AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **28** (2003), Pages 137–148

The domination number of the toroidal queens graph of size $3k \times 3k$

A. P. Burger   C. M. Mynhardt

Department of Mathematics and Statistics
University of Victoria
P. O. Box 3045
Victoria, BC
Canada, V8W 3P4
alewyn@math.uvic.ca
mynhardt@math.uvic.ca

William D. Weakley

Department of Mathematical Sciences
Indiana University– Purdue University
Fort Wayne, IN 46805, U.S.A.
weakley@ipfw.edu

Abstract

Denote the $n \times n$ toroidal queens graph by Q_n^t. We show that $\gamma(Q_{3k}^t) = k + 2$ when $k \equiv 0, 3, 4, 6, 8, 9 \pmod{12}$. This completes the proof that $\gamma(Q_{3k}^t) = 2k - \beta(Q_k^t)$ for all positive integers k.

1 Introduction

The study of combinatorial problems on chessboards dates back to 1848, when German chess player Max Bezzel [2] first posed the n-queens problem, that is, the problem of placing n queens on an $n \times n$ chessboard so that no two queens attack each other. The study of chessboard domination problems dates back to 1862, when C. F. de Jaenisch [7] first considered the queens domination problem, that is, the problem of determining the minimum number of queens required to cover every square on an $n \times n$ chessboard. Since then many papers concerning combinatorial problems on chessboards have appeared in the literature. See [9] for a survey of the topic; recent results not mentioned there can be found in [4–5, 10, 13–15].

The n-queens problem for chessboards drawn on the torus was first considered by Pólya (as cited in [1]) who showed that a placement of n mutually non-attacking queens on an $n \times n$ toroidal chessboard is possible if and only if $n \equiv 1, 5 \pmod{6}$. For other values of n the maximum number of non-attacking queens was determined by Monsky [11]. The study of the queens domination problem on the torus was initiated in [3]. The results obtained in these papers clearly show that the n-queens problem and the queens domination problem on the torus differ substantially from the corresponding problems for plane chessboards.

Consider an $n \times n$ chessboard on the torus and notice that the rows and columns of the chessboard are rings on the torus. We cut the torus along arbitrary lines separating two rows and two columns, and draw the $n \times n$ toroidal chessboard in the plane, numbering its rows and columns from 0 to $n-1$, beginning at the bottom left hand corner. Thus each square has *coordinates* (x, y), where x and y are the column and row numbers of the square, respectively. The *lines* of the board are the rows, columns, *sum diagonals*, abbreviated s-*diagonals* (*i.e.*, sets of squares such that $x + y \equiv k \pmod{n}$, where k is a constant) and *difference diagonals*, abbreviated d-*diagonals* (sets of squares such that $y - x \equiv k \pmod{n}$). The s-diagonal (d-diagonal, respectively) such that $x+y \equiv k \pmod{n}$ ($y-x \equiv k \pmod{n}$, respectively) is denoted by $s = k$ ($d = k$, respectively) and labelled accordingly in the drawing. Note that there are n s-diagonals and n d-diagonals, each of which contains n squares, so that there are $4n$ lines in total. Rows and columns are collectively called *orthogonals.*

Consider any d-diagonal — by symmetry we may assume $d = 0$ — and any s-diagonal $s = k$. These diagonals intersect in a square (x, y) if and only if $x + y \equiv k \pmod{n}$ and $y - x \equiv 0 \pmod{n}$; that is, $2y \equiv k \pmod{n}$. If n is odd, then this congruency has exactly one solution for each k, namely $y = \frac{k}{2}$ if k is even and $y = \frac{n+k}{2}$ if k is odd. If n is even, then the congruency has no solution if k is odd and exactly two solutions if k is even, namely $y = \frac{k}{2}$ and $y = \frac{n+k}{2}$. Therefore, if n is odd, then any d-diagonal intersects any s-diagonal in exactly one square of the toroidal $n \times n$ chessboard, while if n is even, then any d-diagonal and s-diagonal of the same parity intersect in exactly two squares, while diagonals of different parity do not intersect in a square at all.

