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On the size of odd order graphs with no almost perfect matching

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Abstract

A graph G is a $(d, d+1)$-graph if the degree of each vertex of G is either d or $d+1$. If $d \geq 2$ is an integer and G a $(d, d+1)$-graph with exactly one odd component and with no almost perfect matching, then we show in this paper that $|V(G)| \geq 4(d+1)+1$ and $|V(G)| \geq 4(d+1)+3$ when d is odd. This result generalizes corresponding statements by C. Zhao (*J. Combin. Math. Combin. Comput.* 9 (1991), 195–198) and W.D. Wallis (*Ars Combin.* 11 (1981), 295–300) on the size of even order graphs without a perfect matching. Examples will show that the given bounds are best possible, and some related results are also presented.

We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [2]). In this paper, all graphs are finite and simple. The vertex set and edge set of a graph are denoted by $V(G)$ and $E(G)$, respectively. The *neighborhood* $N_G(x) = N(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = d(x) = |N(x)|$ is the *degree* of x. If $d \leq d_G(x) \leq d+k$ for each vertex x in a graph G, then we speak of a $(d, d+k)$-*graph*. If M is a matching in a graph G with the property that every vertex (with exactly one exception) is incident with an edge of M, then M is a *perfect matching* (an *almost perfect matching*). We denote by K_n the complete graph of order n and by $K_{r,s}$ the complete bipartite graph with partite sets A and B, where $|A| = r$ and $|B| = s$. If G is a graph and $A \subseteq V(G)$, then we denote by $q(G-A)$ the number of odd components in the subgraph $G-A$.

The proof of our main theorem is based on the following generalization of Tutte's famous 1-factor theorem [4] by Berge [1] in 1958, and we call it the theorem of Tutte-Berge (for a proof see e.g., [5]).

Theorem of Tutte-Berge (Berge [1] 1958) Let G be a graph of order n. If M is a maximum matching of G, then

$$n - 2|M| = \max_{A \subseteq V(G)} \{q(G-A) - |A|\}.$$

Theorem 1 Let $d \geq 2$ be an integer, and let G be a $(d, d+1)$-graph with exactly one odd component and without any almost perfect matching. Then,

$$|V(G)| \geq 4(d+1)+1, \tag{1}$$

$$|V(G)| \geq 4(d+1)+3, \quad \text{if } d \geq 3 \text{ is odd or } d=2 \text{ and } G \text{ is connected} \tag{2}$$

$$|V(G)| \geq 4(d+1)+5, \quad \text{if } d \geq 3 \text{ is odd and } G \text{ is connected.} \tag{3}$$

Proof. Suppose to the contrary that there exists a $(d, d+1)$-graph G with exactly one odd component and without any almost perfect matching of size $|V(G)| \leq 4(d+1)-1$, $|V(G)| \leq 4(d+1)+1$ when $d \geq 3$ is odd or $d=2$ and G is connected, and $|V(G)| \leq 4(d+1)+3$ when $d \geq 3$ is odd and G is connected, respectively.

By the hypothesis and the theorem of Tutte-Berge, there exists a non-empty set $A \subseteq V(G)$ such that $q(G-A) \geq |A|+3$. We call an odd component of $G-A$ large if it has more than d vertices and small otherwise. If we denote by α and β the number of large and small components, respectively, then we observe that

$$\alpha+\beta \geq |A|+3, \tag{4}$$

$$|V(G)| \geq |A|+\beta+\alpha(d+1), \tag{5}$$

$$|V(G)| \geq |A|+\beta+\alpha(d+2), \quad \text{if } d \geq 3 \text{ is odd.} \tag{6}$$

In the case that $d=2$ and G is connected, the assumption $|V(G)| \leq 13$ and (5) lead to $\alpha \leq 4$. However, if $\alpha=4$, then the connectivity of G implies $|A| \geq 2$, and (4) and (5) yield the contradiction $|V(G)| \geq |A|+\beta+\alpha(d+1) \geq 14$. In all other cases, the assumptions $|V(G)| \leq 4(d+1)-1$ and $|V(G)| \leq 4(d+1)+3$ when $d \geq 3$ is odd, the inequalities (5) and (6) immediately imply $\alpha \leq 3$. Thus, in all cases, it follows from the inequality (4) that $\beta \geq |A|$.

