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Combinatorics and distributions of partial injections

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Abstract

We obtain several combinatorial results about chains, cycles and orbits of the elements of the symmetric inverse semigroup $\mathcal{IS}_n$ and the set T_n of nilpotent elements in $\mathcal{IS}_n$. We also get some estimates for the growth of $|\mathcal{IS}_n|$ and $|T_n|$, and study random products of elements from $\mathcal{IS}_n$.

1 Introduction

Roughly speaking, there are three semigroups, which play a principal role in the theory of transformation semigroups. The first one is the *full transformation semigroup* $\mathcal{T}_M$ of all transformations of a set, M, the second one is the *full partial transformation semigroup* $\mathcal{PT}_M$ of all partial transformations of M, and the third one is the *symmetric inverse semigroup* $\mathcal{IS}_M$ of all partial injective transformations of M. The role of the last semigroup is especially important in the theory of inverse semigroups, where this role is analogous to that of the symmetric group S_n in the group theory.

In the present paper we consider only finite sets. Hence we choose M to be the set $N = \{1, 2, \dots, n\}$. We denote the corresponding semigroups by $\mathcal{T}_n$, $\mathcal{PT}_n$ and $\mathcal{IS}_n$ respectively.

Combinatorial properties of $\mathcal{T}_n$, $\mathcal{PT}_n$ and some related transformation semigroups (for example the semigroup $\mathcal{O}_n = \{\alpha \in \mathcal{T}_n : x \leq y \Rightarrow \alpha(x) \leq \alpha(y)\}$ of all transformations preserving the natural order) were studied in a number of papers by several authors, see for example [10, 11, 13, 7, 15] and references therein. In particular, in the monograph [12] many combinatorial properties of $\mathcal{T}_n$ are collected in a separate big chapter. At the same time the situation with the semigroup $\mathcal{IS}_n$ is completely different. Only few papers, in which nilpotent elements and nilpotent subsemigroups of $\mathcal{IS}_n$ are studied, deal partially with some combinatorial questions, see [8, 2, 3, 6, 4]. A survey on these combinatorial results and some new combinatorial results on $\mathcal{IS}_n$ can be found in [5]. Monographs on semigroups, even those, dedicated completely to inverse semigroups, for example [17, 19], do not go much further than giving a formula for the cardinality of $\mathcal{IS}_n$. No combinatorial results can be found even in the monograph [18], which is dedicated completely to $\mathcal{IS}_n$.

In the present paper we study combinatorial properties of the elements of $\mathcal{IS}_n$ in general. The action of the element $\alpha \in \mathcal{IS}_n$ on N is described by the graph of the action, which leads to the standard combinatorial data, including such notions as *cyclic* and *chain* components of the graph and *orbits* of the elements from N. In Sections 2 and 3 we obtain several combinatorial formulae relating the ingredients of these data with each other, with the cardinality of the semigroup $\mathcal{IS}_n$ itself, and with the cardinality of the set T_n of all nilpotent elements from $\mathcal{IS}_n$.

In Section 4 we concentrate on the study of the set T_n and discover possibly the most surprising result of the paper, namely a strange duality between T_n and $\mathcal{IS}_n$. This duality is incarnated into a number of statements, each consisting of a pair of equalities, dual to each other in the sense, that one of the equalities is obtained from the other one by substituting the combinatorial data, related to $\mathcal{IS}_n$, with the corresponding combinatorial data, related to T_n, and vice versa.

In Section 5 we study the asymptotics of both $|\mathcal{IS}_n|$ and $|T_n|$. We obtain that the growth of both $|\mathcal{IS}_n|$ and $|T_n|$ can be (very) roughly described by $(n+2)!$, in particular, that it is roughly the same. At the same time, it is also shown that the limit value of the ratio $|T_n|/|\mathcal{IS}_n|$ is 0.