The vertices of Q_n^t, the *queens graph obtained from an $n \times n$ chessboard on the torus*, are the n^2 squares of the chessboard, and two squares are adjacent if they are collinear, that is, if they lie on the same line as defined above. It is easy to verify that for any $a, b \in \{0, 1, \ldots, n-1\}$, the mapping defined by $\tau_{a,b}(x, y) = (x + a, y + b)$ is a graph automorphism of Q_n^t, so Q_n^t is vertex-transitive. Also, for any integer m that is relatively prime to n, the mapping defined by $\pi_m(x, y) = (mx, my)$, with reduction modulo n, is a graph automorphism of Q_n^t. Automorphisms of this type will be useful later.

A queen on a square (x, y) of Q_n^t is said to *cover* or *dominate* (x, y) and any square adjacent to (x, y). A set D of squares is a *dominating set* of Q_n^t if every square of Q_n^t is either in D or adjacent to a square in D, *i.e.,* if a set of queens, one on each square in D, covers the board. If no two squares of the dominating set D are adjacent, then D is an *independent dominating set.* As is standard in domination theory (see [8]) we denote the domination number — the minimum cardinality amongst all dominating sets — of Q_n^t by $\gamma(Q_n^t)$.

2 More definitions and previous results

Let S be a set of squares of Q_n^t. A line (row, column, diagonal, orthogonal) which contains (respectively does not contain) a square of S is called an *occupied* (respectively *empty*) line (row, column, diagonal, orthogonal). Let r be any empty row. Each element of S dominates r exactly three times (by column and s- and d-diagonals), hence in at most three squares. A similar statement holds for any empty column. If a square (x, y) is dominated $p \geq 2$ times by squares in S, we say that (x, y) contains $p - 1$ *wastes*. The *waste number* $w(l)$ of an empty line l is the sum of the wastes on the squares in l.

If $|S| \geq k$, then S is said to form a *perfect pattern* on Q_{3k}^t if there is at least one square of S in every third row, column and diagonal, and no queens in any other row, column or diagonal.

We now show that if S forms a perfect pattern on Q_{3k}^t, then S dominates Q_{3k}^t. We may choose coordinates so that the label of each occupied orthogonal is a multiple of 3, and then the labels of the occupied diagonals are also multiples of 3. Consider a square (x, y) of Q_{3k}^t. If (x, y) is not in an occupied row or column, then $x \equiv 1$ or $2 \pmod 3$, and similarly for y. Then either $x \equiv y \pmod 3$ and (x, y) is in an occupied d-diagonal, or $x \not\equiv y \pmod 3$, so $x + y \equiv 0 \pmod 3$ and then (x, y) is in an occupied s-diagonal.

This gives a method (see [3, 12]) of constructing a dominating set S of Q_{3k}^t from a set S_1 of squares of Q_k^t. We wish to begin with a set S_2 of at most k squares of Q_k^t that occupies as many lines as possible. To this end, recall that a set of vertices of a graph G is *independent* if no two are adjacent, and $\beta(G)$ denotes the maximum size of an independent set of vertices of G. Let S_2 be an independent set of Q_k^t containing $\beta = \beta(Q_k^t)$ squares. Then S_2 occupies 4β of the $4k$ lines of Q_k^t. There remain $2k - 2\beta$ unoccupied orthogonals and $2k - 2\beta$ unoccupied diagonals. Pair off these orthogonals with these diagonals; for each pair, adjoin the unique intersection square of the two lines to S_2, thus obtaining a set S_1 of $\beta + (2k - 2\beta) = 2k - \beta$ squares that occupies every line of Q_k^t. Then $S = \{(3x, 3y) : (x, y) \in S_1\}$ forms a perfect pattern on Q_{3k}^t and thus dominates. Therefore $\gamma(Q_{3k}^t) \leq 2k - \beta(Q_k^t)$ for all positive integers k.

