Generalised Bhaskar Rao designs with elements from cyclic groups of even order

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Abstract

A necessary condition is given for the existence of some Generalised Bhaskar Rao designs (GBRDs) with odd block size over cyclic groups of even order. Some constructions are given for GBRDs over cyclic groups of even order with block size 3 and with block size 4.

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1 Introduction

A *design* is a pair (X, B) where X is a finite set of *elements* and B is a finite collection of (not necessarily distinct) subsets (called blocks) of X. A *balanced incomplete block design*, $BIBD(v,b,r,k,\lambda)$, is a design with v elements and b blocks such that

(i) each element appears in exactly r blocks

(ii) each block contains exactly $k(\leq v)$ elements

(iii) each pair of distinct elements appears in exactly λ blocks

Since $r(k-1) = \lambda(v-1)$ and $vr = bk$ are well-known necessary conditions for the existence of a $BIBD(v,b,r,k,\lambda)$ we denote this design by $BIBD(v,k,\lambda)$.

Let G be a finite group with identity e, and let $Z(G)$ be the group ring of G over the ring of integers Z. A *generalised Bhaskar Rao design (with one association class)* with parameters v,b,r,k,λ and G is a $v \times b$ matrix with entries from $G \cup \{0\}$, where $0 \notin G$, such that

$$WW^+ = reI + \frac{\lambda}{|G|}\sum_{g\in G} g(J-I) \tag{1}$$

where W^+ is the transpose of W with the group elements replaced by their inverses, and the product WW^+ is evaluated in $Z(G)$.

Let N be the matrix formed from W by replacing its group element entries by 1; then (1) gives

$$NN^T = (r-\lambda)I + \lambda J$$

that is, N is the incidence matrix of a $BIBD(v,k,\lambda)$. Thus we use the notation $GBRD(v,k,\lambda;G)$ to denote a generalised Bhaskar Rao design with one association class. Note that a necessary condition for the existence of a $GBRD(v,k,\lambda;G)$ is $|G| \mid \lambda$.

Note that the existence of a $GBRD(v,k,\lambda;G)$ for some v,k and G implies the existence of a $GBRD(v,k,c\lambda;G)$ for all c, by concatenation.

In this paper we are concerned only with the case where G is the group Z_n of integers modulo n. We will use $+$ to denote the addition in Z_n (the 'multiplication' of the group ring $Z[Z_n]$) and $\oplus$ to be the addition of the group ring. The zero of the group ring will be denoted by $*$; the identity of Z_n by 0 as usual.

We further restrict attention to the case n even; necessary and sufficient conditions for the existence of a $GBRD(v,3,\lambda;Z_n)$ where n is odd are given in [2].

2 The group $Z_{2t}, t \in N$

Theorem 2.2 below eliminates the possibility of the existence of certain $GBRD(v,k,\lambda;Z_{2t})$s.

Lemma 2.1 *If there is a $GBRD(v,k,\lambda;Z_{mn})$ then there is a $GBRD(v,k,\lambda;Z_n)$.*

Proof: Replace every group element entry x of the $GBRD(v,k,\lambda;Z_{mn})$ by x (mod n). Clearly the resulting matrix is a $GBRD(v,k,\lambda;Z_n)$. □

Theorem 2.2 *A necessary condition for the existence of a $GBRD(v,k,2ct;Z_{2t})$ with t, k and c odd is $v \equiv 0$ or 1 (mod 4).*

Proof: In view of Lemma 2.1 above it is sufficient to prove the result for $t=1$ only.