It is easy to see that there are at least d edges of G joining each small component of $G-A$ with A, and at least one per large component with one possible exception, if G is not connected.

If $\alpha=0$, then, since G is a $(d, d+1)$-graph, we conclude that $d\beta \leq |A|(d+1)$, and (4) implies $\beta \geq |A|+3$. Hence, $d(|A|+3) \leq |A|(d+1)$ and so, it holds $|A| \geq 3d$. In view of (5), we arrive at the contradictions

$$|V(G)| \geq |A|+\beta \geq |A|+|A|+3 \geq 6d+3 \geq 4(d+1)+3$$

and

$$|V(G)| \geq |A|+\beta \geq 6d+3 \geq 4(d+1)+5, \quad \text{if } d \geq 3.$$

Altogether, we see that there remain to investigate the cases

$$1 \leq \alpha \leq 3 \quad \text{and} \quad \beta \geq |A|. \tag{7}$$

As G is a $(d, d+1)$-graph, it follows that

$$\alpha-1+d\beta \leq |A|(d+1) \tag{8}$$

and thus,

$$\beta \leq |A| + \frac{\beta + 1 - \alpha}{d+1}. \tag{9}$$

For the proofs of the bounds (1), (2), and (3), we distinguish two cases.

Case 1. Let $\beta + 1 - \alpha \leq d$. According to (9), this implies $\beta \leq |A|$, and therefore, (7) yields $\beta = |A|$. In view of (4) and (7), it follows that $\alpha = 3$. Let U be a small component. Since $N(x) \subseteq V(U) \cup A$ for $x \in V(U)$, we observe that $|A| + |V(U)| \geq d + 1$. In addition, we deduce from $\beta = |A|$ and inequality (9) that $\beta \geq \alpha - 1 = 2$, and so, we obtain instead of (5) the estimate

$$\begin{aligned} |V(G)| &\geq |A| + |V(U)| + \beta - 1 + \alpha(d+1) \\ &\geq d + 1 + 1 + 3(d+1) = 4(d+1) + 1. \end{aligned}$$

In connection with inequality (1), this is a contradiction to our assumption.

If $d \geq 3$ is odd, then the following estimate leads to a contradiction to our assumption

$$\begin{aligned} |V(G)| &\geq |A| + |V(U)| + \beta - 1 + \alpha(d+2) \\ &\geq d + 1 + 1 + 3(d+2) = 4(d+1) + 4. \end{aligned}$$

If $d = 2$ and G is connected, then necessarily $\beta = |A| \geq 3$, and (5) yields the contradiction $|V(G)| \geq 15$.

Case 2. Let $\beta + 1 - \alpha \geq d + 1$. Consequently, $\beta \geq \alpha + d$ and (8) implies

$$|A| \geq \frac{\alpha - 1 + d\beta}{d+1} \geq \frac{\alpha - 1 + d(\alpha + d)}{d+1}.$$

If $\alpha = 1$, then we deduce that $|A| \geq d$, and if $\alpha \geq 2$ then we obtain $|A| \geq d + 1$. In the case $\alpha \geq 2$, inequality (5) leads to

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+1) \\ &\geq (d+1) + (\alpha + d) + 2(d+1) \geq 4(d+1) + 1. \end{aligned}$$

If $\alpha = 1$, then (3) yields $\beta \geq |A| + 2$. In view of (8), we obtain $d(|A| + 2) \leq |A|(d+1)$ and so, $|A| \geq 2d$. Hence, it follows together with (5) that

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+1) \\ &\geq |A| + (|A| + 2) + (d+1) \geq 5d + 3 \geq 4(d+1) + 1. \end{aligned}$$

With regard to (1), in both cases $\alpha \geq 2$ and $\alpha = 1$, we have a contradiction and thus, the estimate (1) is proved.

Subcase 2.1. Let $d = 2$, and let G be connected. Since G is connected, we deduce that $\alpha + 2\beta \leq 3|A|$, and this implies $\beta \leq |A| + (\beta - \alpha)/3$.

Subcase 2.1.1. Let $\beta = \alpha + 2$. Then, we conclude from (7) that $\beta = |A|$. Combining this with (4), we arrive at $\alpha = 3$. Consequently, (5) yields the contradiction $|V(G)| \geq 19$.

