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An eccentric coloring of trees

Christian Sloper

Department of Informatics
University of Bergen
Norway
sloper@ii.uib.no

Abstract

Eccentric coloring is a new variation of coloring, where higher numbered colors cannot be used as freely as lower numbered colors. In addition there is a correspondence between the eccentricity (max distance) of a vertex and the highest legal color for that vertex.

In this note we investigate eccentric coloring of trees. We give the eccentric chromatic number or a bound on the eccentric chromatic number for several simple classes of trees. In particular we show the eccentric chromatic number for paths ($\chi_e = 3$), spiders ($\chi_e = 3$) and caterpillars ($\chi_e \leq 7$).

Further, we discuss the eccentric chromatic number of complete k-ary trees and show that the complete binary trees have eccentric chromatic number $\chi_e \leq 7$. We also show that *large* binary trees are eccentrically colorable and have $\chi_e \leq 7$. We then conclude by showing that no complete k-ary tree, $k \geq 3$, is eccentrically colorable.

1 Introduction

Coloring is one of the most basic and well studied concepts in graph theory. A myriad of papers have been published and coloring remains a popular area of study in the field.

In its most simple form coloring is simply an assignment of symbols to the vertices in a graph such that vertices that share an edge are given different symbols. We will use the natural numbers $\mathbb{N}$ as symbols and strengthen the requirements on the coloring function. We will require that vertices colored with high numbers must be far apart and we will give a restriction on the maximum color of a vertex.

Definition 1 *Distance*
The distance $d(u, v)$ *between two vertices* u, v *in a graph* G*, is the length (number of edges) of the shortest path between* u *and* v *in* G*.*

Definition 2 *Eccentricity*
The eccentricity *of a vertex* v *in a graph* $G = (V, E)$ *is* $e(v) = \max_{u \in V}\{d(v, u)\}$*.*

Definition 3 *Eccentric coloring*
An eccentric coloring *of a graph $G = (V, E)$ is a function $color : V \rightarrow \mathbb{N}$ such that*

(i) $\forall u, v \in V, \quad \big(color(u) = color(v)\big) \Rightarrow d(u, v) > color(u)$

(ii) $\forall v \in V, \quad color(v) \leq e(v)$

A coloring that adheres to rule (i) is known as a broadcast-coloring*[6].*

Definition 4 *Eccentric chromatic number*
The eccentric chromatic *number $\chi_e \in \mathbb{N}$ for a graph G is the lowest number of colors for which it is possible to eccentrically color G by colors: $V \rightarrow \{1, 2, ..., \chi_e\}$.*

As we understand from the definition of eccentric coloring not all graphs can be colored. In this paper we will study some simple classes of trees (paths, spiders and caterpillars) before we move on to complete binary, ternary and n-ary trees. For each class we will give a bound on the eccentric chromatic number or prove that the class is incolorable.

2 Paths, Spiders and Caterpillars

Graphs with few branchings can usually be eccentrically colored with a finite number of colors when they reach a certain size. This is due to sequences of colors that can be repeated indefinitely without violating the constraints on color-distance. Here we will give the eccentric chromatic number for all paths, and spiders and give a bound on the number of colors needed for caterpillars.

Observation 5 *A path P is eccentrically colorable ($\chi_e = 3$) if and only if $|V(P)| \geq 4$.*

Proof. That paths of length at most three are not eccentrically colorable is easily verifiable. Longer paths can be colored with the color-sequence $3121, 3121, ...$ which can be repeated indefinitely to color any path of length at least four. □

Definition 6 *A* star *is a bipartite graph $K_{1,n}$. A* spider *is a star with subdivided edges.*

Observation 7 *A spider S can be eccentrically colored ($\chi_e = 3$) if and only if S is not a star.*

Proof. It is easy to verify that a star cannot be eccentrically colored. If a spider S is not a star, then S is either a path of length at least 4 (colorable by Observation 5), or has one vertex v with degree 3 or more and a set of paths connected to v. Color v with color 2 and color the paths with the sequence $1312, \cdots$. Shorter paths can safely be colored with parts of the sequence. □

Definition 8 *A* caterpillar *is a path, called the body, where each vertex except the end-vertices in the path may have any number of single vertices, called leaves, connected to it. We say that the number of edges in the body is the length of a caterpillar.*

It is clear that many caterpillars with short bodies cannot be colored, due to very low eccentricity and a possibility for many leaves. However, we will now give a result showing that all caterpillars of length 7 or more can be colored.

