AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **29** (2004), Pages 99–102

A new lower bound for the harmonious chromatic number

Duncan Campbell   Keith Edwards

Division of Applied Computing
University of Dundee
Dundee, DD1 4HN
U.K.

Abstract

A *harmonious colouring* of a simple graph G is a proper vertex colouring such that each pair of colours appears together on at most one edge. The *harmonious chromatic number h(G)* is the least number of colours in such a colouring. We obtain a new lower bound for the harmonious chromatic number of general graphs, in terms of the independence number of the graph, generalizing results of Moser [2].

1 Introduction

A *harmonious colouring* of a simple graph G is a proper vertex colouring such that each pair of colours appears together on at most one edge. Formally a harmonious colouring is a function c from a colour set C to the set $V(G)$ of vertices of G such that for any edge e of G, with endpoints x, y say, $c(x) \neq c(y)$, and for any pair of distinct edges e, e', with endpoints x, y and x', y' respectively, then $\{c(x),\ c(y)\} \neq \{c(x'),\ c(y')\}$. The *harmonious chromatic number h(G)* is the least number of colours in such a colouring. For a survey of results on the harmonious chromatic number, see [1].

It is intuitively clear that if the independence number $\alpha(G)$ of a graph G is very small, then the harmonious chromatic number must be near to the number of vertices of the graph. For example, in an extreme case when the independence number is two, we cannot have two colour classes of size two or more. Indeed, there is at most once edge between two such classes, and thus at least three of the vertices would be independent. Using similar arguments, Moser [2] gave results which imply lower bounds for the harmonious chromatic number of the form $|V(G)|$ minus a constant for the cases of graphs with independence number up to 4.

Here we give a general result, showing that for a class of graphs with bounded independence number, the harmonious chromatic number is at least the number of vertices minus a constant. The constant achieved is exponential in $\alpha(G)$; that this is necessary is shown by modifying an example due to Gyárfás, quoted by Moser.

2 Lower bound

Lemma 2.1 *Let G be a graph with n vertices, and suppose that G has a harmonious colouring with k colours. If $n - k \geq 2^r$, where r is a non-negative integer, then G has an independent set of size at least $r + 2$.*

Proof We prove the result by induction on r. If $r = 0$, then $n - k \geq 1$. It follows that $n \geq 2$, and that G has a harmonious k-colouring, where $k < n$. Therefore there must be a colour class containing at least 2 vertices, which is the required independent set of size 2.

Therefore, suppose that $r \geq 1$. Denote the set of colour classes by S. Let m be the size of the largest colour class, which implies that $m \geq 2$, and choose a colour class C_0 of size m. If $m \geq r + 2$, then this immediately gives an independent set of size $r + 2$, therefore we assume that $m \leq r + 1$. If C_0 is the only colour class, then $m = n \geq 2^r + 1 \geq r + 2$, since $r \geq 1$, hence again there is an independent set of size at least $r + 2$. Thus we may assume that there is at least one other colour class. Let $C_0 = \{v_1, \ldots, v_m\}$, and for $i = 1, \ldots, m$, let

$$A_i = \{C : C \in S \text{ and } (v_i, w) \in E(G) \text{ for some } w \in C\}.$$

Thus A_i is the set of colour classes which are joined to v_i. Also set

$$A_0 = \{C : C \in S \setminus \{C_0\} \text{ and } (v, w) \notin E(G) \text{ for any } v \in C_0, w \in C\}.$$

A_0 is the set of colour classes not joined to any v_i. Note that because the colouring is harmonious, the sets A_i are pairwise disjoint. Now, for $i = 0, \ldots, m$, let

$$B_i = \bigcup A_i.$$

Hence B_i is the union of the colour classes joined to v_i, if $i \geq 1$, and B_0 is the union of the colour classes not joined to any v_i. Let

$$n_i = |B_i| - |A_i|.$$

It follows that

$$\sum_{i=0}^{m} n_i = \sum_{i=0}^{m} |B_i| - \sum_{i=0}^{m} |A_i| = (n - m) - (k - 1).$$

Therefore

$$\sum_{i=1}^{m} n_i = (n - m) - (k - 1) - n_0,$$

hence for some I with $1 \leq I \leq m$, we have

$$n_I \geq \frac{n - m - k + 1 - n_0}{m} \quad .$$

It follows that

$$n_I + n_0 \geq \frac{n - m - k + 1}{m} + \frac{(m-1)n_0}{m} \geq \frac{n - m - k + 1}{m} \quad .$$

