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Strong subtournaments of close to regular multipartite tournaments

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is regular and if $i_g(D) \leq 1$, then D is called almost regular.

Recently, L. Volkmann and S. Winzen showed that every almost regular c-partite tournament D with $c \geq 5$ contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c\}$. In this paper we will investigate multipartite tournaments with $i_g(D) \leq l$ and $l \geq 2$. Treating a problem of L. Volkmann (*Australas. J. Combin.* 20 (1999), 189–196) we will prove that, if D is a c-partite tournament with at least three vertices in each partite set, $i_g(D) \leq l$ and $c \geq l + 2$ with $l \geq 2$, then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c - l + 1\}$.

1 Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X, Y)$, i.e., $d(X, Y) = |\{xy \in E(D) : x \in X, y \in Y\}|$. If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we speak of a *cycle* (*path*), then we mean a directed cycle (directed path), and a cycle of length n is called an *n-cycle*. A cycle (path) of a digraph D is *Hamiltonian*, if it includes all the vertices of D. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices in D, there is a path from x to y. The digraph D with at least $k+1$ vertices is called *k-strong*, if for arbitrary $k-1$ vertices $x_1, x_2, \ldots, x_{k-1}$ of D the digraph $D[V(D) - \{x_1, x_2, \ldots, x_{k-1}\}]$ is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-strong.

There are several measures of how much a digraph differs from being regular. In [8], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}$$

and the *local irregularity* as $i_l(D) = \max |d^+(x) - d^-(x)|$ over all vertices x of D. Clearly, $i_l(D) \leq i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leq 1$, then D is called *almost regular*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D, and we define $\gamma(D) = |V_1|$.

In the last time, the advantage of having many statements about strong tournaments led to a search for strong subtournaments in multipartite tournaments. In 1999, a first result was presented by Volkmann [4].

Theorem 1.1 (Volkmann [4], 1999) *Let D be an almost regular c-partite tournament with $c \geq 4$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c-1\}$.*

Recently, L. Volkmann and S. Winzen [6] settled the conjecture of Volkmann [4] in affirmative that Theorem 1.1 also holds for $p = c$, if $c \geq 5$. This yields the following theorem.

Theorem 1.2 (Volkmann, Winzen [6]) *Let D be an almost regular c-partite tournament with $c \geq 5$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c\}$.*

We now want to deal with the following problem which was posed by L. Volkmann [4] in 1999 (see also [5], Problem 2.32).

Problem 1.3 (Volkmann [4]) *Determine other sufficient conditions for (strongly connected) c-partite tournaments to contain strong subtournaments of order p for some $4 \leq p \leq c$.*

The complexity of the proof of Theorem 1.2 makes it clear that the statement of this theorem becomes false, if we enlarge $i_g(D)$ without changing the rest of the parameters. This also demonstrates the following example.

Example 1.4 *Let* $V_1 = \{z_1\}$, $V_2 = \{z_2\}$, $V_3 = \{z_3, \hat{z}_3\}$, $V_4 = \{x, \hat{x}\}$ *and* $V_5 = \{v', v''\}$ *be the partite sets of a multipartite tournament* D *such that* $x \rightarrow z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow x \rightarrow z_2$, $z_1 \rightarrow z_3$, $x \rightarrow \hat{z}_3 \rightarrow \{z_1, z_2\} \rightarrow \hat{x} \rightarrow z_3$, $\{x, z_1, z_2, z_3\} \rightarrow v' \rightarrow \{\hat{z}_3, \hat{x}\} \rightarrow v'' \rightarrow \{x, z_1, z_2, z_3\}$ *and* $\hat{z}_3 \rightarrow \hat{x}$ *(see also Figure 1). The resulting 5-partite tournament* D *with* $i_g(D) \leq 2$ *does not contain a strong subtournament of order 5.*

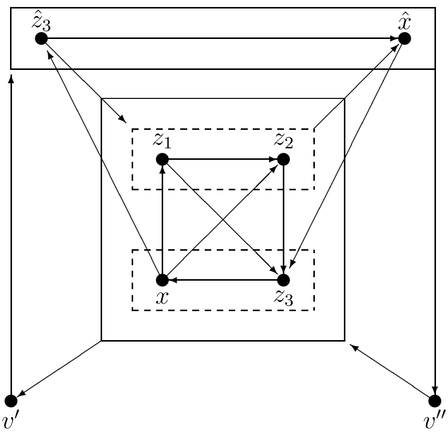

Figure 1: A 5-partite tournament with $i_g(D) = 2$ and without a strong subtournament of order 5.