Observation 9 *All* caterpillars *where the body is of length* 6 *or more have* $\chi_e \leq 7$.

Proof. Coloring the body, except the end-vertices, with the color-sequence

$$243256243257\cdots$$

and all the leaves and end-vertices with color 1, will color a caterpillar of any length 6 or more (use only first part of the sequence for lengths 6 through 12). Note that the sequence cannot be used on caterpillars of length 5 or less due to eccentricity. □

A simple computer analysis shows that a color-sequence of length 34 (2342562342 5326423524 6235243265 2342) using only the colors 2 through 6 exists and can be used for caterpillars with body-length 35 or less, though no such color-sequence using only colors 2 through 6 has length 35 or greater.

3 Binary Trees

In this section we investigate eccentric coloring of binary trees and in particular complete binary trees.

Definition 10 *Binary tree*

A binary tree *is a tree where all vertices have degree* $1, 2,$ *or* 3.

Definition 11 *Complete binary tree*

We inductively define the complete binary tree B_i.

1. $B_1 \stackrel{def}{=}$ *1 vertex, the* root. *This vertex is* Level 1

2. $B_h \stackrel{def}{=}$ *Start with* B_{h-1} *and append* 2 *new leaves to each leaf of* B_{h-1}. *The new leaves are Level* h.

The height of a complete binary tree is $h = d(root, leaf\,) + 1$.

We show that all complete binary trees with height at least 3 can be eccentrically colored with no more than 7 colors, and that the same number applies for *large* binary trees.

We will use induction to prove our result, and in fact we strengthen our induction hypothesis to carry through the proof. We introduce a set of extra rules and show that if a coloring adheres to these rules, then the coloring can be used as a basis for coloring a larger tree. We will call a coloring of a complete binary tree with these extra restrictions an *expandable* eccentric coloring.

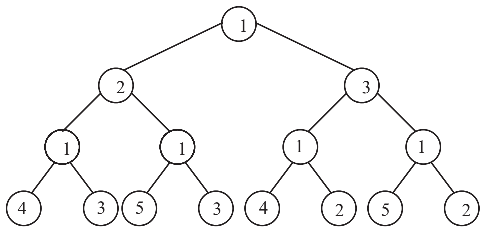

Figure 1: An expandable eccentric coloring of a complete binary tree of height 4.

Definition 12 *Expandable eccentric coloring*
An expandable eccentric coloring *of a complete binary tree $T = (V, E)$ is a coloring such that*

(i) $\forall u, v \in V, \quad \big(color(u) = color(v)\big) \Rightarrow d(u, v) > color(u)$

(ii) $\forall v \in V, \quad color(v) \leq e(v)$

(iii) *The root(level 1) is colored* 1*.*

(iv) *All vertices on odd levels are colored* 1*.*

(v) *Every vertex colored* 1 *has at least one child colored* 2 *or* 3*.*

(vi) $color(v) = 6$ *and* $color(u) = 7 \Rightarrow d(u, v) \geq 5$

(vii) $color(p) \in \{4, 5, 6, 7\} \Rightarrow$ *p's children each have children colored* 2 *and* 3*.*

(viii) $\forall u \in V, \quad color(u) \leq 7$

Observation 13 Base case: *The graph in Figure 1 is an expandable eccentric coloring of height* 4*.*

We now show that given an expandable coloring of a complete binary tree of height n, we can create an expandable coloring of a complete binary tree of height $(n + 1)$ by using the coloring for the tree of height n as a basis.

