Some small large sets of t–designs

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Abstract

We construct large sets of $t-(v,k,\lambda)$ designs for the parameter sets $2-(9,4,3)$, $2-(10,4,2)$, $2-(10,5,4)$, $2-(11,5,2)$, $2-(12,4,3)$, $2-(12,6,5)$, $3-(12,6,2)$, $2-(12,5,20)$ and $3-(12,5,6)$. The existence and non-existence of all possible large sets of $t-(v,k,\lambda)$ designs is now completely determined for $v \leq 12$.

1 Introduction

A $t-(v,k,\lambda)$ design is a pair $(\mathbf{X}, \mathcal{A})$ which satisfies the following properties:

1. $\mathbf{X}$ is a set of v elements (called *points*)
2. $\mathcal{A}$ is a family of subsets of $\mathbf{X}$, each of cardinality k (called *blocks*)
3. every t-subset of distinct points occurs in exactly λ blocks.

A $t-(v,k,\lambda)$ design is called *simple* if it contains no repeated blocks.

By elementary counting, it can be shown that if $s < t$, a $t-(v,k,\lambda)$ design is also an $s-(v,k,\mu)$ design, where

$$\mu = \frac{\lambda\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

Since μ must be an integer, this equation yields a necessary condition for existence of the t-design, for any $s < t$. Given t, k and v, there is a smallest positive integer $\lambda^*(t,k,v)$ such that these conditions are satisfied for all $0 \leq s < t$.

If we complement every block of a $t-(v,k,\lambda)$ design with respect to the point set, we get a $t-(v,v-k,\lambda')$ design, where

$$\lambda' = \frac{\lambda\binom{v-k}{t}}{\binom{k}{t}}.$$

Hence, we shall restrict our attention to the situation where $k \leq v/2$.

Let $\binom{\mathbf{X}}{k}$ denote the set of all $\binom{v}{k}$ k−subsets of a v−set $\mathbf{X}$. Suppose $\lambda = \lambda^*(t,k,v)$. *A large set* of $t-(v,k,\lambda)$ designs is a partition of $\binom{\mathbf{X}}{k}$ into $t-(v,k,\lambda)$ designs. The number of designs in the partition is $N = \binom{v-t}{k-t}/\lambda$. We shall denote a large set of $t-(v,k,\lambda)$ designs by LS $t-(v,k,\lambda)$. Note that all the designs in a large set are simple and we use the term "large set" only when $\lambda = \lambda^*(t,k,v)$.

If we take all the blocks of a $t-(v,k,\lambda)$ design through a point x, and delete x, we get a $(t-1)-(v-1,k-1,\lambda)$ design, called the *derived* design. Further, if $\lambda^*(t,k,v) = \lambda^*(t-1,k-1,v-1)$, then the derived designs of an LS $t-(v,k,\lambda)$ form an LS $(t-1)-(v-1,k-1,\lambda)$.

Under certain conditions, the process of derivation can be reversed. Suppose $(\mathbf{X}, \mathcal{A})$ is a $t-(v,k,\lambda)$ design, where t is even and $v = 2k+1$. Let $\infty \notin \mathbf{X}$, and denote $\mathbf{X}^* = \mathbf{X} \cup \{\infty\}$. Define

$$\mathcal{A}^* = \{A \cup \{\infty\} : A \in \mathcal{A}\} \cup \{\mathbf{X} \backslash A : A \in \mathcal{A}\}.$$

Then, $(\mathbf{X}^*, \mathcal{A}^*)$ is a $(t+1)-(v+1,k+1,\lambda)$ design [1]. This operation is called *extension.*

It is easy to see that if we have an LS $t-(v,k,\lambda)$ (where t is even and $v = 2k+1$), and form the extension of every design in the large set, then we obtain an LS $(t+1)-(v+1,k+1,\lambda)$.

A table of $t-(v,k,\lambda)$ designs has recently been published by Chee, Colbourn and Kreher [6]. They list parameter sets up to $v = 30$, and also include information about the existence of large sets. As well, a survey of large sets of disjoint designs has been written by Teirlinck [24].

In this paper, we find several new examples of large sets of $t-(v,k,\lambda)$ designs when $v = 9, 10, 11$ and 12. The parameter sets are $2-(9,4,3)$, $2-(10,4,2)$, $2-(10,5,4)$, $2-(11,5,2)$, $2-(12,4,3)$, $2-(12,6,5)$, $3-(12,6,2)$, $2-(12,5,20)$ and $3-(12,5,6)$. These large sets and the algorithms used to obtain them are described in the remainder of the paper. We also provide an updated table of large sets of $t-(v,k,\lambda)$ designs for $v \leq 15$ in the Appendix.