Monsky [11] has shown that

$$\beta(Q_k^t) = \begin{cases} k & \text{if } k \equiv 1, 5, 7, 11 \pmod{12} \\ k - 1 & \text{if } k \equiv 2, 10 \pmod{12} \\ k - 2 & \text{if } k \equiv 0, 3, 4, 6, 8, 9 \pmod{12}. \end{cases}$$

The study of the queens domination problem for chessboards on the torus, that is, the problem of determining $\gamma(Q_n^t)$, was initiated in [3], where it was shown that $\gamma(Q_n^t) \geq \lceil \frac{n}{3} \rceil$ for all n, and in particular when n is divisible by 3, that

$$\gamma(Q_{3k}^t) \begin{cases} = k & \text{if } k \equiv 1, 5, 7, 11 \pmod{12} \\ = k + 1 & \text{if } k \equiv 2, 10 \pmod{12} \\ \geq k + 1 & \text{if } k \equiv 0, 3, 4, 6, 8, 9 \pmod{12}. \end{cases}$$

Configurations given in [12] further show that

$$\gamma(Q^t_{3k}) \le k+2 \text{ if } k \equiv 0,3,4,6,8,9 \pmod{12}. \tag{1}$$

In this paper we show that the upper bound in (1) is exact, completing the proof that

$$\gamma(Q^t_{3k}) = 2k - \beta(Q^t_k) \text{ for all positive integers } k.$$

Mynhardt [12] also showed that $\gamma(Q^t_{2k}) \le k$ for all k, and so if n is even and not divisible by 3, then the best known bounds for $\gamma(Q^t_n)$ are

$$\lceil \tfrac{n}{3} \rceil \le \gamma(Q^t_n) \le \tfrac{n}{2}.$$

Denote the graph obtained from the moves of queens on the ordinary (plane) $n \times n$ chessboard by Q_n. It is easy to see that any dominating set of Q_n also dominates Q^t_n and so $\gamma(Q^t_n) \le \gamma(Q_n)$ for all n. Determining upper bounds for $\gamma(Q_n)$ is a difficult problem — see [6] and [15] for recent bounds. In contrast the upper bounds in [12] were easier to obtain, but when $n \equiv 1,5 \pmod 6$, the bounds for $\gamma(Q_n)$ are still the best general bounds for $\gamma(Q^t_n)$.

3 Main theorem

Theorem 1 *If $k \equiv 0,3,4,6,8,9 \pmod{12}$, then $\gamma(Q^t_{3k}) = k+2$.*

Proof. As reported in [3], $\gamma(Q^t_9) = 5$ and $\gamma(Q^t_{12}) = 6$, so we will assume $k \ge 6$. Suppose to the contrary that S is a dominating set of Q^t_{3k} of size $k+1$, where the queens in S occur in rows (in non-decreasing order) $r_1, ..., r_{k+1}$ and columns $c_1, ..., c_{k+1}$. Define Δ_i by $\Delta_1 = 3k + r_1 - r_{k+1}$ and $\Delta_i = r_i - r_{i-1}$ for $i \in \{2, ..., k+1\}$; note that $\Sigma_{i=1}^{k+1} \Delta_i = 3k$. Using columns instead of rows, Δ'_j, $j \in \{1, ..., k+1\}$, is defined similarly.

Since each queen in S dominates each empty orthogonal l exactly three times, l is dominated exactly $3k+3$ times, and since each square in l is dominated at least once, $w(l) = 3$. We see in particular that $\Delta_i, \Delta'_j = 0$ for at most three values of i and at most three values of j. Let α, $0 \le \alpha \le 3$, be the number of values of i for which $\Delta_i = 0$.

If s consecutive rows are occupied (respectively empty), we refer to them as a *row block* (respectively *row blank*) of size s. *Column block* and *column blank* are defined similarly. A row (or column) block (or blank) is *maximal* if not part of a larger one. Clearly, the number of maximal row (column) blocks equals the number of maximal row (column) blanks. Note that for $t \ge 1$, a maximal row blank of size t corresponds to $\Delta_i = t+1$ for some i.

We wish to show that S does not have a row or column blank of size 3. Our proof will require the following two technical lemmas.

Lemma 2 *Suppose there is a column blank of size at least* 3. (*That is, there is a j with $\Delta'_j \ge 4$.*) *Then $\alpha \le 2$, and there are at most $3-\alpha$ maximal row blocks. There is at least one row block of size $\lceil \frac{k+1-\alpha}{3-\alpha} \rceil$ and at least one row blank of size $\lceil \frac{2k-1+\alpha}{3-\alpha} \rceil$.*