Lemma 14 *An expandable eccentric coloring of a complete binary tree of height n can be extended to an expandable eccentric coloring of a complete binary tree of height $(n + 1)$.*

Proof. We will construct the eccentric coloring for the $(n+1)$-height tree by coloring the n-first levels as the n-height tree and show that vertices on the new level always can be colored in such a way that they adhere to the expandable coloring-rules.

If n is even then, as a consequence of rule (iv), all vertices at level $(n - 1)$ are colored 1, and hence any leaf on level $(n + 1)$ can be colored 1.

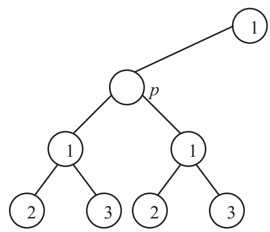

Figure 2: A coloring of the leaves if $color(p) \in \{4, 5, 6, 7\}$.

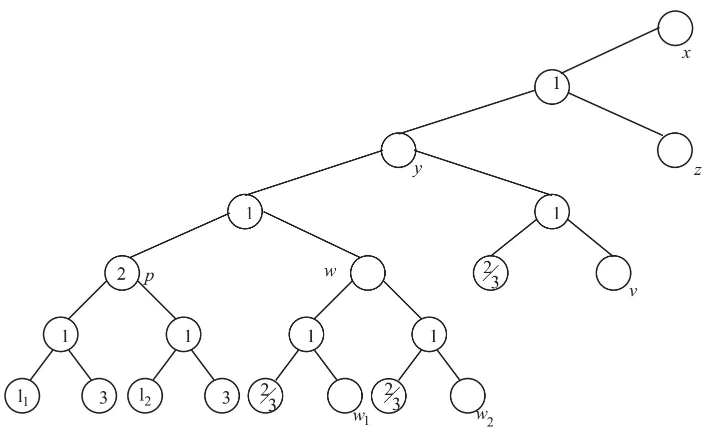

Figure 3: The subgraph examined if $color(p) = 2$.

If n is odd, we will for each leaf consider the value of its grandparent p. If $color(p) \in \{4, 5, 6, 7\}$ then according to rule (vii) we must color its grandchildren $2, 3$ and $2, 3$. Observe that this is always a legal coloring (Figure 2).

The situation is more complicated if $color(p) = 2$ (color 3 will not be argued for but is analogous). The four grandchildren of p can have any color with the exception of 2. We will examine the part of the graph which can affect the coloring of p's grandchildren, consisting of vertices at distance at most 7 on even levels from these grandchildren. This subgraph can be seen in Figure 3, note that the following discussion uses the labeling seen in this figure.

Note that w_1 and w_2(and their siblings) are on level $(n + 1)$ and may or may not have been colored already. If they have not been colored, then they cannot interfere with the coloring of l_1 and l_2. Thus in the critical case, examined here, we can assume they already have been colored according to the expandable coloring rules.

Due to rule (v), we can assume without loss of generality that w_1, w_2 and v's respective siblings are colored 2 or 3. Vertex l_1's and vertex l_2's siblings are colored 3.

We now have, due to rule (i), that $color(l_1) \in \{4, 5, 6, 7\}$ and $color(l_2) \in \{4, 5, 6, 7\}$.

It remains to show that we can always color l_1 and l_2 with these colors. We examine y, p's grandparent, and show by a case analysis on y's color how to color l_1 and l_2.

We say that vertex v *blocks* color c from vertex u if and only if $color(v) = c$ and $d(v,u) \leq c$, i.e. coloring vertex u with color c would violate rule (i).

1. $color(y) \in \{1,2\}$

   This is impossible due to rule (i).

2. $color(y) = 3$

$$\begin{aligned} color(y) = 3 \text{ and } color(p) = 2 \overset{(i)}{\Rightarrow} color(w) \notin \{2,3\} \overset{(vii)}{\Rightarrow} \\ color(w_1) \in \{2,3\}, color(w_2) \in \{2,3\} \end{aligned}$$

   The distance from x, z, and v to l_1 and l_2 is 6, hence x, z, and v can only block colors 6 and 7 from l_1 and l_2. Due to rule (vi) only one of x, z, v, or w can be colored 6 or 7.