2 $G-$invariant large sets

Let $(\mathbf{X}, \mathcal{A})$ be a $t-(v,k,\lambda)$ design, and let π be a permutation of $\mathbf{X}$. If we let π act on $(\mathbf{X}, \mathcal{A})$, then we obtain an isomorphic copy of the design, which we denote $(\mathbf{X}, \mathcal{A}^\pi)$, where $\mathcal{A}^\pi = \{A^\pi : A \in \mathcal{A}\}$ ($A^\pi = \{x^\pi : x \in A\}$ for $A \in \mathcal{A}$). Suppose $\mathcal{F} = \{(\mathbf{X}, \mathcal{A}_i) : 1 \leq i \leq N\}$ is an LS $t-(v,k,\lambda)$. Then, define $\mathcal{F}^\pi = \{(\mathbf{X}, \mathcal{A}_i^\pi) : 1 \leq i \leq N\}$. It is clear that $\mathcal{F}^\pi$ is also an LS $t-(v,k,\lambda)$, and $\mathcal{F}^\pi$ is isomorphic to $\mathcal{F}$.

Let G be a subgroup of $Sym(\mathbf{X})$, the symmetric group on $\mathbf{X}$, and let $\mathcal{F}$ be an LS $t-(v,k,\lambda)$. We say that $\mathcal{F}$ is $G-$*invariant* if $\mathcal{F}^\pi = \mathcal{F}$ for all $\pi \in G$.

Denote the orbits of $\binom{\mathbf{X}}{k}$ under the action of G by $\mathcal{C} = \{\Gamma_i : 1 \leq i \leq s\}$. Similarly, consider the set of all distinct $t-(v,k,\lambda)$ designs on $\mathbf{X}$, and name the orbits of designs under the action of G as $\mathcal{D} = \{\Delta_i : 1 \leq i \leq r\}$. Next, define the $r \times s$ matrix $M = (m_{ij})$ by the rule $m_{ij} = |D \cap \Gamma_j| \times |\Delta_i|/|\Gamma_j|$, where D is any $t-(v,k,\lambda)$ design in Δ_i. (Note that the value m_{ij} is independent of the particular orbit representative D that is chosen.)

We have the following easy observation.

Theorem 2.1 *There exists a $G-$invariant large set of $t-(v,k,\lambda)$ designs if and only if there exists a $0-1$ vector U of dimension r such that $UM = J$, where J is the $s-$dimensional column vector of 1's.*

We remark that any rows of M that contain entries greater than one can be deleted, since the corresponding entry of U must be zero in any solution to $UM = J$.

Suppose that $\mathcal{F}$ is a $G-$invariant LS $t-(v,k,\lambda)$, and let π be a permutation of $\mathbf{X}$. As mentioned above, $\mathcal{F}^\pi$ is an LS $t-(v,k,\lambda)$, but it is not, in general, $G-$invariant. However, if $\pi \in N(G)$ (the normalizer of G in $Sym(\mathbf{X})$), then $\mathcal{F}^\pi$ is $G-$invariant. This observation is of use in determining isomorphism of $G-$invariant LS $t-(v,k,\lambda)$.

3 Large sets of $2-(9,4,3)$ designs

In this section, we discuss the parameter set $2-(9,4,3)$. There are seven designs in a large set. It seems reasonable to look for an LS $2-(9,4,3)$ which is obtained from one "starter design" by applying the seven powers of permutation $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8)(9)$; i.e., we take $G = \langle\sigma\rangle$. In such a large set, the seven designs will all be isomorphic.

A solution U to the equation $UM = J$ will have only one non-zero co-ordinate, so it is simpler in this case to proceed as follows. Let $(\mathbf{X}, \mathcal{A})$ be a $2-(9,4,3)$ design, and let π be a permutation of $\mathbf{X}$. For any such permutation π, it is easy to check if $(\mathbf{X}, \mathcal{A}^\pi)$ is a starter design for a large set. If so, the resulting large set will be denoted $C(\mathcal{A}^\pi)$. If we repeat this process for each of the non-isomorphic $2-(9,4,3)$ designs, then we will obtain all $G-$invariant LS $2-(9,4,3)$.

Clearly, $C(\mathcal{A}^\pi) = C(\mathcal{A}^{\pi\sigma})$. It is also obvious that $C(\mathcal{A}^\pi) = C(\mathcal{A}^{g\pi})$ if $g \in Aut(\mathcal{A})$, where $Aut(\mathcal{A}) = \{g : \mathcal{A}^g = \mathcal{A}\}$ is the automorphism group of $\mathcal{A}$. More generally, it is not difficult to see that $C(\mathcal{A}^\pi) = C(\mathcal{A}^\rho)$ if and only if $\pi = g\rho\sigma^i$, for some $g \in Aut(\mathcal{A})$ and for some $i, 0 \leq i \leq 6$.