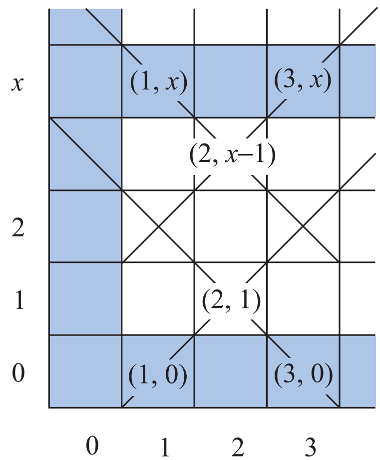

Figure 1: Three adjacent empty columns

Proof. Without loss of generality we may assume that columns 1, 2, and 3 are empty; each of these columns contains α wastes caused by more than one queen per row. Since $k \geq 6$ and the chessboard is drawn on the torus, whenever a maximal row block ends at occupied row r_i, there always exists an occupied row $r_{i+1} \neq r_i$; if $i = k+1$, then $r_{i+1} = r_1$. Assume without loss of generality that a maximal row block ends at occupied row $r_1 = 0$ (see Figure 1). Then square $(2, 1)$ is in an empty row and empty column and is dominated along an s- or d-diagonal; in either case there is a waste in the occupied row $r_1 = 0$ of one of the empty columns 1 (square $(1, 0)$) or 3 (square $(3, 0)$). Say row $r_2 = x \geq 2$ is the next occupied row and consider square $(2, x-1)$ (which may be the same as $(2, 1)$), which is also dominated diagonally. Again there is a waste in the occupied row $r_2 = x$ of one of the empty columns 1 (square $(1, x)$) or 3 (square $(3, x)$). Hence every maximal row block contributes at least two wastes in columns 1 and 3. Since $w(1) + w(3) = 6$, there are at most $3 - \alpha$ maximal row blocks. As there is at least one maximal row block, $\alpha \leq 2$.

The other assertions of the lemma follow from the fact that the number of occupied rows is $k + 1 - \alpha$. □

We will also use Lemma 2 with the roles of rows and columns reversed, which we will refer to as Lemma 2′.

Let l, h be integers with $2 < l \leq h$. Relative to S, an (l, h)-*rectangle* occurs where l consecutive empty columns meet h consecutive rows, of which only the top and bottom rows are empty.

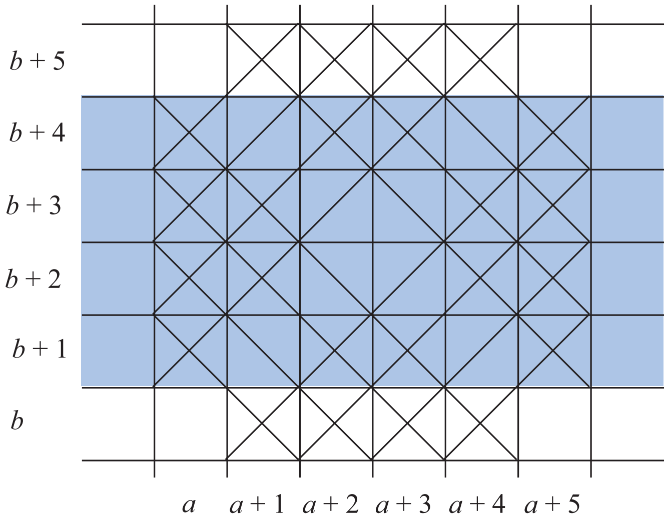

Figure 2: Four occupied rows and six empty columns

Lemma 3 *Suppose we have an (l, h)-rectangle involving empty columns $a, a+1, \ldots,$ $a+l-1$ and occupied rows $b+1, \ldots, b+h-2$, with rows b and $b+h-1$ empty. Then the squares in the set $E = \{(x, y) : x \in \{a, a+l-1\}$ and $b < y < b+h-1\}$ contain at least $2(l-2)$ wastes. Thus one of columns a, $a+l-1$ has waste number at least $l-2$.*

Proof. Every square of E is in an occupied row, so diagonal attacks on these squares give wastes. Consider the $2(l-2)$ squares satisfying $a < x < a+l-1$ and $y = b$ or $b+h-1$. Each of these squares has empty row and column, so must be in a diagonal occupied by S. The diagonals containing these squares are all distinct, and each one contains a square of E, so at least $2(l-2)$ wastes occur at squares of E. The lemma is proved; an example with $l = h = 6$ is shown in Figure 2. □

We can now show that S does not have a row blank or column blank of size 3. It suffices to consider column blanks; for purposes of contradiction, assume there is a column blank of size 3.