Let k and c be odd and suppose there exists a $GBRD(v,k,2c;Z_2)$; denote it by B. Let $\mathcal{B}$ be the set of matrices, of the same size as B, with $*$'s in the same positions as B and elements of Z_2 elsewhere. We associate with every matrix $M \in \mathcal{B}$ a binary $\binom{v}{2}$ -vector $\underline{d}(M)$ as follows. Let $M \in \mathcal{B}$. If (r_i, r_j) is any pair of rows of M define

$$d(r_i, r_j) = a_0 + a_1 + \ldots + a_{2c-1}$$

where the a_i are given by

$$r_i - r_j = a_0 \oplus a_1 \oplus \ldots \oplus a_{2c-1} \qquad (a_l \in Z_2, 0 \le l \le 2c-1)$$

Label the distinct unordered pairs of rows of M by $p_1(M), p_2(M), \ldots, p_{\binom{v}{2}}(M)$ (where the ordering is the same for all $M \in \mathcal{B}$). Define $\underline{d}(M)$ by

$$\underline{d}(M)_i = d(p_i(M)), \qquad \left(1 \le i \le \binom{v}{2}\right).$$

Now consider $M \in \mathcal{B}$ and let M' be the matrix obtained from M by replacing one of its group element entries a by the group element a'. This change can only affect $k-1$ pairs of rows; we have

$$\underline{d}(M') = \underline{d}(M) + \underline{e}$$

where $\underline{e}$ is a vector of weight $k-1$ if $a - a' = 1$ and is the zero vector otherwise.

Let A be the matrix in $\mathcal{B}$ with every group element entry 0, so that $\underline{d}(A)$ is the zero vector. Since B can be obtained from A by repeatedly replacing occurrences of the group element entry 0 by the group element entry 1, it follows that $\underline{d}(B)$ lies in the span of a set of vectors all of weight $k-1$. Since $k-1$ is even, and the set of even weight binary $\binom{v}{2}$-vectors is a subspace of the vector space of binary $\binom{v}{2}$-vectors, $\underline{d}(B)$ has even weight.

Now since B is a $GBRD$ over Z_2 with $\lambda = 2c$, c odd,

$$d(p_i(B)) = c(0+1) = 1 \ , \qquad \left(1 \le i \le \binom{v}{2}\right) \ ,$$

ie. $\underline{d}(B)$ has every component equal to 1, and so has weight $\binom{v}{2}$.

Thus $\binom{v}{2}$ is even, i.e. $v \equiv 0$ or $1 \pmod 4$. □

The rest of the paper gives constructions for $GBRD$s with $k = 3$ and with $k = 4$. S_t will denote the subset $\{1, 2, \ldots, t-1, t+1, \ldots, 2t-1\}$ of Z_{2t} where $t \in N$. (We assume that we add and multiply elements of S_t as elements of Z_{2t}.

The following lemma is a generalisation of Lemma 3.1 [4].

Lemma 2.3 *If there exist*

(i) $k-1$ *permutations* $\pi_1, \pi_2, \ldots, \pi_{k-1}$ *of* S_t, $t \in N$, *such that for all* $p, q \in \{1, \ldots, k-1\}$, $p \ne q$,

$$\{\pi_p(i) - \pi_q(i) \mid i \in S_t\} = S_t \ , \qquad (2)$$

(ii) a $GBRD(v, k, 2; Z_2)$,

(iii) a $BIBD(v, k, 1)$,

then there exists a $GBRD(v, k, 2t; Z_{2t})$.

Proof: In the $GBRD(v, k, 2; Z_2)$ replace each 1 by t, to give a matrix A. Let B be the incidence matrix of the $BIBD(v, k, 1)$, with entries $*$ and 0. Form a new incidence matrix C from B by, replacing in each column of B,

(i) each $*$ by a row of $2t-2$ $*$'s ,

(ii) the j^{th} 0 by $\pi_j(1), \ldots, \pi_j(2t-1)$ $\quad (j = 1, \ldots, k-1)$,

(iii) the k^{th} 0 by a row of $2t-2$ 0's .