Subcase 2.1.2. Let $\beta \geq \alpha + 3$. It follows from $\alpha + 2\beta \leq 3|A|$ that

$$|A| \geq \left(\frac{\alpha + 2\beta}{3}\right) \geq \frac{\alpha + 2\alpha + 6}{3} = \alpha + 2.$$

In the case $\alpha \geq 2$, inequality (5) implies the contradiction $|V(G)| \geq 2\alpha + 5 + 3\alpha \geq 15$. If $\alpha = 1$, then, inequality (4) implies $\beta \geq |A| + 2$. Because of $2\beta \leq 3|A| - 1$, we obtain $|A| \geq 5$ and therefore, we arrive at the contradiction $|V(G)| \geq |A| + \beta + 3 \geq 15$.

Subcase 2.2. Let $d \geq 3$ odd.

In the case $\alpha = 3$, we deduce from inequality (6) that

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+2) \\ &\geq (d+1) + (d+3) + 3(d+2) \geq 4(d+1) + 5. \end{aligned}$$

This is a contradiction to our assumption, and thus, the estimates (2) and (3) are completely proved for $d \geq 3$ odd and $\alpha = 3$.

If $\alpha = 2$, then, with regard to the bound (2), inequality (6) yields the contradiction

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+2) \\ &\geq (d+1) + (d+2) + 2(d+2) = 4(d+1) + 3. \end{aligned}$$

If in additional, G is connected, then we have $\alpha + d\beta \leq |A|(d+1)$, and hence, we obtain

$$|A| \geq \frac{\alpha + d\beta}{d+1} \geq \frac{\alpha + d(\alpha + d)}{d+1} = \frac{d(d+2)+2}{d+1} > d + 1.$$

Since $|A|$ is an integer, it follows that $|A| \geq d + 2$. Combining this with $\beta \geq |A| + 1$ and inequality (6), we arrive at the contradiction

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+2) \geq |A| + |A| + 1 + 2(d+2) \\ &\geq 2(d+2) + 1 + 2(d+2) = 4(d+1) + 5. \end{aligned}$$

If $\alpha = 1$, then, (4) leads to $\beta \geq |A| + 2$. Together with (8), we conclude that $d(|A|+2) \leq |A|(d+1)$, and thus, $|A| \geq 2d$. With regard to the bound (2), inequality (6) yields the contradiction

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+2) \\ &\geq 2|A| + 2 + (d+2) \geq 5d + 4 \geq 4(d+1) + 3. \end{aligned}$$

If in additional, G is connected, then we have $\alpha + d\beta \leq |A|(d+1)$. This implies $d(|A|+2) \leq d\beta \leq |A|(d+1) - 1$ and hence, $|A| \geq 2d + 1$. This finally gives the desired contradiction

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+2) \\ &\geq 2|A| + 2 + (d+2) \geq 4(d+1) + 5. \quad \square \end{aligned}$$

Note that each (1,2)-graph with exactly one odd component contains an almost perfect matching. The following examples will show that the bounds in Theorem 1 are best possible.

Example 2 *Case 1.* Let d be even. Let H_1, H_2, H_3, and H_4 be four copies of the complete graph K_{d+1} with the vertex sets $V(H_i) = \{x_1^i, x_2^i, \ldots, x_{d+1}^i\}$ for $i = 1, 2, 3, 4$, and let u be a further vertex. Now we define G' as the disjoint union of H_1, H_2, H_3, H_4 and the vertex u.

If $d = 2$, then let G be the graph G' together with the three edges ux_1^1, ux_1^2, ux_1^3.

If $d = 4p$, then let G consists of the graph G' together with the edge sets $\{ux_1^i, ux_2^i, \ldots, ux_p^i\}$ for $i = 1, 2, 3, 4$.

If $d = 4p + 2$, then let G consists of the graph G' together with the edge sets $\{ux_1^i, ux_2^i, \ldots, ux_{p+1}^i\}$ for $i = 1, 2$ and $\{ux_1^i, ux_2^i, \ldots, ux_p^i\}$ for $i = 3, 4$.

In all cases, G is a $(d, d + 1)$-graph of order $|V(G)| = 4(d + 1) + 1$ with exactly one odd component and without an almost perfect matching. Thus, inequality (1) is best possible.