First consider the case $k \geq 9$. Here Lemma 2 gives a row block of size 4 and a row blank of size 6. Using the latter and Lemma 2′, we get a column blank of size 6. Thus there is some $h \geq 6$ such that there is a $(6, h)$-rectangle. By Lemma 3 there is an empty column c with $w(c) \geq 4$, a contradiction.

Now let $k = 8$. We claim that there cannot be a row block of size more than 4. If there is, we can assume without loss of generality that rows 0, 1, 2, 3, 4 are occupied. Then $\pi_5(S)$ is a dominating set of size 9 for Q_{24}^t and $\pi_5(S)$ occupies at least the rows 0, 5, 10, 15, 20. Going around the torus, the five differences we get from these numbers are 5, 5, 5, 5, 4; all at least 4. Since $|\pi_5(S)| = 9$, there are at most four other rows occupied by $\pi_5(S)$; thus $\pi_5(S)$ has some $\Delta_i \geq 4$, so has a row blank of size 3. Then Lemma $2'$ implies $\pi_5(S)$ has a column blank of size 5, whence Lemma 2 gives a row blank of size 5, so for some d, $\pi_5(S)$ has $\Delta_d \geq 6$, which clearly does not happen.

So S cannot have a row block of size more than 4, which implies that any (l, h)-rectangle for S has $h \leq 6$. By Lemma 2, S has a row block of size 3 and a row blank of size 5, and then Lemma $2'$ gives a column blank of size 5. These facts imply there is an (l, h)-rectangle with $l = 5$ and $h = 5$ or 6. Then by Lemma 3, some empty columns c, c' have $w(c) + w(c') \geq 2(l - 2) = 6$ (from diagonal covers of squares in occupied rows). Since $w(c), w(c') \leq 3$, we may conclude $\alpha = 0$. Since $|S| = 9$, there are at least five occupied rows not yet discussed, which fall into at most two maximal row blocks (Lemma 2). Thus there is another row block of size 3; using this with the previously employed column blank to make another rectangle, we get more wastes in columns c_1 and c_2, a contradiction.

Finally, consider $k = 6$. By Lemma 2, there is a row block of size 3 and a row blank of size 4, and then Lemma $2'$ gives a column blank of size 4. Then for some $h \geq 5$, there is a $(4, h)$-rectangle, and by Lemma 3, $w(c) + w(c') \geq 4$ for some empty columns c, c'.

Let z be the size of a largest row block. There are $7 - \alpha - z$ occupied rows not in this block, and by Lemma 2 there are at most $2 - \alpha$ other maximal row blocks; if any of these row blocks has size two or more, then for some $h' \geq 4$ we have a $(4, h')$-rectangle involving the same columns but different rows, thus giving at least four additional wastes in columns c and c' jointly, so that $w(c) + w(c') \geq 8$, a contradiction. Therefore $\lceil \frac{7-\alpha-z}{2-\alpha} \rceil \leq 1$, which implies $z \geq 5$. We may assume without loss of generality that S occupies rows $0, 1, 2, 3, 4$. Then the automorphic image $\pi_7(S)$ is a dominating set of size 7 for Q^t_{18}, and the rows occupied by $\pi_7(S)$ include 0, 3, 7, 10, 14. Going around the torus, the successive differences between these numbers are 3, 4, 3, 4, 4. From the fact that $\pi_7(S)$ has at most two more occupied rows, we may derive two conclusions. First, $\pi_7(S)$ has at least four maximal row blanks, and second, at least one of these has size 3. However, these lead to a contradiction, because the first implies by Lemma 2 that $\pi_7(S)$ has no column blank of size 3 or more (the number of maximal row blanks being equal to the number of maximal row blocks), while the second implies by Lemma $2'$ that $\pi_7(S)$ has a column blank of size at least $\lceil \frac{11+\alpha}{3-\alpha} \rceil \geq 4$.

This concludes the proof that S does not have a row or column blank of size 3.