Then $B||C$ is a $GBRD(v, k, 2t; Z_{2t})$ (where $||$ denotes concatenation). □

Lemma 2.4 *For all* $t \ge 4$ *there exist at least two permutations* π_1, π_2 *of* S_t *satisfying (2) of Lemma 2.3.*

Proof: We consider separately the four cases according to the congruency modulo 4 of t. In each case π_1 is taken to be the identity, and we exhibit a construction for a particular π_2. Recall that operations are modulo $2t$.

(i) $t \equiv 0 \pmod 4$

Put $u = \frac{1}{2}t$, and take

$$\pi_2(i) = \begin{cases} 4u - i - 1 & (i = 1, \ldots, u-1) \\ u - 1 & (i = u) \\ 4u - i & (i = u+1, u+2, \ldots, 2u-1, 2u+1, \ldots, 3u-1) \\ 4u - i - 3 & (i = 3u, 3u+2, \ldots, 4u-2) \\ 4u - i + 1 & (i = 3u+1, 3u+3, \ldots, 4u-1) \end{cases}$$

Then $\pi_2(S_t)$ is the union of the following sets

$$\begin{array}{lcl}
\{\pi_2(i) \mid i = 1, 2, \ldots, u-1\} & = & \{4u-2, 4u-3, \ldots, 3u\} \\
\{\pi_2(i) \mid i = u\} & = & \{u-1\} \\
\{\pi_2(i) \mid i = u+1, \ldots, 2u-1, 2u+1, \ldots, 3u-1\} & = & \{3u-1, 3u-2, \ldots, 2u+1, \\
 & & 2u-1, \ldots, u+1\} \\
\{\pi_2(i) \mid i = 3u, 3u+4, \ldots, 4u-2\} & = & \{u-3, u-5, \ldots, 3, 1, 4u-1\} \\
\{\pi_2(i) \mid i = 3u+1, 3u+3, \ldots, 4u-3, 4u-1\} & = & \{u, u-2, \ldots, 4, 2\}
\end{array}$$

which is S_t, and $\{i - \pi_2(i) \mid i \in S_t\}$ is the union of the following sets

$$\begin{array}{lcl}
\{i - \pi_2(i) \mid i = 1, 2, \ldots, u-1\} & = & \{3, 5, \ldots, 2u-3, 2u-1\} \\
\{i - \pi_2(i) \mid i = u\} & = & \{1\} \\
\{i - \pi_2(i) \mid i = u+1, \ldots, 2u-1, 2u+1, \ldots, 3u-1\} & = & \{2u+2, 2u+4, \ldots, 4u-2, \\
 & & 2, \ldots, 2u-2\} \\
\{i - \pi_2(i) \mid i = 3u, 3u+2, 3u+4, \ldots, 4u-2\} & = & \{2u+3, 2u+7, \ldots, 4u-1\} \\
\{i - \pi_2(i) \mid i = 3u+1, 3u+3, \ldots, 4u-3, 4u-1\} & = & \{2u+1, 2u+5, \ldots, 4u-3\}
\end{array}$$

which again is S_t.

(ii) $t \equiv 1 \pmod 4$

Put $u = \frac{1}{2}(t-1)$ and take

$$\pi_2(i) = \begin{cases} 4u - i + 1 & (i = 1, \ldots, u-1) \\ u - 1 & (i = u) \\ 4u - i + 2 & (i = u+1, u+2, \ldots, 2u, 2u+2, \ldots, 3u+1) \\ 4u - i - 1 & (i = 3u+2, 3u+4, \ldots, 4u) \\ 4u - i + 3 & (i = 3u+3, 3u+5, \ldots, 4u+1) \end{cases}$$