It is very probably that the size of strong subtournaments decreases, if the global irregularity increases. In this paper we will present a result that guarantees strong subtournaments of a size depending on the global irregularity $i_g(D)$.

Theorem 1.5 *Let* D *be a* c*-partite tournament with at least 3 vertices in each partite set,* $i_g(D) \leq l$, $c \geq l + 2$ *and* $l \geq 2$. *Then* D *contains a strongly connected subtournament of order* p *for every* $p \in \{3, 4, \ldots, c - l + 1\}$.

Neglecting a finite family of multipartite tournaments, this result enlarges Theorem 1.1 to classes of multipartite tournaments with $i_g(D) \leq l$.

2 Preliminary results

The following results play an important role in our investigations.

Theorem 2.1 (Moon [2], 1966) *If* T *is a strongly connected tournament, then every vertex of* T *is contained in a cycle of order* m *for all* $3 \leq m \leq |V(T)|$.

Theorem 2.2 (Bondy [1], 1976) *Each strongly connected c-partite tournament contains a cycle of order m for each $m \in \{3, 4, \ldots, c\}$.*

Theorem 2.3 (Yeo [7], 1998) *If D is a multipartite tournament, then*

$$\kappa(D) \geq \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3}.$$

Lemma 2.4 (Volkmann [4], 1999) *Let T be a strongly connected tournament of order $|V(T)| \geq 4$. Then there exists a vertex $u \in V(T)$ of maximum outdegree such that for all $x \in V(T) - \{u\}$, the subtournament $T - \{x\}$ has a Hamiltonian path with the initial vertex u.*

Lemma 2.5 (Tewes, Volkmann, Yeo [3], 2002) *If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D, then $||V_i| - |V_j|| \leq 2i_g(D)$ for $1 \leq i, j \leq c$.*

The following lemma is a generalization of a result of Volkmann [4] and can be proved similarly.

Lemma 2.6 *If X is a vertex set of a digraph D with $i_g(D) \leq l$, then*

$$|d(X, V(D) - X) - d(V(D) - X, X)| \leq l|X|.$$

Proof. We consider the following sum $S = \sum_{x \in X}(d^+(x) - d^-(x))$. Every arc with both ends in X is added once and subtracted once. Furthermore, every arc going out of X is added once, and every arc going into X is subtracted once. Therefore, we obtain $S = d(X, V(D) - X) - d(V(D) - X, X)$. Since $i_g(D) \leq l$, each term in the sum is between minus and plus l, and hence the desired estimation $|S| = |d(X, V(D) - X) - d(V(D) - X, X)| \leq l|X|$ follows. □

Lemma 2.7 *If D is a c-partite tournament with $r \geq 2$ vertices in each partite set, then there are vertices $x, y \in V(D)$ such that $d^-(x), d^+(y) \geq c - 1$.*

Proof. For every vertex $x \in V(D)$ we observe that

$$d^-(x) + d^+(x) = |V(D)| - |V(x)| \geq (c-1)r \geq 2(c-1).$$

Counting all outdegrees and indegrees in D we obtain that

$$2 \sum_{x \in V(D)} d^+(x) = 2 \sum_{x \in V(D)} d^-(x) = \sum_{x \in V(D)} (d^+(x) + d^-(x)) \geq |V(D)| 2(c-1),$$

which immediately implies the statement of this lemma. □

For vertices that are contained in large partite sets the following lemma presents a good result.