Case 2. Let $d = 2$, and let G be connected. Let $w_1w_2w_3w_1$, $x_1x_2x_3x_1$, $y_1y_2y_3y_1$ be three triangles, and let $u_1v_1u_2v_2u_3v_3u_1$ be a cycle of length six. Let G be the disjoint union of these four graphs together with the three edges u_1w_1, u_2x_2, and u_3y_3. Then, G is a connected $(2, 3)$-graph of order $|V(G)| = 15$ without an almost perfect matching. Consequently, inequality (2) is best possible in this case.

Case 3. Let $d \geq 3$ be odd. Let $K_{d+1,d+2}$ be the complete bipartite graph with the partite sets $\{x_1, x_2, \ldots, x_{d+1}\}$ and $\{y_1, y_2, \ldots, y_{d+2}\}$. If we delete in the graph $K_{d+1,d+2}$ the edges $x_1y_1, x_2y_2, \ldots, x_{d+1}y_{d+1}$ and x_1y_{d+2}, then we denote the resulting graph by F. Furthermore, let $H = K_{d+2} - wz$, where wz is an arbitrary edge in the graph K_{d+2}. If G is the disjoint union of F, H, and K_{d+2} together with the edge wx_1, then G is a $(d, d+1)$-graph of order $|V(G)| = 4(d+1)+3$ with exactly one odd component and without an almost perfect matching. Thus, inequality (2) is best possible for $d \geq 3$ odd.

Case 4. Let $d \geq 3$ be odd, and let G be connected. Let H_1, H_2, H_3, and H_4 be four copies of the graph $K_{d+2} - M$, where M is an almost perfect matching of the complete graph K_{d+2}, and let u be a further vertex. We denote the vertex sets of H_i by $V(H_i) = \{x_1^i, x_2^i, \ldots, x_{d+2}^i\}$ such that $d_{H_i}(x_{d+2}^i) = d + 1$ for $i = 1, 2, 3, 4$. Now we define G' as the disjoint union of H_1, H_2, H_3, H_4 and the vertex u.

If $d = 4p + 1$, then let G consists of the graph G' together with the edge sets $\{ux_1^1, ux_2^1, \ldots, ux_{p+1}^1\}$ and $\{ux_1^i, ux_2^i, \ldots, ux_p^i\}$ for $i = 2, 3, 4$.

If $d = 4p + 3$, then let G consists of the graph G' together with the edge sets $\{ux_1^i, ux_2^i, \ldots, ux_{p+1}^i\}$ for $i = 1, 2, 3$ and $\{ux_1^4, ux_2^4, \ldots, ux_p^4\}$.

In both cases, G is a connected $(d, d + 1)$-graph of order $|V(G)| = 4(d + 1) + 5$ without an almost perfect matching. Thus, inequality (3) is best possible.

Corollary 3 (Zhao [8] 1991) Let $d \geq 2$ be an integer. If a $(d, d + 1)$-graph G has no odd component and no perfect matching, then

$$|V(G)| \geq 3d + 4.$$

Proof. Suppose to the contrary that there exists a graph G with no odd component and no perfect matching of size $|V(G)| \leq 3d + 3$.

If d is even, then the disjoint union $H = G \cup K_{d+1}$ is a $(d, d+1)$-graph with exactly one odd component, but H has no almost perfect matching. Because of $|V(H)| \le 4(d+1)$, this is a contradiction to inequality (1).

If d is odd, then the disjoint union $H = G \cup K_{d+2}$ is a $(d, d+1)$-graph with exactly one odd component, but H has no almost perfect matching. Because of $|V(H)| \le 4(d+1)+1$, this is a contradiction to inequality (2). □

Corollary 4 (Wallis [6] 1981) Let $d \ge 3$ be an integer. If a d-regular graph G has no odd component and no perfect matching, then $|V(G)| \ge 3d+4$.

Note that each 1-regular and 2-regular graph without an odd component has a perfect matching. Furthermore, if d is odd or $d = 4$ in Corollary 4, then Wallis [6], [7] even has presented the better bounds $|V(G)| \ge 3d+7$ or $|V(G)| \ge 3d+10 = 22$, respectively.

If G is a regular graph, then, in two special cases, we are able to improve Theorem 1 slightly.