   This implies that from the set of colors $\{4,5,6,7\}$, vertex w will block one color and x, z, and v will block at most one other color. This leaves at least two colors for l_1 and l_2.

3. $color(y) \in \{4,5\}$

   Note that w, x, z, and v do not block $4, 5, 6$, or 7 from l_1 and l_2 since:

$$\begin{aligned} color(y) \in \{4,5\}, color(p) = 2 &\overset{(vii)}{\Rightarrow} color(w) = 3 \\ color(y) \in \{4,5\} &\overset{(vii)}{\Rightarrow} color(v) \in \{2,3\} \\ color(y) \in \{4,5\} &\overset{(v)}{\Rightarrow} color(z) \in \{2,3\} \end{aligned}$$

   Also note that since $color(y) \in \{4,5\}$, y's grandparent x cannot be in $\{4,5,6,7\}$ as this would violate rule (vii). Hence, $color(x) \in \{2,3\}$.

   We can now see that w_1 and w_2 cannot block both color 6 and color 7 from l_1 and l_2 because of rule (vi), and y blocks either color 4 or color 5. This leaves at least two colors for l_1 and l_2.

4. $color(y) \in \{6,7\}$

   From rule (vii) we have:

$$color(y) \in \{6,7\}, color(p) = 2 \overset{(vii)}{\Rightarrow} color(w) = 3$$

   Since x, z, v, w_1, and w_2 cannot block colors 4 and 5 from l_1 and l_2 and $color(w) = 3$, we can always use color 4 and color 5 as a valid coloring for l_1 and l_2.

It is easy to verify that the coloring of the leaf adheres to the expandable coloring rules.

We have shown that it is possible to color any leaf such that the new coloring adheres to expandable coloring rules. This is true as long as the other leaves are each colored according to the expandable rules or are uncolored. Thus, we can color level $(n+1)$. □

Theorem 15 *Any complete binary tree of height of three or more is eccentrically colorable with* 7 *colors or less.*

Proof. For height 3, use Observation 13 without leaves. For larger heights the proof is by induction on the height. For height 4 use Observation 13 as a base and for the inductive step use Lemma 14. □

Corollary 16 *Any tree T with degrees $1, 2$ and 3 and with diameter greater or equal to* 14 *can be colored using no more than* 7 *colors.*

Proof. Let C be a complete binary tree such that T is a subtree of C. By Theorem 15 we can color C with 7 colors. Remove vertices from C to obtain T, T is now legally colored as the diameter of 14 ensures that any vertex in T can be colored with 7 without violating the eccentricity. □

4 Ternary trees

In Section 3 we proved that all complete binary trees can be eccentrically colored using a constant number of colors. Now we show the surprising result that complete *ternary* trees cannot be eccentrically colored. We even prove the stronger result that ternary trees cannot be *eccentrically broadcast-colored*.

Definition 17 *Ternary tree*

A ternary tree *is a tree where all vertices have degree* $1, 2, 3,$ *or* 4.

Definition 18 *Complete ternary tree*

We inductively define the complete ternary tree T_i.

1. $T_1 \stackrel{def}{=}$ *1 vertex, the* root*. This vertex is* Level 1

2. $T_h \stackrel{def}{=}$ *Start with* T_{h-1} *and append* 3 *new leaves to each leaf of* T_{h-1}*. The new leaves are Level* h.

The height of a complete ternary tree is $h = d(root, leaf) + 1$*. We define* T_h^- *as a* T_h *with one leaf missing.*

Definition 19 *Eccentric Broadcast-coloring*

An eccentric broadcast-coloring *of a graph* $G = (V, E)$ *is a function* $color : V \to \mathbb{N}$ *such that*

(i) $\forall u, v \in V, \quad \big(color(u) = color(v)\big) \Rightarrow d(u, v) > color(u)$

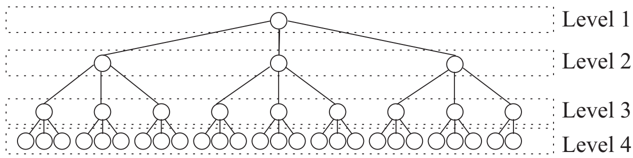

Figure 4: An example of T_4^-.