Define $H = Aut(\mathcal{A})$ and $H\rho G = \{h\rho g : h \in H, g \in G\}$. We shall obtain exactly $|H\rho G|$ copies of each large set. In fact, it turns out that $|H\rho G| = |H| \times |G|$. We see this as follows. Suppose that $h\rho g = h'\rho g'$, where $h, h' \in H, g, g' \in G$. Then, $\rho^{-1}h^{-1}h'\rho = g(g')^{-1}$. Now, $\rho^{-1}h^{-1}h'\rho \in H^\rho$ and $g(g')^{-1} \in G$. But, it is easy to see that no non-identity element of G can be an automorphism of any $2-(9,4,3)$ design. Hence, $h = h'$ and $g = g'$, and thus $|H\rho G| = |H| \times |G|$.

The non-isomorphic $2-(9,4,3)$ designs have been enumerated in [23], [12] and [3]; there are precisely 11 non-isomorphic designs. We find that only two of the 11 designs admit large sets that are constructed in this fashion, and each of these two designs gives rise to a unique large set up to isomorphism.

Large Set #1

{1,9,2,5}	{1,9,8,3}	{1,9,4,6}	{1,2,8,3}	{1,2,4,6}	{1,7,5,8}
{1,7,5,4}	{1,7,3,6}	{9,2,7,3}	{9,2,7,6}	{9,7,8,4}	{9,5,8,6}
{9,5,3,4}	{2,7,8,4}	{2,5,8,6}	{2,5,3,4}	{7,5,3,6}	{8,3,4,6}

Large Set #2

{6,8,1,2}	{6,8,2,7}	{6,8,5,4}	{6,1,7,3}	{6,1,7,9}	{6,2,5,4}
{6,3,5,9}	{6,3,9,4}	{8,1,3,5}	{8,1,9,4}	{8,2,3,9}	{8,7,3,4}
{8,7,5,9}	{1,2,3,4}	{1,2,5,9}	{1,7,5,4}	{2,7,3,5}	{2,7,9,4}

The underlying design for Large Set # 1 has an automorphism group of order 8, and we find that exactly 672 permutations give rise to a large set. Since $672 = 8 \times 7 \times 12$, we know that there are exactly 12 distinct large sets among the 672. In fact, these 12 large sets are all isomorphic. The isomorphisms are the 12 permutations in the group $\langle\alpha, \beta\rangle$, where $\alpha = (1\ 3\ 2\ 6\ 4\ 5)(7)(8)(9)$ and $\beta = (1)(2)(3)(4)(5)(6)(7)(8\ 9)$. Note that $\langle\alpha, \beta\rangle$ is a subgroup of the normalizer $N(G)$.

A similar situation arises with Large Set # 2. The underlying design has an automorphism group of order 32, and we find that exactly 1344 permutations give rise to a large set. Since $1344 = 32 \times 7 \times 6$, there are exactly 6 distinct large sets among the 1344. The 6 large sets are all isomorphic. This large set has $\alpha^3\beta$ as an automorphism, so $\langle\alpha\rangle$ will permute the six distinct isomorphic copies of Large Set # 2.

Hence, we have the following.

Theorem 3.1 *Let $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8)(9)$ and $G = \langle\sigma\rangle$. Then there are precisely two non-isomorphic $G-$invariant LS* $2-(9,4,3)$.

4 Large sets of $2-(11,5,2)$ designs

There is precisely one non-isomorphic $2-(11,5,2)$ design [14]. We shall use the following $2-(11,5,2)$ design $(\mathbb{X}, \mathcal{A})$, where $\mathbb{X} = \{1,\ldots,11\}$.

$$\begin{array}{llll} \{1,2,3,7,10\} & \{1,2,6,9,11\} & \{1,3,4,5,9\} & \{1,4,6,7,8\} \\ \{1,5,8,10,11\} & \{2,3,4,8,11\} & \{2,4,5,6,10\} & \{2,5,7,8,9\} \\ \{3,5,6,7,11\} & \{3,6,8,9,10\} & \{4,7,9,10,11\} & \end{array}$$

Its automorphism group is $PSL(2,11)$ and has order 660. Therefore, there are $11!/660 = 60480$ distinct $2-(11,5,2)$ designs on $\mathbb{X}$. Note that we can construct these 60480 designs by computing the 60480 coset representatives of $PSL(2,11)$ in S_{11}. We can obtain the 60480 coset representatives by the following easy trick. $PSL(2,11)$ is a subgroup of index 12 of the Mathieu group M_{11}, which is in turn a subgroup of index 7! of S_{11}. It is not difficult to find 12 coset representatives of $PSL(2,11)$ in M_{11}. To find 7! coset representatives of M_{11} in S_{11} is also easy: take the 7! permutations that fix the points $1-4$. These 7! permutations are in different cosets since M_{11} is sharply 4-transitive, and hence no non-identity element fixes four points.