Then $\pi_2(S_t)$ is the union of the following sets

$$\begin{array}{lcl}
\{\pi_2(i) \mid i = 1, 2, \ldots, u-1\} & = & \{4u, 4u-1, \ldots, 3u+2\} \\
\{\pi_2(i) \mid i = u\} & = & \{u-1\} \\
\{\pi_2(i) \mid i = u+1, u+2, \ldots, 2u, 2u+2, \ldots, 3u+1\} & = & \{3u+1, 3u, \ldots, 2u+2, \\
 & & 2u, \ldots, u+1\} \\
\{\pi_2(i) \mid i = 3u+2, 3u+4, \ldots, 4u\} & = & \{u-3, u-5, \ldots, 1, 4u+1\} \\
\{\pi_2(i) \mid i = 3u+3, 3u+5, \ldots, 4u+1\} & = & \{u, u-2, \ldots, 2\}
\end{array}$$

which is S_t, and $\{i - \pi_2(i) \mid i \in S_t\}$ is the union of the following sets

$$
\begin{array}{l|lcl}
\{i-\pi_2(i) & i = 1,2,\ldots,u-1\} & = & \{3,5,\ldots,2u-1\} \\
\{i-\pi_2(i) & i = u\} & = & \{1\} \\
\{i-\pi_2(i) & i = u+1, u+2,\ldots,2u, 2u+2,\ldots,3u+1\} & = & \{2u+2, 2u+4,\ldots,4u, \\
 & & & 2,4,\ldots,2u\} \\
\{i-\pi_2(i) & i = 3u+2, 3u+4,\ldots,4u\} & = & \{2u+5, 2u+9,\ldots,4u+1\} \\
\{i-\pi_2(i) & i = 3u+3, 3u+5,\ldots,4u+1\} & = & \{2u+3, 2u+7,\ldots,4u-1\}
\end{array}
$$

which again is S_t.

(iii) $t \equiv 2 \pmod 4$

Put $u = \frac{1}{2}(t+2)$, and take

$$
\pi_2(i) = \begin{cases}
4u-i-5 & (i = 1,\ldots,u-1) \\
u-1 & (i = u) \\
4u-i-4 & (i = u+1, u+2,\ldots,2u-3, 2u-1,\ldots,3u-5) \\
u & (i = 3u-4) \\
u-3 & (i = 3u-3) \\
4u-i-7 & (i = 3u-2, 3u,\ldots,4u-6) \\
4u-i-3 & (i = 3u-1, 3u+1,\ldots,4u-5)
\end{cases}
$$

Then $\pi_2(S_t)$ is the union of the following sets

$$
\begin{array}{l|lcl}
\{\pi_2(i) & i = 1,\ldots,u-1\} & = & \{4u-6, 4u-7,\ldots,3u-4\} \\
\{\pi_2(i) & i = u+1,\ldots,2u-3, 2u-1,\ldots,3u-5\} & = & \{3u-5, 3u-6,\ldots,2u-1, \\
 & & & 2u-3,\ldots,u+1\} \\
\{\pi_2(i) & i = 3u-2, 3u,\ldots,4u-6\} & = & \{u-5, u-7,\ldots,1, 4u-5\} \\
\{\pi_2(i) & i = 3u-1, 3u+1,\ldots,4u-5\} & = & \{u-2, u-4,\ldots,4,2\} \\
\{\pi_2(i) & i = u, 3u-4, 3u-3\} & = & \{u-1, u, u-3\}
\end{array}
$$

which is S_t, and $\{i - \pi_2(i) \mid i \in S_t\}$ is the union of the following sets

$$
\begin{array}{l|lcl}
\{i-\pi_2(i) & i = 1,2,\ldots,u-1\} & = & \{3,5,\ldots,2u-1\} \\
\{i-\pi_2(i) & i = u+1,\ldots,2u-3, 2u-1,\ldots,3u-5\} & = & \{2u+2, 2u+4,\ldots,4u-6, \\
 & & & 2,\ldots,2u-6\} \\
\{i-\pi_2(i) & i = 3u-2, 3u,\ldots,4u-6\} & = & \{2u+3, 2u+7,\ldots,4u-5\} \\
\{i-\pi_2(i) & i = 3u-1, 3u+1,\ldots,4u-7, 4u-5\} & = & \{2u+1, 2u+5,\ldots,4u-7\} \\
\{i-\pi_2(i) & i = u, 3u-4, 3u-3\} & = & \{1, 2u-4, 2u\}
\end{array}
$$

which again is S_t.