(ii) $\forall v \in V, \quad color(v) \leq diameter(G)$

A coloring that adheres to rule (i) *is known as a* broadcast-coloring *[6].*

Note that any eccentric coloring is trivially an eccentric broadcast-coloring. We seek to prove the following theorem.

Theorem 20 *The trees* $T_h, h \geq 4$*, cannot be eccentrically broadcast-colored.*

To prove this, we will use induction on the height of the tree. However, first we must establish a suitable base case.

Lemma 21 T_4^- *cannot be eccentrically broadcast-colored (See Figure 4).*

Proof. Maximum eccentricity in T_4^- is 6, but as we will see six colors is insufficient. To prove this we will use a case analysis on the placement of vertices colored 1 on level 1 and 2. First we will establish some facts:

Claim 1: Only one vertex on level 1 and level 2 can be colored 4 or greater.

Proof. Assume in contradiction to the claim that there exists two vertices x and y on level 1 and level 2 such that $color(x) > color(y) \geq 4$. Let $c \geq 4, c \notin \{color(x), color(y)\}$. Color c can be used at most three times on level 3 and level 4, no more than once on a non-leaf. This implies that there exists a subtree T_3 where every vertex but the root is colored $1, 2,$ and 3 and a single leaf l is colored c. Then, there exists at least one T_2 in this T_3 where 3 is used as a root, but this is impossible as we have to use a 3 to color a neighbor of l. □

Claim 2: There are at least four vertices colored 1 on levels $1, 2,$ and 3.

Proof. We try to color as many vertices on level 1 through level 3 without using color 1 as possible. We can then use one 6, one 5, one 4, three 3's and three 2's, a total of 9 vertices. Levels $1, 2,$ and 3 have 13 vertices in total, thus at least four vertices must be colored 1. □

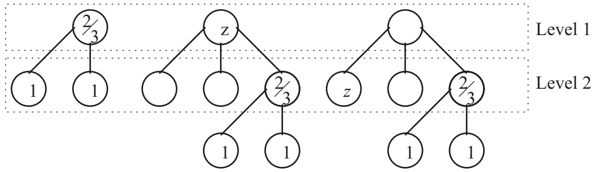

Figure 5: Illegal colorings by Claim 3 if $z \in \{4, 5, 6\}$.

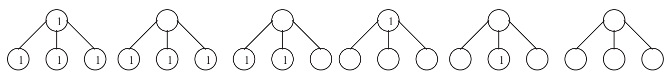

Figure 6: The six different cases we investigate in Lemma 21.

Claim 3: If a vertex v is colored 2 or 3 and v is on level 1 *or* there exists a vertex z on level 1 or level 2, where $color(z) \in \{4, 5, 6\}$, and v is on level 2 then v cannot be the parent of two vertices x and y colored 1. (See Figure 5)

Proof. If v is colored 2 (color 3 is analogous) and x and y are colored 1 we cannot color the children of x and y. If v is on level 1 then x and y has 6 children which must be colored using $\{3, 4, 5, 6\}$ which is impossible. Otherwise, we can assume without loss of generality that $color(z) = 6$, we now have only the colors $\{3, 4, 5\}$ to color the 5 or 6 children of x and y, which cannot be done either. $\square$

We will now begin our case analysis on which vertices on level 1 and level 2 which are colored 1. We have six different cases to investigate (see Figure 6).

1. Four vertices colored 1.
2. Three vertices colored 1.
3. Two vertices colored 1.
4. The root colored 1.
5. One vertex on level 2 colored 1.
6. No vertices colored 1.