A large set will consist of 42 designs. It seems reasonable to search for large sets generated from six starter designs using the permutation $\sigma = (1)(2)(3)(4)(5\ 6\ 7\ 8\ 9\ 10\ 11)$. Such a large set will be G-invariant, where $G = \langle\sigma\rangle$.

There will be $60480\ /\ 7 = 8640$ G-orbits of $2-(11,5,2)$ designs. We can obtain a list of orbit representatives by taking the 8640 coset representatives that fix the point 5. It turns out that 2160 rows of the matrix M contain an entry exceeding one, so we are left with a 6480×66 matrix M'. We proceed to find all binary solutions U to the equation $UM' = J$ using our binary knapsack solver *Synth*. Note that any solution contains exactly six non-zero entries. The solutions were enumerated in about three weeks time on a SPARCstation 1. The resulting solutions were tested for isomorphism using Brendan McKay's graph isomorphism program *Nauty*. It was found that there were five large sets, up to isomorphism. Hence, we have the following.

Theorem 4.1 *Let* $\sigma = (1)(2)(3)(4)(5\ 6\ 7\ 8\ 9\ 10\ 11)$ *and* $G = \langle\sigma\rangle$. *Then there are precisely five non-isomorphic G-invariant LS* $2-(11,5,2)$.

The six starter designs in a large set are described by letting suitable permutations act on the design $(\mathbb{X}, \mathcal{A})$. The permutations used to generate the five large sets are as follows.

Large Set #1

1. (1 2 5 4 6 10 8 3 7 9 11)
2. (1 4 8 11 10 5 7)(2 6)(3 9)

3. (1)(2)(3)(4)(5)(6 11 7 10 8)(9)
4. (1 3 2 8 4 6)(5 7)(9)(10)(11)
5. (1)(2)(3)(4)(5)(6 8 9 7 10)(11)
6. (1 3 11)(2 9 5 7 10)(4)(6 8)

Large Set #2

1. (1)(2)(3)(4)(5)(6 10)(7)(8 11)(9)
2. (1)(2)(3)(4)(5)(6 8 9)(7)(10)(11)
3. (1 11)(2 7 4 5 3 9 10 6)(8)
4. (1 4 6 2 7)(3 9)(5 11 8 10)
5. (1 3 2 6)(4 7 5 11 9 8)(10)
6. (1 3 8 7 6 11)(2 10)(4)(5 9)

Large Set #3

1. (1 4 6 2 7)(3 9)(5 10)(8)(11)
2. (1 2 5 4 8 3 9 10 11)(6)(7)
3. (1 3 2 6 4 10 5 8 11 9 7)
4. (1 4 10 5 11 8 7)(2 6)(3 9)
5. (1 10 9 7 4 2 8 5 3)(6)(11)
6. (1 2)(3)(4)(5)(6 7 10 8)(9 11)

Large Set #4

1. (1 4 9 7 3 11 5 8 6)(2)(10)
2. (1 10 8 5 3 2 6 7 4)(9 11)
3. (1 4 7 2 10 5 11 9 3 6)(8)
4. (1 7 6 10 5)(2 4 3 11 8 9)
5. (1 11 2 10 7 4 5 3 9 6)(8)
6. (1 6 7 4)(2 9 8 5 3)(10)(11)

1. (1 6 3 11 9 7 10 4 8)(2)(5)
2. (1 2 5 4 8 3 7 9 11)(6)(10)
3. (1 3 2 7)(4 6)(5 10)(8 11)(9)
4. (1 8 7 10 5 2 4 3 9)(6)(11)
5. (1 6 3 9 8)(2)(4 10)(5)(7 11)
6. (1 3 11)(2 8 10)(4)(5 7 6 9)

5 Large sets of $2-(10,4,2)$ designs

There are precisely three non-isomorphic $2-(10,4,2)$ designs [19]. The design D_1 has automorphism group $G_1 = \langle\alpha_1, \beta_1\rangle$ of order 720, where $\alpha_1 = (1\ 3\ 2)(5\ 8\ 6\ 9\ 7\ 10)$ and $\beta_1 = (1\ 7\ 9\ 3\ 2)(4\ 8\ 6\ 10\ 5)$. D_1 is obtained from the starter block $\{1,2,3,4\}$ under the action of G_1.

The design D_2 has automorphism group $G_2 = \langle\alpha_2, \beta_2\rangle$ of order 48, where $\alpha_2 = (1\ 9\ 4)(2\ 3\ 5\ 10\ 7\ 6)$ and $\beta_2 = (1\ 8\ 9\ 4)(3\ 5\ 7\ 6)$. D_2 is obtained from the two starter blocks $\{1,2,3,4\}$ (orbit of length 12) and $\{2,3,7,10\}$ (orbit of length 3).