Lemma 2.8 *Let D be a multipartite tournament with $i_g(D) \leq l$ and $\gamma(D) = r$. For any $x \in V(D)$ with $|V(x)| = r + 2l - k$ $(0 \leq k \leq 2l)$ it follows that*

$$\max\{d^+(x), d^-(x)\} \leq \frac{|V(D)| - r - 2l + 2k}{2} = \frac{|V(D)| - |V(x)| + k}{2}.$$

Proof. Suppose that $d^+(x) \geq \frac{|V(D)|-r-2l+2k+1}{2}$. Then we conclude that $d^-(x) \leq |V(D)| - r - 2l + k - \frac{|V(D)-r-2l+2k+1}{2} = \frac{|V(D)|-r-2l-1}{2}$. Let $y \in V(D)$ such that $|V(y)| = r$. Because of $i_g(D) \leq l$, it follows that $d^+(y), d^-(y) \leq d^-(x) + l \leq \frac{|V(D)|-r-1}{2}$, and we arrive at the contradiction

$$|V(D)| = d^+(y) + d^-(y) + r \leq |V(D)| - 1.$$

Hence, the assertion for $d^+(x)$ holds. The assertion for $d^-(x)$ follows analogously. This completes the proof of the lemma. □

3 Proof of Theorem 1.5

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D and let $r = \gamma(D)$. Because of Lemma 2.5 we obtain $3 \leq r \leq |V_i| \leq r + 2l$ for all $i \in \{1, 2, \ldots, c\}$, and thus we have $|V(D)| = cr + k$ with $0 \leq k \leq 2l(c-1)$. We proceed the proof by induction on the order p of strongly connected subtournaments. Theorem 2.3 yields that

$$\begin{aligned} \kappa(D) &\geq \frac{cr + k - \alpha(D) - 2i_l(D)}{3} \geq \frac{(c-1)r - 2i_g(D)}{3} \\ &\geq \frac{(l+1)r - 2l}{3} \geq \frac{3l + 3 - 2l}{3} = 1 + \frac{l}{3} > 1. \end{aligned}$$

This implies that D is strongly connected. Hence, according to Theorem 2.2, there exists a 3-cycle in D, which is a strong subtournament of order 3.

Now, let $c \geq l + 3$ and T_p be a strong subtournament of order p with $3 \leq p \leq c - l$. Suppose that D does not contain a strong subtournament of order $p + 1$. Without loss of generality we assume that $T_p = D[\{v_1, v_2, \ldots, v_p\}]$ with $v_i \in V_i$ for $i = 1, 2, \ldots, p$. If there is a vertex $z \in V_{p+1}, V_{p+2}, \ldots, V_c$ such that z has an inner and an outer neighbor in T_p, then $D[\{z, v_1, v_2, \ldots, v_p\}]$ is a strong subtournament of order $p + 1$, a contradiction. If such a vertex does not exist, then let $V_i' \subseteq V_i$ and $V_i'' = V_i - V_i'$ such that $V(T_p) \to V_i'$ when $V_i' \neq \emptyset$, and $V_i'' \to V(T_p)$ when $V_i'' \neq \emptyset$, for $i = p+1, p+2, \ldots, c$. In addition, we define $V' = V_{p+1}' \cup V_{p+2}' \cup \ldots \cup V_c'$ and $V'' = V_{p+1}'' \cup V_{p+2}'' \cup \ldots \cup V_c''$. Now we distinguish two cases.

Case 1. Let $V' \neq \emptyset$ and $V'' \neq \emptyset$. If there is an arc xy with $x \in V'$ and $y \in V''$, then $D[\{x, y, v_1, v_2, \ldots, v_p\}]$ is a strong subtournament of order $p + 2$. As a consequence of Theorem 2.1, we see immediately that there also exists a strong subtournament of order $p + 1$, a contradiction. Hence, we conclude that $V'' \leadsto V'$. Furthermore, let $R = V(D) - (V' \cup V'' \cup V(T_p))$ and $|V_i'| = t_i$ for $p + 1 \leq i \leq c$, and without loss of generality, we assume that $t_{p+1} \geq t_{p+2} \geq \ldots \geq t_c$.