1. If all four vertices of level 1 and level 2 are colored 1 we contradict the requirements on distance.

2. If three vertices on level 1 and level 2 are colored 1, we have only one possibility, all three vertices on level 2 are colored 1. This is not viable as we must then color level 3 without 1s. The nine vertices can then only be colored $((2, 3, 4), (2, 3, 5), (2, 3, 6))$, parentheses group siblings, which leaves no free color for the root.

3. If two vertices on level 1 and level 2 are colored 1, we again have only one possibility, two level 2 vertices are colored 1. By Claim 3 we cannot have color

2 or 3 as root, thus the root has color 4 or greater. This implies, by Claim 1, that 2 or 3 is used on vertex v on level 2. Claim 2 implies that v has at least two children colored 1, but this contradicts Claim 3.

4. If the root is colored 1 then the three level 2 vertices must be colored $(2, 3, c), c \geq 4$. We can assume without loss of generality that $c = 6$. Let v be the vertex colored 2 and w be the vertex colored 3. Vertex v's children can choose from the colors $\{1, 4, 5\}$, but because of Claim 3 we cannot select more than one 1. This implies that v's children is colored $(1, 4, 5)$. Unfortunately, this leaves only $\{1, 2\}$ for w's children, and again due to Claim 3 we cannot select more than one 1. Thus we cannot color w's children.

5. If we have one vertex v colored 1 on level 2, we must, by Claim 1, use the colors $\{2, 3, c\}, c \geq 4$, to color the other vertices on level 1 and level 2. We can without loss of generality assume that $c = 6$. We now have several sub-cases.

   If c is used on level 1 then level 2 is colored $(2, 1, 3)$ and due to Claim 2, either the vertex colored 2 or the vertex colored 3 has two children colored 1, violating Claim 3.

   If c is used on level 2, we have either color 2 or color 3 on the root. If the root is colored 2 then it is impossible to color the children of v (vertex colored 1). If the root is colored 3 then v's children must be colored $\{4, 5, 2\}$, but this leaves only the color 1 for the children of the vertex w colored 2. Due to Claim 3, we cannot color more than one child of w with 1.

6. If no vertices on level 1 and level 2 is colored 1 then we must use four colors greater or equal to 2 contradicting Claim 1.

As we have demonstrated, it is not possible to color T_4^- with six colors, and since the maximum eccentricity of T_4^- is 6 we have that T_4^- cannot be eccentrically broadcast-colored. This completes the proof of Lemma 21. □

Lemma 22 T_5 *cannot be broadcast-colored with less than* 10 *colors.*

Proof. We first prove the following general facts about T_5.

Claim 1: In any eccentric broadcast-coloring of T_5 there exists at least one vertex colored 7 or more in each of the roots three T_4 subtrees.

Proof. Assume in contradiction to the claim that there exists a broadcast-coloring of T_5 with a T_4-subtree T where T has no vertex colored 7 or more. Then we can obtain an eccentric broadcast-coloring of T_4^- by removing an arbitrary leaf. This contradicts Lemma 21. □

Claim 2: In any broadcast-coloring of T_5 at least one T_4-subtree has either no vertex colored 6 or one leaf colored 6.

Proof. The claim is trivially true since 6 is the maximum distance between two non-leaf vertices in T_5. □

Assume in contradiction to the stated lemma that T_5 can be broadcast-colored with 9 colors. We will do a case analysis where we will split the problem into two cases.

1. At most one vertex colored 7 in T_5.
2. More than one vertex colored 7 in T_5.

1. By Claim 1 each T_4-subtree has at least one vertex colored at least 7 and since only one vertex is colored 7, all three subtrees must have not more than one vertex v colored at least 7. By Claim 2 at least one of these subtrees contain either no vertex colored 6 or a leaf colored 6. If no vertex is colored 6, then recolor v with 6 and remove an arbitrary leaf. If a leaf is colored 6, then remove this leaf and recolor v with 6. In both cases we obtain an eccentric broadcast-coloring of T_4^-. This contradicts Lemma 21.