The design D_3 has automorphism group $G_3 = \langle\alpha_3, \beta_3\rangle$ of order 24, where $\alpha_3 = (1\ 8\ 4)(2\ 6\ 3\ 7\ 5\ 10)$ and $\beta_3 = (2\ 3\ 6)(4\ 9\ 8)(5\ 7\ 10)$. D_3 is obtained from the three starter blocks $\{1,2,3,6\}$ (orbit of length 8), $\{1,2,4,7\}$ (orbit of length 6) and $\{1,4,8,9\}$ (fixed block).

We searched for G–invariant large sets, where $G = \langle\sigma\rangle$ and $\sigma = (1)(2)(3)(4\ 5\ 6\ 7\ 8\ 9\ 10)$. A large set must contain exactly 14 designs, and since a $2-(10,4,2)$ design has no automorphisms of order seven, a large set is comprised of exactly two orbits of designs under G.

Since σ cannot fix a 4-set or a $2-(10,4,2)$ design, it follows that G–orbits of 4-sets and of 2-(10,4,2) designs all have length seven. Hence, there are exactly $\binom{10}{4}/7 = 30$ orbits of 4-sets under G. There are altogether $|S_{10}|/|G_1| = 10!/720 = 5040$ $2-(10,4,2)$ designs isomorphic to D_1, fused into $5040/7 = 720$ orbits of designs isomorphic to D_1. We proceed to compute the matrix $M = M_1$, having dimensions 720×30, as in Section 2. Here it turns out that every row of M_1 has at least one entry which exceeds 1, so that there can be no binary solutions U to the matrix equation $UM_1 = J$. In fact, no G–invariant large set can involve a $2-(10,4,2)$ design isomorphic to D_1.

There are a total of $|S_{10}|/|G_2| = 10!/48 = 75600$ designs isomorphic to D_2, comprising exactly $75600/7 = 10800$ orbits of designs of type D_2 under G. Thus, the matrix $M = M_2$ has dimensions 10800×30. After removing those rows of M_2 which contain entries greater than 1, we obtain a submatrix M_2' of dimensions 444×30. It required about one minute computing time on a SPARCstation 1 for our program *Synth* to determine that there are no binary solutions U to the system $UM_2' = J$.

There are a total of $|S_{10}|/|G_3| = 10!/24 = 151200$ designs isomorphic to D_3, and these fuse into $151200/7 = 21600$ orbits under G. Here the matrix $M = M_3$ has dimensions 21600×30, but after removal of the rows with entries greater than 1, we obtain a submatrix M_3' of dimensions 1104×30. A *Synth* run required about six minutes to determine the complete set of binary solutions to the system $UM_3' = J$. There are presicely 36 solutions, yielding G–invariant LS $2-(10,4,2)$ in which all 14 designs are isomorphic to D_3. These 36 solutions are all isomorphic.

There remains the possibility that there could exist G–invariant LS $2-(10,4,2)$ in which one G–orbit of designs is isomorphic to D_2, and the other G–orbit of designs is isomorphic to D_3. By concatenating the matrices M_2' and M_3', we construct a matrix M_4 of dimensions 1548×30. We determined that there are a total of 84 binary solutions to the system $UM_4 = J$. 36 of these solutions comprise two orbits of designs isomorphic to D_2 and were discussed above; the remaining 48 solutions split into exactly two further isomorphism classes.

We now present representatives of the above three classes of large sets. In the first example, both starter designs are isomorphic to D_3; in the second and third examples, one starter design is isomorphic to D_2 and the other is isomorphic to D_3.

Large Set #1

1. ${D_3}^\pi$, $\pi = (1\ 4\ 9\ 3\ 8\ 7\ 6)(2\ 5)$
2. ${D_3}^\rho$, $\rho = (1\ 7\ 3\ 5\ 6)(2\ 9\ 8\ 4)$

Large Set #2

1. ${D_2}^\pi$, $\pi = (1\ 5\ 7\ 8\ 6)(2\ 9)(3\ 10\ 4)$
2. ${D_3}^\rho$, $\rho = (1\ 7\ 3\ 8\ 4\ 9\ 10\ 5)$

Large Set #3

1. ${D_2}^\pi$, $\pi = (1\ 7\ 9\ 10\ 2\ 6\ 8\ 5\ 3)$
2. ${D_3}^\rho$, $\rho = (1\ 10\ 3\ 8\ 4\ 7\ 2\ 6\ 9\ 5)$

Summarizing, we have the following result.

Theorem 5.1 *Let $\sigma = (1)(2)(3)(4\ 5\ 6\ 7\ 8\ 9)$ and $G = \langle\sigma\rangle$. Then there are precisely three non-isomorphic G–invariant LS* $2-(10,4,2)$.