It follows that $\Delta'_j \leq 3$ for each $j \in \{1, ..., k+1\}$ and $\Delta_i \leq 3$ for each $i \in \{1, ..., k+1\}$. But $\Sigma_{i=1}^{k+1} \Delta_i = 3k$ and the only possibilities are

(*i*) $\Delta_i = 3$ for $k - 2$ values of i and $\Delta_i = 2$ for three values of i

(*ii*) $\Delta_i = 3$ for $k - 1$ values of i, $\Delta_i = 2$ for one i and $\Delta_i = 1$ for one i

(*iii*) $\Delta_i = 3$ for k values of i and $\Delta_i = 0$ for one i.

A similar statement holds for the Δ'_j. Suppose that for some j, $\Delta'_j \geq 2$, $\Delta'_{j+1} = 2$ and $\Delta'_{j+2} \geq 2$. Choose i such that $\Delta_i = 3$ and consider the intersection of the sections of

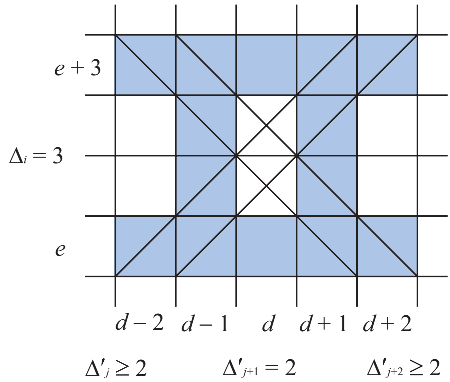

Figure 3: Dominating squares in empty rows in column d creates wastes in columns $d \pm 2$

rows and columns concerned (see Figure 3). Say columns d and $d \pm 2$ are empty and columns $d \pm 1$ are occupied, while rows e and $e+3$ are occupied with rows $e+1$ and $e+2$ empty. The squares in the empty rows in column d are dominated diagonally, and we see that at least two wastes are created in columns $d \pm 2$. Since $k \geq 6$, there are at least four values of i such that $\Delta_i = 3$, and in each case at least two wastes occur, giving $w(d-2) + w(d+2) \geq 8$, which is impossible.

Therefore possibility (i) does not occur for either rows or columns. If possibility (ii) occurs, say $\Delta_i = 2$ ($\Delta'_j = 2$), then (without loss of generality) $\Delta_{i+1} = 1$ ($\Delta'_{j+1} = 1$, respectively). Thus there are queens in every third row and every third column; without loss of generality in the sets of rows and columns $\mathcal{R}_3 = \mathcal{C}_3 = \{0, 3, 6, ..., 3k-3\}$. Depending on whether possibility (ii) occurs for rows and/or columns, there may also be a queen in an additional row u and/or column v (see Figure 4). This implies that at most two of the squares of S have diagonal labels not divisible by 3.

We next show that each diagonal with label divisible by 3 is occupied by a queen in S. Suppose that s-diagonal $s = 3h$ is empty. Note that $2k$ of the $3k$ squares of s have neither row nor column label divisible by 3. At most two of them are covered orthogonally if row u or column v is occupied. Hence at least $2k-2$ of these squares must be covered by d-diagonals, none of which has label divisible by 3. Since at most two such d-diagonals are occupied, and each of them intersects s in at most two squares, we see that $2k - 2 \leq 2 \cdot 2$, contradicting $k \geq 6$. Thus every s-diagonal and, similarly, every d-diagonal, with label divisible by 3 is occupied by S. Denote these sets of s- and d-diagonals by $\mathcal{S}_3$ and $\mathcal{D}_3$, respectively. (There may be additional occupied diagonals.)

Observe that every square (t, u) in row u is dominated exactly once along a line $q_t \in \mathcal{C}_3 \cup \mathcal{S}_3 \cup \mathcal{D}_3$, which intersects row $u+1$ only in squares in $\mathcal{C}_3 \cap \mathcal{S}_3 \cap \mathcal{D}_3$. We may thus move the queen on square (t', u) along $q_{t'}$ to row $u+1$ to obtain a dominating set of Q^t_{3k} in which there are queens in every third row only. Similarly, we may move