Theorem 5 Let G be d-regular graph with exactly one odd component and without any almost perfect matching. Then,

$$|V(G)| \ge 4(d+1)+7, \quad \text{if } \ d=4,$$

$$|V(G)| \ge 4(d+1)+9, \quad \text{if } \ d=6 \ \text{ and } \ G \ \text{ is connected.}$$

Proof. We use the same notations as in the proof of Theorem 1. Suppose to the contrary that there exist such graphs G with $|V(G)| \le 4(d+1)+5 = 25$ when $d=4$ and $|V(G)| \le 4(d+1)+7 = 35$ when $d=6$ and G is connected.

Let $d=4$. One the one hand, inequality (5) shows that $\alpha \le 4$. On the other hand, the handshaking lemma implies $2(\alpha-1)+4\beta \le 4|A|$ and (3) says that $\beta \ge |A|+3-\alpha$. Therefore,

$$4(|A|+3-\alpha) \le 4\beta \le 4|A| - 2\alpha + 2.$$

This leads to the contradiction $\alpha \ge 5$.

Let $d=6$ and let G be connected. On the one hand, inequality (5) shows that $\alpha \le 4$. On the other hand, the handshaking lemma implies $2\alpha + 6\beta \le 6|A|$ and (3) says that $\beta \ge |A|+3-\alpha$. Therefore,

$$6(|A|+3-\alpha) \le 6\beta \le 6|A| - 2\alpha.$$

This leads to the contradiction $\alpha \ge 5$. □

Theorem 5 implies, analogously to the proof of Corollary 3, the above mentioned special case of Wallis [6].

Corollary 6 (Wallis [6] 1981) If a 4-regular graph G has no odd component and no perfect matching, then $|V(G)| \ge 22$.

The following examples will show that the presented bounds in Theorem 5 are also best possible.

Example 7 *Case 1.* Let $d = 4$. Let $H_i = K_5 - x_iy_i$ for $i = 1, 2, 3, 4$, where x_iy_i is an arbitrary edge of the complete graph K_5, $H_5 = K_5$, and let u and v be two further vertices. If G is the disjoint union of H_1, H_2, H_3, H_4, H_5 and the vertices u, v together with the eight edges ux_i and vy_i for $i = 1, 2, 3, 4$, then G is a 4-regular graph with exact one odd component of order $|V(G)| = 4(d + 1) + 7 = 27$ without an almost perfect matching. Thus, Theorem 5 is best possible for $d = 4$.

Case 2. Let $d = 6$. Let $H_i = K_7 - x_iy_i$ for $i = 1, 2, 3, 4$, where x_iy_i is an arbitrary edge of the complete graph K_7, $H_5 = K_7 - \{x_5y_5, x_6y_6\}$, where x_5y_5 and x_6y_6 are two independent edges of the complete graph K_7, and let u and v be two further vertices. If G is the disjoint union of H_1, H_2, H_3, H_4, H_5 and the vertices u, v together with the edges ux_i and vy_i for $i = 1, 2, 3, 4, 5, 6$, then G is a connected 6-regular graph of order $|V(G)| = 4(d+1)+9 = 37$ without an almost perfect matching. Consequently, Theorem 5 is also best possible for $d = 6$.

Similarly to Theorem 1, one can prove the following corresponding result for $(d, d + 2)$-graphs.

Theorem 8 If G is a $(d, d + 2)$-graph with exactly one odd component and without any almost perfect matching, then

$$|V(G)| \geq 3d + 3.$$

The next examples will show that Theorem 8 is also best possible.

Example 9. The disjoint union of the graphs K_{d+1} and $K_{d,d+2}$ if d is even and the disjoint union of the graphs K_{d+2} and $K_{d,d+2}$ if d is odd, are $(d, d + 2)$-graphs with exactly one odd component and without an almost perfect matching of order $3d + 3$ and $3d + 4$, respectively.

Theorem 8 implies, analogously to the proof of Corollary 3, that a $(d, d+2)$-graph G with no odd component and with no perfect matching, satisfies $|V(G)| \geq 2d + 2$. However, if $|V(G)| \leq 2d$, then, by the classical theorem of Dirac [3], the graph G is Hamiltonian, and this statement is stronger than the property that G contains a perfect matching.

The same is valid, if we consider a $(d, d + k)$-graph G of odd order with $k \geq 3$ and without any almost perfect matching. In this case Dirac's theorem leads to $|V(G)| \geq 2d + 3$, and this bound is best possible, as the complete bipartite graph $K_{d,d+3}$ shows.

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