Finally, in Section 6 we study random products of k elements from $\mathcal{IS}_n$ under the assumption of the uniform distribution of original probabilities. We give both, a precise formula and some estimates, for the probability of such product to equal some fixed element from $\mathcal{IS}_n$, and show that for all k big enough almost all products of k elements from $\mathcal{IS}_n$ are zero. The distribution of probabilities we calculate is controlled by a square upper triangular matrix with non-negative integer entries. We show that the eigenvectors of this matrix can be computed purely combinatorially, in terms of the combinatorial data of $\mathcal{IS}_n$, and derive that the corresponding transformation matrix transforms the vector $(1, 1, \dots, 1)$ into the vector $(|\mathcal{IS}_n|, |\mathcal{IS}_{n-1}|, \dots, |\mathcal{IS}_0|)$.

2 Preliminary combinatorics

Throughout the paper for two sets, X and Y, by $X \subset Y$ we mean that $x \in X$ implies $x \in Y$ for every element x (in particular, $X = Y$ implies $X \subset Y$).

From the definition of $\mathcal{IS}_n$ it follows immediately that every element $a \in \mathcal{IS}_n$ is uniquely determined by its domain $\mathrm{dom}(a)$, its range $\mathrm{im}(a)$ and a bijection from dom(a) to im(a). Hence

$$|\mathcal{IS}_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k!.$$

The number $\mathrm{rank}(a) = |\operatorname{dom}(a)| = |\operatorname{im}(a)|$ is called the *rank* of a and the number $\operatorname{def}(a) = n - \operatorname{rank}(a)$ is called the *defect* of a.

For elements from $\mathcal{IS}_n$ one can use their regular table presentation

$$a = \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix},$$

where $\operatorname{dom}(a) = \{i_1, \dots, i_k\}$ and $\operatorname{im}(a) = \{j_1, \dots, j_k\}$. However, sometimes it is more convenient to use the so-called *chain* (or *chart*) decomposition of a, which is analogous to the cyclic decomposition for usual permutations. We refer the reader to [18] for rigorous definitions, however, this decomposition is very easy to explain on the following example. The element

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 7 & 9 \\ 7 & 4 & 5 & 1 & 10 & 2 & 6 \end{pmatrix} \in \mathcal{IS}_{10}$$

has the following graph of the action on $\{1, 2, \dots, 10\}$:

$$\begin{array}{ccc} 1 & \rightarrow & 7 \\ \uparrow & & \downarrow \\ 4 & \leftarrow & 2 \end{array} \qquad 3 \to 5 \to 10 \qquad 9 \to 6 \qquad 8,$$

and hence it is convenient to write it as $a = (1, 7, 2, 4)[3, 5, 10][9, 6][8]$. We call $(1, 7, 2, 4)$ a *cycle* and $[3, 5, 10]$ (as well as $[9, 6]$ and $[8]$) a *chain* of the element a. We remark that chains of length 1 correspond to those elements $x \in N$, which do not belong to $\operatorname{dom}(a) \cup \operatorname{im}(a)$. It is obvious that $\operatorname{def}(a)$ equals the number of chains in the chain decomposition of a. For $a \in \mathcal{IS}_n$ and $i \in N$ let c_i and d_i denote respectively the number of cycles and the number of chains of length i in the chain decomposition of a. The vector $(c_1, \dots, c_n, d_1, \dots, d_n)$ is called the *chain type* of a, see [4, 18].

Proposition 1. *([14, Lemma V.1.9]) The set $E(\mathcal{IS}_n)$ of idempotents in $\mathcal{IS}_n$ is a semigroups, isomorphic to the semigroup $\mathfrak{B}_n = \{A : A \subset N\}$ with the intersection of sets as the corresponding binary operation. In particular, $|E(\mathcal{IS}_n)| = 2^n$.*

The semigroup $\mathcal{IS}_n$ contains the zero element 0, which is the unique transformation such that $\operatorname{dom}(0) = \varnothing$. Recall that if S is a semigroup with zero 0, then the element $a \in S$ is called *nilpotent* provided that $a^k = 0$ for some $k > 0$. We will denote by T_n the set of all nilpotent elements in $\mathcal{IS}_n$ and remark that T_n is not a subsemigroup of $\mathcal{IS}_n$ (the product of two nilpotent elements is not nilpotent in general).