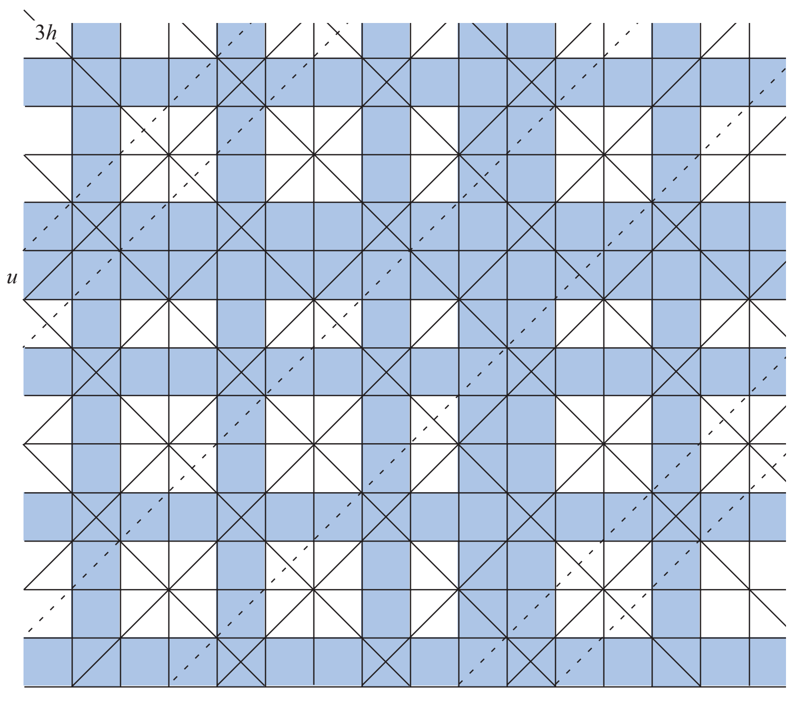

Figure 4: Diagonals required to dominate squares in empty rows and columns

the queen in column v to column $v+1$ to obtain a dominating set S' that forms a perfect pattern on Q^t_{3k}.

The $k+1$ queens in S' have coordinates $(3x_i, 3y_i)$ for $0 \le i \le k$, where $\{x_i : 0 \le i \le k\} = \{y_i : 0 \le i \le k\} = \{0, ..., k-1\}$. Consider the set X of queens on Q^t_k with $X = \{(x_i, y_i) : (3x_i, 3y_i) \in S'\}$. Let $s_i \equiv (x_i + y_i) \pmod{k}$ and $d_i \equiv (y_i - x_i) \pmod{k}$ be the s- and d-diagonals containing (x_i, y_i). Then there are queens in every row, column, s-diagonal and d-diagonal of Q^t_k (S' forms a perfect pattern on Q^t_{3k}), so that there are one row, one column and one diagonal of each type that contain exactly two queens. By symmetry we may assume that row 0, column 0, s-diagonal p and d-diagonal q contain two queens. Now,

$$\begin{array}{lrcll} & s_i & \equiv & (x_i + y_i) \pmod{k} & \text{for each } i \\ \text{i.e.} & \sum_{i=0}^{k} s_i & \equiv & \sum_{i=0}^{k} (x_i + y_i) \pmod{k} & \text{(summing over all queens in } X) \\ \text{i.e.} & \sum_{i=0}^{k-1} i + p & \equiv & 2\sum_{i=0}^{k-1} i \pmod{k} & \left\{ \begin{array}{l} \text{since row 0, column 0 and} \\ \text{s-diagonal } p \text{ contain two queens} \end{array} \right. \\ \text{i.e.} & \frac{k(k-1)}{2} + p & \equiv & k(k-1) \pmod{k} & \\ \text{i.e.} & p & \equiv & \frac{k(k-1)}{2} \pmod{k}. & \end{array}$$

Similarly, $q \equiv \frac{k(k-1)}{2} \pmod{k}$. Now if k is odd, then $\frac{k-1}{2}$ is integral and so $p = q = 0$. If k is even, then $p = q = \frac{k}{2}$. Define ω to be 1 if k is odd, 2 if k is even. It is easy to check that $a \equiv b \pmod{k}$ implies that $a^2 \equiv b^2 \pmod{\omega k}$. We thus have

$$s_i^2 \equiv (x_i + y_i)^2 \pmod{\omega k} \equiv (x_i^2 + 2x_i y_i + y_i^2) \pmod{\omega k}$$

and

$$d_i^2 \equiv (y_i - x_i)^2 \pmod{\omega k} \equiv (x_i^2 - 2x_i y_i + y_i^2) \pmod{\omega k},$$

therefore

$$s_i^2 + d_i^2 \equiv 2(x_i^2 + y_i^2) \pmod{\omega k} \quad \text{for each } i.$$