Now let G' be the subgraph of G induced by the vertices of $B_I \cup B_0$. Then the colouring of G induces a harmonious colouring of G' using the colours of $A_I \cup A_0$. Let $n' = |V(G')| = |B_I| + |B_0|$, and let $k' = |A_I| + |A_0|$, hence G' has a harmonious k'-colouring. Then

$$n' - k' = n_I + n_0 \geq \left\lceil \frac{n - m - k + 1}{m} \right\rceil.$$

Now $n - k \geq 2^r$, hence

$$n' - k' \geq \left\lceil \frac{2^r - m + 1}{m} \right\rceil.$$

Then since $m \leq r + 1$, we have $r - m + 1 \geq 0$ and hence 2^{r-m+1} is an integer. Therefore

$$\left\lceil \frac{2^r - m + 1}{m} \right\rceil \geq 2^{r-m+1}$$

if

$$\frac{2^r - m + 1}{m} > 2^{r-m+1} - 1.$$

This follows if

$$\frac{2^r + 1}{m} > \frac{2^r}{2^{m-1}}.$$

But this is true provided that $2^{m-1} \geq m$, which is true since $m \geq 1$. Hence we conclude that $n' - k' \geq 2^{r-m+1}$. Since $m \geq 2$, by induction it follows that G' has an independent set of size at least $r - m + 3$. Now observe that none of the $m - 1$ vertices of $C_0 \setminus \{v_I\}$ is joined to any vertex of G'. Thus adding these gives an independent set in G of size at least $r + 2$, as required.

We can now prove the lower bound on the harmonious chromatic number.

Theorem 2.2 *Let G be a graph with n vertices and independence number $\alpha(G)$. Then*

$$h(G) \geq n - 2^{\alpha(G)-1} + 1.$$

Proof Consider any harmonious colouring of G, and suppose it has k colours. Set $r = \alpha(G) - 1$. Now if $n - k \geq 2^r$, then by Lemma 2.1, G would have an independent set of size at least $r + 2 = \alpha(G) + 1$, which is impossible. Hence $n - k \leq 2^r - 1$, or $k \geq n - 2^r + 1$. Since the colouring was arbitrary, we have

$$h(G) \geq n - 2^{\alpha(G)-1} + 1.$$

3 Example

We now give an example to show that the exponential term involving the independence number cannot be avoided.

Theorem 3.1 *Let m be a positive integer. Then there exist arbitrarily large graphs G, with n vertices, satisfying $\alpha(G) \leq 4m$, and*

$$h(G) \leq n - 2^{\alpha(G)/4} + 1.$$

Proof We start with an example, due to Gyárfás, given by Moser [2]. For $m \geq 1$, Gyárfás gives a graph G_m with $2^{m+1} - 2$ vertices, and with $h(G_m) = 2^m - 1$ and $\alpha(G_m) \leq 4m - 1$. Now let $G = G_m \cup K_N$, where N is a positive integer. Then we have $h(G) \leq h(G_m) + N$, and $\alpha(G) = \alpha(G_m) + 1 \leq 4m$. Also n, the number of vertices of G, is $2^{m+1} - 2 + N$.

Hence

$$\begin{aligned} n - h(G) &= 2^{m+1} - 2 + N - h(G) \\ &\geq 2^{m+1} - 2 + N - (h(G_m) + N) \\ &= 2^{m+1} - 2 - (2^m - 1) \\ &= 2^m - 1 \\ &\geq 2^{\alpha(G)/4} - 1. \end{aligned}$$

Hence $h(G) \leq n - 2^{\alpha(G)/4} + 1$. Since N is arbitrary, the result follows.

References

[1] K.J. Edwards, The harmonious chromatic number and the achromatic number, In: R.A. Bailey, ed., Surveys in Combinatorics 1997 (Invited papers for 16th British Combinatorial Conference) (Cambridge University Press, Cambridge, 1997) 13–47.

[2] D.E. Moser, Mixed Ramsey numbers: harmonious chromatic number versus independence number, Journal of Combinatorial Mathematics and Combinatorial Computing 25 (1997) 55–63

(Received 27 Sep 2002)