Subcase 1.1. Let $V''_c \neq \emptyset$. In this case, we choose the index s such that

$$\begin{cases} t_s \geq 2 \wedge t_{s+1} \leq 1, & \text{if} \quad t_{p+1} \geq 2 \wedge t_c \leq 1 \\ s = c-1 \quad\quad\quad , & \text{if} \quad t_c \geq 2 \\ s = p+1 \quad\quad\quad , & \text{if} \quad t_{p+1} \leq 1 \end{cases}.$$

Let $v \in V(D')$ with $D' = D[V'_{p+1} \cup V'_{p+2} \cup \ldots \cup V'_s]$ such that v is of maximum indegree in D'. Furthermore let $w \in V(D'')$ with $D'' = D[V''_{s+1} \cup V''_{s+2} \cup \ldots \cup V''_c]$ a vertex of maximum outdegree in D''. Since each of the vertex-sets $V'_{s+1}, V'_{s+2}, \ldots, V'_c$ consists of at most one vertex (for the case that $s \neq c-1$), and because of $r \geq 3$, each of the vertex-sets $V''_{s+1}, V''_{s+2}, \ldots, V''_c$ (for $s \neq c-1$) has to consist of at least two vertices. Hence, according to the choice of the parameter s, Lemma 2.7 yields that $d^-_{D'}(v) \geq s-p-1$ and $d^+_{D''}(w) \geq c-s-1$ (even if $s = c-1$). Let $v \in V_i$ and $w \in V_j$. If $|V_j| = r+b$ and $d^-(w) = \frac{|V(D)|-r-a}{2}$, then it follows that $d^+(w) = |V(D)| - r - b - d^-(w) = \frac{|V(D)|-r+a-2b}{2}$ and because of $i_g(D) \leq l$ we arrive at $d^+(v) \geq \frac{|V(D)|-r+a-2(b+l)}{2}$. Summarizing our results we observe that

$$|N^+(v) \cap R| \geq \max\left\{0, \frac{|V(D)|-r+a-2(b+l)}{2} - \sum_{\substack{m=p+1 \\ m\neq i}}^{c} (t_m) + s - p - 1\right\} \quad (1)$$

and

$$|N^-(w) \cap R| \geq \max\left\{0, \frac{|V(D)|-r-a}{2} - \sum_{\substack{m=p+1 \\ m\neq j}}^{c} (r - t_m) - s_1 + c - s - 1\right\} \quad (2)$$

with $0 \leq s_1 \leq \min\{k-b, 2l(c-p-1)\}$ such that $|(V' \cup V'') - V_j| = (c-p-1)r + s_1$. If $|R| = pr - p + s_2$, then we observe that $0 \leq s_2 \leq \min\{k-b, 2lp\}$ and $s_1 + s_2 \leq k-b$. Because of $t_i \geq t_j$, $s_1 + s_2 \leq k-b$, $p \geq 3$ and $c - p \geq l$, (1) and (2) imply that

$$\begin{aligned} |N^+(v) \cap R| + |N^-(w) \cap R| &\geq (c-1)r + k - b - l + t_i - t_j - s_1 - (c-p-1)r \\ &\quad +c - s - 1 + s - p - 1 \\ &\geq pr + k - b - s_1 - 2 \geq pr + s_2 - 2 \\ &\geq pr - p + s_2 + 1 = |R| + 1. \end{aligned}$$

Hence, there exists a vertex $x \in ((N^+(v) \cap R) \cap (N^-(w) \cap R))$. Without loss of generality, let $x \in V_1$. Since $V(T_p) \to v$ and $w \to V(T_p)$, and since v and w are in different partite sets, we conclude that $D[\{v, x, w, v_3, v_4, \ldots, v_p\}]$ is a strongly connected tournament of order $p+1$, a contradiction.