As before, summation over all queens in X gives

$$\begin{array}{rrcl} & \sum_{i=0}^{k} (s_i^2 + d_i^2) & \equiv & 2\sum_{i=0}^{k} (x_i^2 + y_i^2) \pmod{\omega k} \\ \text{that is,} & 2\sum_{i=0}^{k-1} i^2 + p^2 + q^2 & \equiv & 4\sum_{i=0}^{k-1} i^2 \pmod{\omega k}. \end{array} \qquad (2)$$

If k is even, (2) gives

$$\begin{array}{lrcl} & \frac{2k(k-1)(2k-1)}{6} + \frac{k^2}{2} & \equiv & \frac{4k(k-1)(2k-1)}{6} \pmod{2k} \\ \text{i.e.} & \frac{2k(k-1)(2k-1)}{6} - \frac{k^2}{2} & \equiv & 0 \pmod{2k} \\ \text{i.e.} & 2k\left[\frac{(4k-1)(k-2)}{12}\right] & \equiv & 0 \pmod{2k}. \end{array}$$

Thus $\frac{(4k-1)(k-2)}{12}$ is an integer, which is impossible if $k \equiv 0, 4, 6, 8 \pmod{12}$. If k is odd, (2) gives

$$\begin{aligned} \frac{2k(k-1)(2k-1)}{6} &\equiv 0 \pmod{k} \\ \text{i.e.}\quad k\left[\frac{(k-1)(2k-1)}{3}\right] &\equiv 0 \pmod{k}. \end{aligned}$$

Thus $\frac{(k-1)(2k-1)}{3}$ is an integer, which is impossible if $k \equiv 3, 9 \pmod{12}$. ■

Acknowledgement

This paper was written while A. P. Burger and C. M. Mynhardt were employed in the Department of Mathematics, Applied Mathematics and Astronomy of the University of South Africa. Financial support from the University and the South African National Research Foundation is gratefully acknowledged.

References

[1] W. Ahrens, *Mathematische Unterhalten und Spiele*, B. G. Teubner, Leipzig-Berlin, 1910.

[2] M. Bezzel, Schachfreund, *Berliner Schachzeitung* **3** (1848), 363.

[3] A. P. Burger, E. J. Cockayne and C. M. Mynhardt, Queens graphs for chessboards on the torus, *Australas. J. Combin.* **24** (2001), 231–246.

[4] A. P. Burger and C. M. Mynhardt, Symmetry and domination in queens graphs, *Bull. ICA* **29** (2000), 11–24.

[5] A. P. Burger and C. M. Mynhardt, Properties of dominating sets of the queens graph Q_{4k+3}, *Utilitas Math.* **57** (2000), 237–253.

[6] A. P. Burger and C. M. Mynhardt, An improved upper bound for queens domination numbers, *Discrete Math.* **266** (2003), 119–131.

[7] C. F. de Jaenisch. *Applications de l'Analyse Mathematique au Jeu des Echecs.* Petrograd, 1862.

[8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[9] S. M. Hedetniemi, S. T. Hedetniemi, and R. Reynolds, Combinatorial problems on chessboards: II. In T. W. Haynes, S. T. Hedetniemi and P. J. Slater, editors, *Domination in Graphs: Advanced Topics.* Marcel Dekker, New York, 1998.

[10] M. D. Kearse and P. B. Gibbons, Computational methods and new results for chessboard problems, *Australas. J. Combin.* **23** (2001), 253–284.

[11] P. Monsky, Problem E3162, *Amer. Math. Monthly* **96** (1989), 258–259.

[12] C. M. Mynhardt, Upper bounds for the domination number of toroidal Queens graphs, *Discussiones Math.* **23** (2003), 163-175.

[13] P. R. J. Östergård and W. D. Weakley, Values of domination numbers of the queen's graph, *Electron. J. Combin.* **8** (2001), *no.* 1, Research paper 29, 19 pp.

[14] W. D. Weakley, A lower bound for domination numbers of the queen's graph, *J. Combin. Math. Combin. Comput.* **43** (2002), 231-254.

[15] W. D. Weakley, Upper bounds for domination numbers of the queen's graph, *Discrete Math.* **242** (2002), 229–243.

(Received 1 May 2002)