Proposition 2. *([5]) The element $a \in \mathcal{IS}_n$ is nilpotent if and only if the chain decomposition of a contains only chains. The number of nilpotent elements in $\mathcal{IS}_n$ with the given defect k equals the signless Lah number $L'(n, k) = \frac{n!}{k!}\binom{n-1}{k-1}$.*

By the *permutational part* of the element $(a_1,\dots,a_p)\dots(b_1,\dots,b_q)[c_1,\dots,c_s][d_1,\dots,d_t]$ we will mean the element $(a_1,\dots,a_p)\dots(b_1,\dots,b_q)[c_1][c_2]\dots[d_t]$. The rank of the permutational part of $\alpha\in\mathcal{IS}_n$ is called the *stable rank* of α and is denoted by $\mathrm{st.rank}(\alpha)$. This notion is analogous to the corresponding notion for $\mathcal{T}_n$, see [12]. It is obvious that $\mathrm{st.rank}(\alpha)=\mathrm{st.rank}(\alpha^i)$ for all $i\in\mathbb{N}$.

Taking the inverse element defines an anti-involution, $\alpha\mapsto\alpha^{-1}$, on $\mathcal{IS}_n$. The action of this anti-involution on α can be described as follows: one takes the graph of the action of α and reverses all arrows in it. It follows that this map does not change the chain type of α, in particular, nilpotent elements are sent to nilpotent elements. Since this map switches $\mathrm{im}(\alpha)$ and $\mathrm{dom}(\alpha)$, it allows one to transfer all statements about the ranges of the elements (in particular, of nilpotents) to the dual statements about the domains, and vice versa.

Studying probabilistic characteristics of various parameters of elements in $\mathcal{IS}_n$ it is natural to assume that the original distribution of probabilities of the elements in $\mathcal{IS}_n$ is uniform. An unexpected difficulty in this case is the fact that for two fixed $x,y\in N$ the random events "$x\in\mathrm{dom}(\alpha)$" and "$y\in\mathrm{dom}(\alpha)$" are not independent in general. For example, in $\mathcal{IS}_3$ we have

$$\Pr\big(1\in\mathrm{dom}(\alpha)\big)=\Pr\big(2\in\mathrm{dom}(\alpha)\big)=21/34,\quad\text{but}$$
$$\Pr\big((1\in\mathrm{dom}(\alpha))\text{ and }(2\in\mathrm{dom}(\alpha))\big)=6/17\neq(21/34)^2.$$

Furthermore, the random events "$x\in\mathrm{dom}(\alpha)$" and "$y\in\mathrm{dom}(\alpha)$" are not independent if we consider them for T_n instead of $\mathcal{IS}_n$ either.

For $k=0,\dots,n$ denote

$$\begin{aligned}R_{n,k}&=|\{\alpha\in\mathcal{IS}_n:\mathrm{rank}(\alpha)=k\}|,\\ D_{n,k}&=|\{\alpha\in\mathcal{IS}_n:\mathrm{def}(\alpha)=k\}|,\\ St_{n,k}&=|\{\alpha\in\mathcal{IS}_n:\mathrm{st.rank}(\alpha)=k\}|.\end{aligned}$$

Then we have

$$R_{n,k}=\binom{n}{k}^2\cdot k!\qquad\text{and}\qquad|\mathcal{IS}_n|=\sum_{k=0}^{n}R_{n,k}.$$

As $\mathrm{rank}(\alpha)+\mathrm{def}(\alpha)=n$, we have

$$D_{n,k}=R_{n,n-k}=\binom{n}{k}^2\cdot(n-k)!\qquad\text{and}\qquad|\mathcal{IS}_n|=\sum_{k=0}^{n}D_{n,k}.$$

From Proposition 2 we have $|T_n|=\sum_{k=1}^{n}L'(n,k)$. Now from

$$L'(n,k)=\binom{n-1}{k-1}\frac{n!}{k!}=\frac{k}{n}\binom{n}{k}^2(n-k)!=\frac{k}{n}D_{n,k}$$

it follows that

$$|T_n|=\sum_{k=1}^{n}\frac{k}{n}D_{n,k}=\sum_{k=1}^{n}\frac{n-k}{n}R_{n,k}. \tag{1}$$