6 A large set of $2-(10,5,4)$ designs

There are precisely 21 non-isomorphic $2-(10,5,4)$ designs [27]. We found a large set, consisting of 14 designs, generated from two starter designs using the permutation $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8)(9)(10)$. The two starter designs are obtained by letting the two permutations $(1\ 2\ 5\ 9\ 3)(4\ 10)(6\ 7)(8)$ and $(1\ 8\ 6\ 9\ 10\ 7\ 3\ 4\ 5)(2)$ act on the following $2-(10,5,4)$ design.

$\{1,2,3,4,5\}$	$\{1,2,3,4,6\}$	$\{1,2,6,7,8\}$	$\{1,2,8,9,10\}$
$\{1,3,5,8,10\}$	$\{1,3,7,9,10\}$	$\{1,4,5,7,10\}$	$\{1,4,6,8,9\}$
$\{2,3,5,8,9\}$	$\{2,3,6,7,10\}$	$\{2,4,5,7,9\}$	$\{2,4,7,8,10\}$
$\{3,4,6,9,10\}$	$\{3,4,7,8,9\}$	$\{1,5,6,7,9\}$	$\{2,5,6,9,10\}$
$\{3,5,6,7,8\}$	$\{4,5,6,8,10\}$		

We suspect that it would be computationally feasible to perform an enumeration of all non-isomorphic $\langle\sigma\rangle$–invariant LS $2-(10,5,4)$. Since it would be quite time-consuming, we contented ourselves with one example.

7 A large set of $2-(12,4,3)$ designs

For reasons not entirely clear, the search for an LS $2-(12,4,3)$ was frustratingly long. A short description of these efforts may interest the reader. We say that a set of mutually disjoint designs is of *type* (σ,ρ) if each of the designs has σ as an automorphism and ρ permutes the designs amongst themselves. If the set is a large set, it will be G–invariant where $G = \langle\sigma,\rho\rangle$.

Let $\sigma_1 = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)(12)$ and let $\sigma_2 = (1\ 4\ 5\ 9\ 3)(2\ 8\ 10\ 7\ 6)(11)(12)$. Consider the following sets of blocks:

$$\begin{aligned} M_1 &= \{\{1,2,3,5\},\{1,2,6,8\},\{1,4,7,12\}\} \\ M_2 &= \{\{1,2,3,7\},\{1,2,4,10\},\{1,4,8,12\}\} \\ N_1 &= \{\{1,2,3,6\},\{1,2,5,10\},\{1,3,8,12\}\} \\ N_2 &= \{\{1,2,3,6\},\{1,2,5,10\},\{1,3,7,12\}\} \end{aligned}$$

Applying powers of σ_1 to any one of M_1, M_2, N_1, or N_2 produces a $2-(12,4,3)$ design. Then, applying powers of σ_2 to any one of these designs gives a set of five disjoint designs of type (σ_1,σ_2). We label these sets FM_1, FM_2, FN_1 and FN_2, respectively. Each FM_i is disjoint from each FN_j and this gives all non-isomorphic sets of ten disjoint $2-(12,4,3)$ designs with automorphism σ_1 on each design.

We attempted to obtain an additional set of five disjoint $2-(12,4,3)$ designs by searching for a "transversal" across the 33 orbits of length five of four-sets in the complementary set of 165 blocks. There are only two nonisomorphic sets of 165 blocks disjoint from the four initial sets of $10 \times 33 = 330$ blocks. An exhaustive search showed that no such transversal exists. Serious efforts to decompose these 165 blocks in other ways were not made.

Another attempt focused on σ_2. There are 99 orbits of four-sets, each of length five, under the action of σ_2. Several $2-(12,4,3)$ designs (taken from Constable [8]) were hit with random permutations. Roughly one in fifteen of these random copies of a $2-(12,4,3)$ design forms a "transversal" of 33 of the 99 orbits and hence gives rise to five disjoint $2-(12,4,3)$ designs. Over 30,000 such sets of five disjoint $2-(12,4,3)$ designs were found but very few sets of ten mutually disjoint $2-(12,4,3)$ designs were found. Searches for a final "transversal" of the remaining 33 orbits were unsuccessful whenever tried.

The next attempts used alternate numerology. Observe that three divides $v = 12$, $b = 33$, and 15 (the number of designs in an LS $2-(12,4,3)$). Assume $\sigma_3 = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$ and $\sigma_4 = (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)(10)(11)(12)$ are automorphisms of our LS. An LS of 15 designs might arise in a mixture of ways. For example, an LS might have some sets of three mutually disjoint $2-(12,4,3)$ designs of type (σ_3, σ_4) or of type (σ_4, σ_3). Alternatively, there might be "transversal" designs across orbits of size three under σ_3 or across orbits of size three under σ_4. This approach (though very promising) was not seriously pursued since a large set was found by a different method.

