext(G)$. We are going to establish such a connection when an n-extendable graph is also a brick.

2 1-extendable graphs and bricks

A graph G is **1-extendable** if every edge $e \in E(G)$ appears in some perfect matching of G. A graph G is **bicritical** if G contains an edge and $G - u - v$ has a perfect matching, for every pair of distinct vertices $u, v \in V(G)$. Obviously, every bicritical graph is also 1-extendable and no bipartite graph is bicritical. A 3-connected bicritical graph is called a **brick**.

In Chapter 5 of [7] it is described how every 1-extendable graph can be decomposed into (or built from) simpler building blocks. The number of bricks among these building blocks gives us a lower bound on the number of perfect matchings in 1-extendable graphs.

Theorem 1
For every 1-extendable graph G,

$$\Phi(G) \geq q - p + 2 - k,$$

where k is the number of bricks of G. ■

For the proof, we refer the reader to pages 296–302 of [7].

When a graph G is itself a brick, the following lower bound holds.

Theorem 2
If G is a brick, then $\Phi(G) \geq \frac{p}{2} + 1$.

Proof
This follows from Theorem 1 and the fact that $q - p \geq \frac{3}{2}p - p = \frac{p}{2}$, since the degree of each vertex in G is at least 3. ■

It is known that every brick different from K_4, $\overline{C_6}$ and the Petersen graph contains an edge whose removal leaves a 1-extendable graph. This was proved by Lovász in [6]. He also conjectured that this 1-extendable graph has exactly one brick in its brick decomposition. This conjecture was proved in [1]. As we are considering only simple graphs here, we cite their result in the following form.

Theorem 3
Every brick G different from K_4, $\overline{C_6}$ and the Petersen graph has an edge $e^\star$ whose deletion yields a 1-extendable graph with exactly one brick in its brick decomposition. ■

Let us call such an edge $e^\star$ **terminal**. The name is justified by the fact that this edge serves as the last ear in an ear decomposition of G. For more details about ear decompositions, see the Section 5.4 of [7].

From now on, we will consider only bricks satisfying the conditions of Theorem 3. We call such bricks **ordinary**.

Corollary 4

Every ordinary brick G has an edge $e^\star$ such that $\Phi(G - e^\star) \geq q - p$. ■

3 n-extendable bricks

The following simple result holds for all graphs with perfect matchings.

Lemma 5

Let e be an edge in a graph G with the endpoints u and v. Then

$$\Phi(G) = \Phi(G - e) + \Phi(G - u - v).$$

■

We will also need the following properties of n-extendable graphs. (See [9].)

Lemma 6

Let n be a positive integer. An n-extendable graph G is $(n-1)$-extendable and $(n+1)$-connected. Hence, the minimal degree of a vertex in an n-extendable graph is at least $(n+1)$. ■

As any brick is 1-extendable, we will consider only the bricks of extendability 2 or more.

Before we state our main result, recall that $k!!$ is defined by the relation

$$k!! = \prod_{i=0}^{\lfloor (k-2)/2 \rfloor} (k - 2i),$$

for all $k \in \mathbb{N}$.

Theorem 7

Let n be a positive integer. An ordinary n-extendable brick contains at least $q - p + (n-1)!!$ perfect matchings.

Proof

Let G be an ordinary n-extendable brick, and $e^\star \in E(G)$ a terminal edge of G. Then, by Corollary 4, $\Phi(G - e^\star) \geq q - p$.

Consider now the graph $G' = G - u^\star - v^\star$, where $u^\star, v^\star$ are the endpoints of $e^\star$. Then, by Lemma 6, this graph is at least $(n-1)$-connected, and contains a perfect matching. If G' is itself bicritical, then, by Theorem 8.6.1 of [7], it contains at least $(n-1)!!$ perfect matchings. If G' is not bicritical, we invoke Theorem 8.6.2 from [7], which states that every k-connected non-bicritical graph with a perfect matching contains at least $k!$ perfect matchings, and put $k = n-1$. Our claim follows by noting that every perfect matching of G' is, at the same time, also a perfect matching of G containing the edge $e^\star$, and applying Lemma 5. ■

The following result gives a better lower bound for p big enough.

Theorem 8
Let n be a positive integer. Let G be an ordinary n-extendable brick. Then

$$\Phi(G) \geq q - p + \min\left\{\frac{p}{2}, (n-1)!\right\}.$$

Proof
In the same way as in the proof of Theorem 7 we conclude that $\Phi(G - e^\star) \geq q - p$, where $e^\star$ is the terminal edge of G.

By considering the graph G', defined in the same way as in the proof of Theorem 7, we conclude that, if G' is itself bicritical, it contains at least $\frac{p-2}{2} + 1 = \frac{p}{2}$ perfect matchings, and if G' is not bicritical, then it must have at least $(n-1)!$ different perfect matchings. ■

For $ext(G) = 2$, a better lower bound is possible.

Theorem 9
Let G be an ordinary 2-extendable brick. Then G contains at least $q-p+2$ different perfect matchings.

Proof
Since G is a brick, it is 3-connected and hence $G \neq K_2$. Let $e^* = u^*v^*$ be a terminal edge in G. Then $G - e^*$ is 1-extendable and hence by Corollary 4, $\Phi(G - e^*) \geq q - p$. Let G' denote $G - u^* - v^*$. Then by Lemma 5, $\Phi(G) = \Phi(G - e^*) + \Phi(G')$. So it will suffice to show that $\Phi(G') \geq 2$.