2. Now we examine the case where more than one vertex is colored 7. In this case the vertices colored 7 are leaves and we must have at least one subtree T_4 T with no vertex colored 8 or 9. Remove the leaf colored 7 in T to obtain an eccentric broadcast-coloring of T_4^-. This contradicts Lemma 21.

We reached a contradiction in each case, thus the assumption must be false, completing the proof of Lemma 22. □

Proof of Theorem 20. An easy consequence of Lemma 21 is that T_4 is not eccentrically broadcast-colorable. To prove the result for higher h we will use induction on the height of the trees. In fact we will prove the stronger result, that any broadcast-coloring of $T_h, h \geq 5$, will have at least two vertices u, v such that $\mathit{color}(u) > \mathit{color}(v) > \mathit{diameter}(T_h)$.

Proof by induction on the height of the tree.

Base: The diameter of T_5 is 8, by Lemma 22 we have that no broadcast-coloring of T_5 has less than 10 colors. That is, $\exists u, v \in T_5$, $\mathit{color}(u) > \mathit{color}(v) > \mathit{diameter}(T_5) = 8$.

Inductive Hypothesis: We assume that any broadcast-coloring of a $T_{h-1}, h-1 \geq 5$, has at least two vertices u, v such that $\mathit{color}(u) > \mathit{color}(v) > \mathit{diameter}(T_{h-1}) = (2h-4)$.

Inductive Step: T_h is formed by a root vertex and three T_{h-1}. By IH each of the T_{h-1} subtrees has at least two vertices u, v colored greater than $diameter(T_{h-1}) = (2h-4)$. Each subtree potentially has one vertex colored $(2h-3)$ that is not in violation with other vertices, but this leaves three vertices with color at least $(2h-2)$. The diameter of T_h is $(2h-2)$, thus only one vertex in T_h has the color $(2h-2)$, leaving two vertices with color greater than $(2h-2)$, the diameter of T_h.□

Corollary 23 *No complete ternary tree is eccentrically colorable.*

Proof. For complete ternary trees of height at most 3 we can easily verify by hand that no eccentric coloring exists. For higher trees we prove the result by contradiction.

Assume in contradiction to the stated corollary that there exists a complete ternary tree T_h, $h \geq 4$, with a valid eccentric coloring. The coloring is a valid eccentric broadcast-coloring of T_h, this contradicts Theorem 20. □

5 Complete k-ary trees

We now extend our results from Section 4 to complete k-ary trees. We then wrap things up by showing that no complete trees other than the binary trees and paths (unary complete trees) are eccentrically colorable.

Definition 24 *k-ary tree*

A k-ary tree *is a tree T where $\forall v \in V(T), deg(v) \leq k+1$.*

Definition 25 *Complete k-ary tree*

We inductively define the complete k-ary tree T_i.

1. $T_1 \stackrel{def}{=}$ *1 vertex, the* root.

2. $T_i \stackrel{def}{=}$ *Start with T_{i-1} and append k new leaves to each leaf of T_{i-1}.*

The height of a complete k-ary tree is $h = d(root, leaf) + 1$.

Theorem 26 *No complete k-ary tree, $k \geq 3$, of height h, $h \geq 4$ is eccentrically broadcast-colorable.*

Proof. Assume in contradiction to the stated theorem that there exists a complete k-ary tree C with $k \geq 3$ of height h, $h \geq 4$, with a valid eccentric broadcast-coloring. Then the complete 3-ary tree T of height h will be a subtree of C. Color C and remove vertices from C to obtain T, the remaining vertices are properly eccentrically broadcast-colored since the maximum eccentricity has not changed. This contradicts Theorem 20. □

Theorem 27 *No complete k-ary tree, $k \geq 3$, is eccentrically colorable.*

Proof. Assume in contradiction to the stated theorem that there exists a complete k-ary tree C, $k \geq 3$, of some height h that can be eccentrically colored. The complete 3-ary tree T of height h will be a subtree of C. Eccentrically color C and remove vertices from C to obtain T. The remaining vertices are now properly eccentrically colored since the eccentricity of the vertices has not changed. This contradicts Corollary 23. □

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