Let $\sigma = \sigma_3 = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$ and $\rho = (1\ 4\ 7\ 10)(2\ 5\ 8\ 11)(3\ 6\ 9\ 12)$. Then σ and ρ generate a cyclic group G of order 12. If G acts on our LS, there must be some designs in the large set fixed by ρ, since four does not divide 15. We assumed there would be three mutually disjoint designs of type (ρ, σ) that would cover all orbits of lengths 1 and 2 under the action of ρ. The other 12 designs might partition into three disjoint sets where each set consists of four mutually disjoint $2-(12,4,3)$ designs of type (σ, ρ).

Such sets of 12 disjoint designs were easy to create, but efforts to decompose the remaining 99 blocks, in the intended way, failed. In reverse order, we started with a set of three disjoint designs of type (ρ, σ) and tried decomposing the remaining blocks. One attempt ran for a week but no large set resulted. In frustration, about 80 non-isomorphic sets of three designs of type (ρ, σ) were generated.

It turned out that three of these 80 sets of 99 blocks had automorphism groups of order 24 (rather than order $12 = |G|$). After one of these three "special" sets of three disjoint designs of type (ρ, σ) was selected, a simple hill-climbing algorithm was used to find $2-(12,4,12)$ designs from the unused blocks. A design was saved if it decomposed into a type (σ, ρ) set of four disjoint $2-(12,4,3)$ designs. This process was repeated on the remaining blocks.

After several futile runs a fortuitous overnight run produced an LS. Consider the following sets of blocks.

	$\{1,2,4,5\}$	$\{1,2,8,12\}$	$\{1,2,9,11\}$	$\{1,4,10,11\}$
S_1	$\{1,5,6,9\}$	$\{1,5,10,12\}$	$\{1,6,7,8\}$	$\{1,7,9,12\}$
	$\{4,5,9,10\}$	$\{4,7,9,12\}$	$\{4,7,10,11\}$	

	$\{1,2,3,7\}$	$\{1,4,5,7\}$	$\{1,4,8,10\}$	$\{1,4,11,12\}$
S_2	$\{1,5,6,8\}$	$\{1,5,6,10\}$	$\{1,6,11,12\}$	$\{1,7,9,10\}$
	$\{1,9,11,12\}$	$\{4,7,8,11\}$	$\{4,8,9,10\}$	

	$\{1,2,5,11\}$	$\{1,2,8,9\}$	$\{1,2,11,12\}$	$\{1,4,5,9\}$
S_3	$\{1,4,5,12\}$	$\{1,6,7,9\}$	$\{1,6,7,10\}$	$\{1,6,8,12\}$
	$\{4,5,7,12\}$	$\{4,7,10,12\}$	$\{7,8,11,12\}$	

Applying powers of σ to each S_i produces the 33 blocks of a $2-(12,4,3)$ design. Then, applying powers of ρ to each such design produces a total of twelve mutually disjoint $2-(12,4,3)$ designs. We need three more disjoint designs that are disjoint from these twelve.

Define the following set of blocks T.

T	$\{3,6,9,12\}$	$\{1,3,7,9\}$	$\{2,6,8,12\}$	$\{1,2,3,4\}$	$\{1,2,3,10\}$
	$\{1,2,5,7\}$	$\{1,4,9,12\}$	$\{1,5,8,12\}$	$\{1,6,8,11\}$	$\{2,3,8,12\}$

Now apply powers of ρ to the blocks in T to produce orbits of lengths 1, 2, 2, 4, 4, 4, 4, 4, 4 and 4 (respectively). Thes 33 blocks give another $2-(12,4,3)$ design. Now, applying powers of σ to this design gives a total of three $2-(12,4,3)$ designs which complete the large set. This large set is G–invariant where $G = \langle \sigma, \rho \rangle$.

8 A large set of $3-(12,5,6)$ designs

We also found a large set of $3-(12,5,6)$ designs. There are six designs in a large set. The key here is to recognize that if one can get five disjoint designs then the sixth one follows. Hence, we might define $\rho = (1\ 4\ 5\ 9\ 3)(2\ 8\ 10\ 7\ 6)(11)(12)$ and hope to find a G–invariant large set with $G = \langle \rho \rangle$. This will require that one design be fixed by G, and the others cycle through an orbit of size five.

Define $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)(12)$ and let $\langle \sigma, \rho \rangle$ act on the following four starter blocks:

$$\{1,2,3,4,5\} \quad \{1,2,4,5,12\} \quad \{1,2,3,5,8\} \quad \{1,2,3,7,10\}$$

The resulting set of 132 blocks forms a $3-(12,5,6)$ design.

Next, let σ act on the following set of twelve blocks.