Now if G' is 2-connected, then if it is bicritical it must contain a perfect matching and so by Theorem 8.6.1 of [7], it contains at least two perfect matchings. On the other hand, if G' is not bicritical, then by Theorem 8.6.2 from [7], G' contains at least two perfect matchings. So in any case, $\Phi(G') \geq 2$.

So it remains only to show that G' is 2-connected. Suppose, to the contrary, that G' has a cutvertex w^*. Then $S = \{u^*, v^*, w^*\}$ is a cutset in G and since G is 3-connected, S is a minimum cut. Moreover, by parity, $G - S$ must contain at least one odd component. Let C_o be such an odd component and let C be any other component of $G - S$. Since S is minimum, there must be a vertex x^* in C which is adjacent to w^*. But then the matching $\{u^*v^*, w^*x^*\}$ does not extend to a perfect matching, contradicting the fact that G is 2-extendable and the proof is complete. ■

We conclude our review by a lower bound in n-extendable bricks whose girth is not too small. (The **girth** of a graph G is the length of a shortest cycle in G, if G has a cycle. Otherwise, $girth(G) = +\infty$.)

Corollary 10
Let n be a positive integer. Let G be an ordinary n-extendable brick of girth at least 5. Then

$$\Phi(G) \geq q - p + n^{n-1}.$$

Proof
Let us consider one endpoint of a terminal edge e^* in an ordinary n-extendable brick G of girth at least 5. Denote this vertex by u^*. The vertex u^* has, by Lemma 6, at least $n+1$ neighbors. One of them, v^*, is the other endvertex of e^*. Let $u_1, \ldots, u_{n-1}$

be any other $n-1$ neighbors of u^* and set $U = \{u_1, \ldots, u_{n-1}\}$. The set U is an independent set in G. Moreover, the only common neighbor two vertices $u_i, u_j \in U$ can have is u^*. So, each vertex $u_i \in U$ is incident to at least n edges not connecting it to any other vertex from the set $\{u_1, \ldots, u_{n-1}, u^*\}$. It is obvious that, choosing one such edge for every vertex from U, we get a matching of size $n-1$. There are n^{n-1} such matchings, and each of them, taken together with the edge e^*, can be extended to a perfect matching in G that contains e^*. The claim now follows from Corollary 4 and Lemma 5. ■

Corollary 11
Let n be a positive integer. Let G be an ordinary $(n+1)$-regular n-extendable brick of girth at least 5. Then

$$\Phi(G) \geq q - p + n^n.$$

Proof
Let U be a set of neighbors of u^* different from v^*. Now for each $u_i \in U$, choose an edge e_i incident with u_i which is not incident with u^* and such that $M_j = \{e_1, \ldots, e_n\}$ is a matching. Clearly, there are n^n such matchings M_j. Then $|M_j| = n$ and so M_j extends to a perfect matching F_j in G. Furthermore, since M_j covers all neighbors of u^*, except vertex v^*, perfect matching F_j must contain the edge $e^* = u^*v^*$. Again the proof follows from Corollary 4 and Lemma 5. ■

As an interesting consequence of Corollary 11, we cite the best currently known lower bound for number of perfect matchings in fullerene graphs ([3]). A **fullerene graph** is a 3-regular, 3-connected planar graph, twelve of whose faces are pentagons, and any of the remaining faces are hexagons. (For more on fullerene graphs, see, e.g. [2], [4], [5].)

Corollary 12
Every fullerene graph G on p vertices contains at least $\frac{p}{2} + 4$ perfect matchings.

Proof
It is shown in [3] that every fullerene graph is 2-extendable. The claim now follows from Corollary 11 and the definition of fullerene graphs. ■

Acknowledgments
The author is indebted to the referee for the corrected proof of Theorem 9 and for many other useful suggestions.

References

[1] M.H. Carvalho, C.L. Lucchesi and U.S.R. Murty, On a conjecture of Lovász concerning bricks, *J. Combin. Theory Ser. B*, to appear

[2] T. Došlić, On lower bound of number of perfect matchings in fullerene graphs, *J. Math. Chem.* **24** (1998), 359–364

[3] T. Došlić, On some structural properties of fullerene graphs, *J. Math. Chem.* **31** (2002), 187–195.

[4] D.J. Klein et al., *J. Amer. Chem. Soc.* **108** (1986), 1301.

[5] D.J. Klein and X. Liu, *J. Math. Chem.* **11** (1992), 199.

[6] L. Lovász, Matching structure and matching lattice, *J. Combin. Theory Ser. B* **43** (1987), 187.

[7] L. Lovász and M.D. Plummer, Matching Theory, Ann. Discrete Math. **29**, North-Holland, Amsterdam, The Netherlands, 1986.

[8] P. Maschlanka and L. Volkmann, Independence number in n-extendable graphs, *Discrete Math.* **154** (1996), 167–178.

[9] M.D. Plummer, On n-extendable graphs, *Discrete Math.* **31** (1980), 201–210.

[10] M.D. Plummer, Matching extension and connectivity in graphs, *Congr. Numer.* **63** (1988), 147–160.

[11] M.D. Plummer, Extending matchings in graphs: A survey, *Discrete Math.* **127** (1994), 277–292.

[12] L.G. Valiant, The complexity of computing the permanent, *Theoret. Comput. Sci.* **8** (1979), 189–201.

(Received 5/7/2001)