(iv) $t \equiv 3 \pmod 4$

There are two subcases:

(a) $t \equiv 1,2 \pmod 3$

Let

$$
\pi_2(i) = \begin{cases}
4i & (i = 1,2,\ldots,t-1) \\
t-2i & (i = t+1, t+2,\ldots,2t-1)
\end{cases}
$$

Because t is odd, $\{4i \mid i = 1, 2, \ldots, t-1\}$ is the set of even elements of S_t and $\{t - 2i \mid i = t+1, t+2, \ldots, 2t-1\}$ is the set of odd elements of S_t. Therefore π_2 is a permutation of S_t.

We must show that $\pi_1 - \pi_2$ is a permutation of S_t. Now for $i \in S_t$

$$\begin{cases} i - 4i = -3i \\ i - (t - 2i) = 3i + t \end{cases}$$

Since $2t$ is not divisible by 3, for $i, j \in S_t$,

$$-3i = -3j \iff 3(i-j) = 0 \iff i - j = 0 \iff i = j$$

Similarly, for $i, j \in S_t$,

$$3i + t = 3j + t \iff i = j$$

To prove that $\pi_1 - \pi_2$ is a permutation of S_t it remains to show that

$$-3i \neq 3j + t \qquad (i \in \{1, \ldots, t-1\}, \quad j \in \{t+1, \ldots, 2t-1\})$$

ie. that

$$-3i \neq 3j \qquad (i, j \in \{1, \ldots, t-1\})$$

But this follows since for $i, j \in S_t, -3i = 3j \iff i = -j$ similarly to above.

(b) $t \equiv 0 \pmod 3$

Put $t = 12u + 3$, and take

$$\pi_2(i) = \begin{cases} 2 & (i = 1) \\ 12u + 5 - 2i & (i = 2, 4, \ldots, 6u) \\ 12u + 6 - 2i & (i = 3, 5, \ldots, 6u+1) \\ 24u + 6 - \pi(12u + 3 - i) & (i = 6u+2, 6u+3, \ldots, 12u+2) \\ 12u + 1 - 2i & (i = 12u+4, 12u+6, \ldots, 16u+4) \\ 24u + 4 - 2i & (i = 12u+5, 12u+7, \ldots, 14u+1) \\ 12u + 2 - 2i & (i = 14u+3, 14u+5, \ldots, 16u+3) \\ 12u - 1 - 2i & (i = 16u+5, 16u+7, \ldots, 18u+1) \\ 24u + 6 - 2i & (i = 16u+6, 16u+8, \ldots, 18u+2) \\ 24u + 5 & (i = 18u+3) \\ 12u + 4 & (i = 18u+4) \\ 24u + 6 - \pi(12u + 3 - i) & (i = 18u+5, 18u+6, \ldots, 24u+5) \end{cases}$$