Subcase 1.2. Let $V''_c = \emptyset$. This implies $V'_c = V_c$ and $t_{p+1} \geq t_{p+2} \geq \ldots \geq t_c = |V_c| \geq r$. If $|V'| = (c-p)r + l_1$ and $|V''| = l_2$, then it follows that $1 \leq l_1 + l_2 \leq \min\{k, 2l(c-p)\}$. Let $w \in V''_{j_{max}}$ with $j_{max} \in \{p+1, p+2, \ldots, c\}$ such that $|V''_{j_{max}}| =: t''_{max}$ is maximal. According to Lemma 2.7, there is a vertex $v \in V' - V''_{j_{max}}$ such that $d^+_{D[V']}(v) \geq c - p - 2 \geq l - 2$. If $|V(v)| = r + b$ and $d^+(v) = \frac{|V(D)|-r-a}{2}$, then, analogously as in Subcase 1.1, we see that $d^-(w) \geq \frac{|V(D)|-r+a-2(b+l)}{2}$, and we conclude that

$$|N^+(v) \cap R| \geq \max\left\{0, \frac{|V(D)|-r-a}{2} - (c-p-1)r - l_1 + b - t''_{max} + l - 2\right\} \quad (3)$$

and

$$|N^-(w) \cap R| \geq \max\left\{0, \frac{|V(D)|-r+a-2(b+l)}{2} - l_2 + t''_{max}\right\}. \quad (4)$$

If $|R| = pr - p + s_2$, then it follows that $0 \leq s_2 \leq \min\{k, 2lp\}$ and $s_2 + l_1 + l_2 \leq k$. Analogously as in Subcase 1.1, we obtain by (3) and (4) that $|N^+(v) \cap R| + |N^-(w) \cap R| > |R|$. Hence, again there exists a vertex $x \in ((N^+(v) \cap R) \cap (N^-(w) \cap R))$. If, without loss of generality, $x \in V_1$, then, since v and w are in different partite sets, $D[\{v, x, w, v_3, v_4, \ldots, v_p\}]$ is a strong subtournament of order $p+1$, a contradiction.

Case 2. Let $V' = \emptyset$ or $V'' = \emptyset$. Without loss of generality, we discuss the case that $V'' = \emptyset$. This implies that $V'_i = V_i$ for $p+1 \leq i \leq c$, and we write V instead of V'. Let U contain all vertices of $V(D) - (V \cup V(T_p))$ that are dominated by at least one vertex of V, and let W be the set of vertices from $V(D) - (V \cup V(T_p))$ which are not dominated by any vertex from V. Thus, $W \to V$ and hence it follows that $d(V, V(D)-V) \leq |V||U|$ and $d(V(D)-V, V) \geq |V||V(D)-(U \cup V)|$. Now Lemma 2.6 yields that $l|V| \geq d(V(D)-V, V) - d(V, V(D)-V) \geq |V|(|V(D)| - |V| - 2|U|)$, and this implies that

$$|U| \geq \frac{|V(D)| - |V| - l}{2}. \quad (5)$$

We now consider the following two subcases.

Subcase 2.1. Let $p = 3$. Consider that V consists of $c - p \geq l$ partite sets.

Suppose firstly that there is a vertex $w \in W$ that dominates two vertices of $V(T_p)$. This implies $w \leadsto U$, since otherwise let $v' \in V$ and $u \in U$ such that $v' \to u \to w$. In this case v', u, w and the vertex of $V(T_p)$, which is dominated by w and is in another partite set than u induce a strong tournament of order 4, a contradiction. According to Lemma 2.7, there is a vertex $v' \in V$ such that $d^-_{D[V]}(v') \geq l-1$. Hence, we arrive at $d^-(v') \geq l - 1 + |W| + |V(T_p)| = |W| + l + 2$ and $d^-(w) \leq |W - \{w\}| = |W| - 1$, a contradiction to $i_g(D) \leq l$. Since each vertex $w \in W$ has exactly two neighbors in $V(T_p)$, we conclude that $d(W, V(T_p)) \leq d(V(T_p), W)$.

Now let there be a vertex $u \in U$ that dominates two vertices in $V(T_p)$. This yields that u, a vertex of v which dominates u and the two vertices of $V(T_p)$ not belonging to the same partite set as u induce a strongly connected subtournament of order 4, a contradiction. Analogously as for the set W, we conclude that $d(U, V(T_p)) \leq d(V(T_p), U)$.