$\{1,2,3,5,7\}$	$\{1,2,3,6,7\}$	$\{1,2,3,6,9\}$	$\{1,2,3,8,10\}$
$\{1,2,3,8,12\}$	$\{1,2,3,10,12\}$	$\{1,2,4,5,7\}$	$\{1,2,4,5,8\}$
$\{1,2,4,6,10\}$	$\{1,2,4,8,12\}$	$\{1,2,6,10,12\}$	$\{1,3,6,9,12\}$

This produces another $3-(12,5,6)$ design. Finally, let the powers of ρ act on this design, obtaining a total of six $3-(12,5,6)$ designs. This set of six $3-(12,5,6)$ designs is a large set.

Note that this large set implies the existence of a large set of $2-(12,5,20)$ designs.

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A Tables of large sets

In the following tables, N denotes the number of designs in the large set and ? indicates that the large set is unknown. Also, * Denotes that the design does not exist.

Table 1: Existence of large sets of $t-(v,k,\lambda)$ designs, $6 \leq v \leq 12$

Parameters	N	Existence	Remarks
$2-(6,3,2)$	2	yes	Bhattacharya [2]
$2-(7,3,1)$	5	no	Cayley [4]
$2-(8,4,3)$	5	yes	Sharry and Street [22]
$3-(8,4,1)$	5	no	LS $2-(7,3,1)$ does not exist
$2-(9,3,1)$	7	yes	Kirkman [15]
$2-(9,4,3)$	7	yes	this paper
$2-(10,3,2)$	4	yes	Teirlinck [25]
$2-(10,4,2)$	14	yes	this paper
$3-(10,4,1)$	7	no	Kramer and Mesner [16]
$2-(10,5,4)$	14	yes	this paper
$3-(10,5,3)$	7	yes	extension of LS $2-(9,4,3)$
$2-(11,3,3)$	3	yes	Teirlinck [26]
$2-(11,4,6)$	6	yes	Chee, Colbourn, Furino, Kreher [5]
$3-(11,4,4)$	2	yes	derivation of LS $4-(12,5,4)$
$2-(11,5,2)$	42	yes	this paper
$3-(11,5,2)$	14	no *	Oberschelp [20] and Dehon [9]
$4-(11,5,1)$	7	no	LS $3-(10,4,1)$ does not exist
$2-(12,3,2)$	5	yes	Schreiber [21]
$2-(12,4,3)$	15	yes	this paper
$3-(12,4,3)$	3	yes	Teirlinck [26]
$2-(12,5,20)$	6	yes	LS $3-(12,5,6)$ as 2−designs
$3-(12,5,6)$	6	yes	this paper
$4-(12,5,4)$	2	yes	Denniston [11]
$2-(12,6,5)$	42	yes	LS $3-(12,6,2)$ as 2−designs
$3-(12,6,2)$	42	yes	extension of LS $2-(11,5,2)$
$4-(12,6,2)$	14	no *	LS $3-(11,5,2)$ does not exist
$5-(12,6,1)$	7	no	LS $3-(10,4,1)$ does not exist

Table 2: Existence of large sets of $t-(v,k,\lambda)$ designs, $13 \le v \le 15$

Parameters	N	Existence	Remarks
$2-(13,3,1)$	11	yes	Denniston [10]
$2-(13,4,1)$	55	yes	Chouinard [7]
$3-(13,4,2)$	5	yes	Magliveras and O'Brien (unpublished)
$2-(13,5,5)$	33	?	
$3-(13,5,15)$	3	yes	Chee, Colbourn, Furino, Kreher [5]
$4-(13,5,3)$	3	?	
$2-(13,6,5)$	66	?	
$3-(13,6,20)$	6	?	
$4-(13,6,6)$	6	?	
$5-(13,6,4)$	2	yes	derivation of LS $6-(14,7,4)$
$2-(14,3,6)$	2	yes	Hanani [13]
$2-(14,4,6)$	11	?	
$3-(14,4,1)$	11	?	
$2-(14,5,20)$	11	?	
$3-(14,5,5)$	11	?	
$2-(14,6,15)$	33	?	
$3-(14,6,5)$	33	?	
$4-(14,6,15)$	3	yes	Chee, Colbourn, Furino, Kreher [5]
$5-(14,6,3)$	3	?	
$2-(14,7,6)$	132	?	
$3-(14,7,5)$	66	?	
$4-(14,7,20)$	6	?	
$5-(14,7,6)$	6	?	
$6-(14,7,4)$	2	yes	Kreher and Radziszowski [17]
$2-(15,3,1)$	13	yes	Denniston [10]
$2-(15,4,6)$	13	?	
$2-(15,5,2)$	143	?	
$3-(15,5,6)$	11	?	
$4-(15,5,1)$	11	no *	Mendelsohn and Hung [18]
$2-(15,6,5)$	143	?	
$3-(15,6,20)$	11	?	
$4-(15,6,5)$	11	?	
$2-(15,7,3)$	429	?	
$3-(15,7,15)$	33	?	
$4-(15,7,5)$	33	?	
$5-(15,7,15)$	3	?	
$6-(15,7,3)$	3	?	