Then $\pi_2(S_t)$ is the union of the following sets

$$
\begin{array}{lll}
\{\pi_2(i) \mid i = 1\} & = & \{2\} \\
\{\pi_2(i) \mid i = 2, 4, \ldots, 6u\} & = & \{12u+1, 12u-3, \ldots, 5\} \\
\{\pi_2(i) \mid i = 3, 5, \ldots, 6u+1\} & = & \{12u, 12u-4, \ldots, 4\} \\
\{\pi_2(i) \mid i = 6u+2, 6u+3, 12u+2\} & = & \{24u+4\} \\
 & & \cup\{24u+1, 24u-3, \ldots, 12u+5\} \\
 & & \cup\{24u+2, 24u-2, \ldots, 12u+6\} \\
\{\pi_2(i) \mid i = 12u+4, 12u+6, \ldots, 16u+4\} & = & \{12u-1, 12u-5, \ldots, 4u-1\} \\
\{\pi_2(i) \mid i = 12u+5, 12u+7, \ldots, 14u+1\} & = & \{24u, 24u-4, \ldots, 20u+8\} \\
\{\pi_2(i) \mid i = 14u+3, 14u+5, \ldots, 16u+3\} & = & \{8u+2, 8u-2, \ldots, 4u+2\} \\
\{\pi_2(i) \mid i = 16u+5, 16u+7, \ldots, 18u+1\} & = & \{4u-5, 4u-9, \ldots, 3\} \\
\{\pi_2(i) \mid i = 16u+6, 16u+8, \ldots, 18u+2\} & = & \{16u, 16u-4, \ldots, 12u+8\} \\
\{\pi_2(i) \mid i = 18u+3, 18u+4\} & = & \{24u+5, 12u+4\} \\
\{\pi_2(i) \mid i = 18u+5, 18u+6, \ldots, 24u+5\} & = & \{20u+7, 20u+3, \ldots, 12u+7\} \\
 & & \cup\{4u-2, 4u-6, \ldots, 6\} \\
 & & \cup\{20u+4, 20u, \ldots, 16u+4\} \\
 & & \cup\{24u+3, 24u-1, \ldots, 20u+11\} \\
 & & \cup\{12u-2, 12u-6, \ldots, 8u+6\} \\
 & & \cup\{12u+2, 1\}
\end{array}
$$

which is S_t, and $\{i - \pi_2(i) \mid i \in S_t\}$ is the union of the following sets

$$
\begin{array}{lll}
\{i-\pi_2(i) \mid i = 1\} & = & \{24u+5\} \\
\{i-\pi_2(i) \mid i = 2, 4, \ldots, 6u\} & = & \{12u+7, 12u+13, \ldots, 24u+1, \\
 & & 1, 7, \ldots, 6u-5\} \\
\{i-\pi_2(i) \mid i = 3, 5, \ldots, 6u+1\} & = & \{12u+9, 12u+15, \ldots, 24u+3, \\
 & & 3, 9, \ldots, 6u-3\} \\
\{i-\pi_2(i) \mid i = 6u+2, 6u+3, 12u+2\} & = & \{12u+4\} \\
 & & \cup\{6u+8, 6u+14, \ldots, 24u+2\} \\
 & & \cup\{6u+6, 6u+12, \ldots, 24u\} \\
\{i-\pi_2(i) \mid i = 12u+4, 12u+6, \ldots, 16u+4\} & = & \{5, 11, \ldots, 12u+5\} \\
\{i-\pi_2(i) \mid i = 12u+5, 12u+7, \ldots, 14u+1\} & = & \{12u+11, 12u+17, \ldots, 18u-1\} \\
\{i-\pi_2(i) \mid i = 14u+3, 14u+5, \ldots, 16u+3\} & = & \{6u+1, 6u+7, \ldots, 12u+1\} \\
\{i-\pi_2(i) \mid i = 16u+5, 16u+7, \ldots, 18u+1\} & = & \{12u+10, 12u+16, \ldots, 18u-2\} \\
\{i-\pi_2(i) \mid i = 16u+6, 16u+8, \ldots, 18u+2\} & = & \{6, 12, \ldots, 6u-6\} \\
\{i-\pi_2(i) \mid i = 18u+3, 18u+4\} & = & \{18u+4, 6u\} \\
\{i-\pi_2(i) \mid i = 18u+5, 18u+6, \ldots, 24u+5\} & = & \{18u+10, 18u+16, \ldots, 24u-2\} \\
 & & \cup\{2, 8, \ldots, 6u+2\} \\
 & & \cup\{24u+4, 4, 10, \ldots, 12u-2\} \\
 & & \cup\{18u+11, 18u+17, \ldots, 24u-1\} \\
 & & \cup\{6u+9, 6u+15, \ldots, 12u-3\} \\
 & & \cup\{6u+3, 18u+5\}
\end{array}
$$

which again is S_t.