Together with Lemma 2.6 we obtain

$$\begin{aligned}
3l = l|V(T_p)| &\geq d(V(T_p), V(D) - V(T_p)) - d(V(D) - V(T_p), V(T_p)) \\
&= d(V(T_p), V) + d(V(T_p), U) + d(V(T_p), W) \\
&\quad - d(V, V(T_p)) - d(U, V(T_p)) - d(W, V(T_p)) \\
&\geq d(V(T_p), V) = |V(T_p)||V| \geq 9l,
\end{aligned}$$

a contradiction.

Subcase 2.2 Let $p \geq 4$. According to Lemma 2.4, there is a vertex $v \in V(T_p)$ such that for all $y \in V(T_p) - \{v\}$ the subtournament $T_p - \{y\}$ contains a Hamiltonian path with the initial vertex v. If there is a vertex $u \in U$ such that $u \to v$, then let $w \in V$ with $w \to u$. If $u \in V_t$, then w, u and v_j with $1 \leq j \leq p$ and $j \neq t$ induce a strongly connected subtournament of order $p+1$, a contradiction. If otherwise, there is no such vertex u, then clearly $v \leadsto U$. By Lemma 2.4, the vertex v is of maximum outdegree in T_p and thus $d^+_{D[V(T_p)]}(v) \geq 2$. If $v \in V_i$ with $|V_i| = r + 2l - m$ $(0 \leq m \leq 2l)$, then, because of $|V| \geq lr$, $r \geq 3$, (5) and Lemma 2.8, we arrive at the contradiction

$$
\begin{aligned}
\frac{|V(D)| - |V_i| + m}{2} &\geq d^+(v) \geq |V| + |U - (V_i - \{v\})| + d^+_{D[V(T_p)]}(v) \\
&\geq |V| + \frac{|V(D)| - |V| - l}{2} - |V_i| + 3 \\
&= \frac{|V(D)| - |V_i| + m + 1}{2} + \frac{|V| - |V_i| - m - l + 5}{2} \\
&\geq \frac{|V(D)| - |V_i| + m + 1}{2} + \frac{(l-1)r - 3l + 5}{2} \\
&> \frac{|V(D)| - |V_i| + m + 1}{2} + \frac{(l-1)(r-3)}{2} \\
&\geq \frac{|V(D)| - |V_i| + m + 1}{2}.
\end{aligned}
$$

This completes the proof of the theorem. □

If we omit the condition of Theorem 1.5 that there are at least three vertices in each partite set, then the proof becomes much more complicated. Nevertheless, we believe that also in this case the theorem remains valid, if D is strongly connected and the number of partite sets is sufficiently large.

Conjecture 3.1 *Let D be a strongly connected c-partite tournament with c sufficiently large and $i_g(D) \leq l$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c - l + 1\}$.*

If Conjecture 3.1 is valid, then the bounds for c and p as in Theorem 1.5 would be best possible as the following example demonstrates.

Example 3.2 *Let $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$ and $V_4 = \{v_4, v_4'\}$ be the partite sets of the multipartite tournament D such that $v_1 \to v_2 \to v_3 \to v_4' \to v_2 \to v_4 \to v_1 \to v_3$, $v_1 \to v_4'$ and $v_4 \to v_3$ (see also Figure 2). Then we observe that D is a strongly connected c-partite tournament with $i_g(D) = l = 2$, $c = 4 = l + 2$ and without any strong subtournament of order $4 = c - l + 2$.*

Even if we enlarge the number c of partite sets of a multipartite tournament D, then there is not always a strong subtournament of order $c - i_g(D) + 2$, which can be seen in Example 1.4.

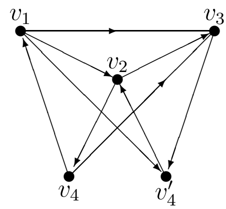

Figure 2: A 4-partite tournament with $i_g(D) = 2$ and without a strong subtournament of order 4.

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