□

Theorem 2.5 *There exists a $GBRD(v, 3, 2t; Z_{2t})$ for all $t \geq 4$ and all $v \equiv 1$ or 9 (mod 12).*

Proof: Let v satisfy the above condition. From [3], a $GBRD(v, 3, 2; Z_2)$ exists, and the existence of a $BIBD(v, 3, 1)$ is well known. The result then follows from Lemmas 2.3 and 2.4.

□

Note. Results about the existence of $GBRD(v,3,2t;Z_{2t})$s with $t<4$ are given in [4].

For some values of v, for example $v=6$, there exists a $GBRD(v,3,2t;Z_{2t})$ for even t but not for odd t:

Theorem 2.6 *There exists a $GBRD(v,3,2t;Z_{2t})$ for t even and $v=6$.*

Proof: In the following incidence matrix of a $BIBD(6,3,2)$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

replace:

each 0 by t $*$'s
the 1^{st} 1 of each column by t 0's
the 2^{nd} 1 of columns 1, 2 and 4 by $0,1,2,\ldots,t-1$
the 2^{nd} 1 of columns 3, 7, 8 and 9 by $t,t+1,\ldots,2t-1$
the 2^{nd} 1 of columns 5 and 10 by $1,3,\ldots,2t-1$
the 2^{nd} 1 of column 6 by $1,2,\ldots,t$
the 3^{rd} 1 of columns 1, 2, 3 and 4 by $0,2,\ldots,2t-2$
the 3^{rd} 1 of columns 5 and 10 by $t,t+1,\ldots,2t-1$
the 3^{rd} 1 of columns 6 and 7 by $2t-1,2t-2,\ldots,t$
the 3^{rd} 1 of column 8 by $t-1,t-2,\ldots,0$
the 3^{rd} 1 of column 9 by $1,3,\ldots,2t-1$

It is easy to check that the resulting matrix is a $GBRD(6,3,2t;Z_{2t})$. □

We now construct $GBRD$s with $k=4$.

Lemma 2.7 *If $t\equiv 1$ or $5 \pmod 6$ then there exist at least three permutations π_1,π_2,π_3 of S_t satisfying (2) of Lemma 2.3.*

Proof: Let π_1 and π_2 be the permutations given in the proof of Lemma 2.4, (part (iv)(a)), and let $\pi_3(i)=\pi_2(i)-i \quad (i\in S_t)$.

The proof of Lemma 2.4 (part (iv)(a)) remains valid for all $t\equiv 1$ or $5 \pmod 6$, so we have that π_1,π_2 and $\pi_1-\pi_2$ are permutations of S_t. Now

$$\pi_3(i)=-(i-\pi_2(i))$$

so clearly π_3 is a permutation of S_t. Also

$$i-\pi_3(i)=\begin{cases} -2i & (i=1,\ldots,t-1) \\ t+4i & (i=t+1,\ldots,2t-1) \end{cases} \qquad = t+\pi_2(i+t)$$

so $\pi_1-\pi_3$ is a permutation of S_t. Finally

$$\pi_2(i)-\pi_3(i)=i \qquad (i\in S_t)$$

so $\pi_2-\pi_3$ is a permutation of S_t. □

Theorem 2.8 *There exists a $GBRD(v, 4, 2; Z_2)$ for all $t \equiv 1$ or $5 \pmod 6$ and $v \equiv 1 \pmod{12}$, v a prime power (or $v = 85$ or 133).*

Proof: From [1] there exists a $GBRD(v, 4, 2t; Z_{2t})$ for all v satisfying the above condition, and the existence of a $BIBD(v, 4, 1)$ is well-known. The result then follows from Lemmas 2.3 and 2.7. □

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