Remark 1. There is a purely combinatorial way to show that the sets

$$M_1 = \{(\alpha, x) : \alpha \in T_n, \operatorname{def}(\alpha) = k, x \in N\} \quad \text{and}$$
$$M_2 = \{(\beta, l) : \beta \in \mathcal{IS}_n, \operatorname{def}(\beta) = k, l \text{ is a chain of } \beta\}$$

have the same cardinality, which implies $nL'(n, k) = kD_{n,k}$. Indeed, for $(\alpha, x) \in M_1$ we define $f((\alpha, x)) = (\beta, l) \in M_2$ in the following way: let $N_\alpha = \{y \in N : \alpha^i(x) = y \text{ for some } i \in \mathbb{N}\} = \{t_1, t_2, \dots, t_s\}$, $t_1 < t_2, \dots < t_s$, then $\operatorname{dom}(\beta) = (\operatorname{dom}(\alpha) \cup N_\alpha) \setminus \{x\}$, $\beta(y) = \alpha(y)$ for all $y \in \operatorname{dom}(\alpha) \setminus (\{x\} \cup N_\alpha)$, $\beta(\alpha^i(x)) = t_i$ for all $i = 1, \dots, s$; and l is the chain of β, containing x. One easily checks that $(\beta, l) \in M_2$ and that f is a bijection.

For $x \in \mathbb{R}$ and $k \in \{0, 1, \dots\}$ we denote by $[x]_k$ that *k-th decreasing factorial* $[x]_k = x(x-1)\dots(x-k+1)$.

Proposition 3. $St_{n,k} = [n]_k \sum_{i=1}^{n-k} L'(n-k, i)$.

Proof. We partition $\mathcal{IS}_n$ into classes with respect to the domain $A \subset N$ of the permutational part of the element $\alpha \in \mathcal{IS}_n$. The element α acts as a permutation on A and as a nilpotent on $N \setminus A$. Choosing A such that $|A| = k$, a permutation on A, and a nilpotent on $N \setminus A$ in all possible ways, and taking Proposition 2 into account, we get

$$St_{n,k} = \binom{n}{k} \cdot k! \cdot \sum_{i=1}^{n-k} L'(n-k, i) = [n]_k \sum_{i=1}^{n-k} L'(n-k, i).$$

□

Denote by $C_{n,k}$ the number of cycles of length k and by $L_{n,k}$ the number of all chains of length k in all elements in $\mathcal{IS}_n$.

Proposition 4. $L_{n,k} = [n]_k \cdot |\mathcal{IS}_{n-k}|$ *and* $C_{n,k} = \frac{1}{k}[n]_k \cdot |\mathcal{IS}_{n-k}|$.

Proof. The number of those elements in $\mathcal{IS}_n$, whose chain decomposition contains a fixed cycle (chain) of length k, equals $|\mathcal{IS}_{n-k}|$. On the other hand, given k elements from N, we can form $k!$ different chains and $(k-1)!$ different cycles of length k. Now the remark that $\binom{n}{k} \cdot k! = [n]_k$ completes the proof. □

Invertible elements in $\mathcal{IS}_n$ are exactly permutations, that is elements of the symmetric group S_n. Hence $b_n = \frac{|\mathcal{IS}_n|}{|S_n|} = \frac{|\mathcal{IS}_n|}{n!}$ characterizes (in some sense) the non-invertability of the elements of $\mathcal{IS}_n$, or how far $\mathcal{IS}_n$ is from being a group.

Corollary 1. *The average number c_n of components in the chain decomposition of the element $\alpha \in \mathcal{IS}_n$ equals*

$$c_n = b_n^{-1} \sum_{k=1}^{n} \left(1 + \frac{1}{k}\right) b_{n-k}.$$

Proof. From Proposition 4 it follows that

$$c_n = \frac{1}{|\mathcal{IS}_n|}\sum_{k=1}^{n}[n]_k\left(1+\frac{1}{k}\right)|\mathcal{IS}_{n-k}| = \frac{n!}{|\mathcal{IS}_n|}\sum_{k=1}^{n}\left(1+\frac{1}{k}\right)\frac{|\mathcal{IS}_{n-k}|}{(n-k)!}.$$

□

Remark 2. One can compare the last result with S_n and $\mathcal{T}_n$: the average number of components (that is cycles) in the cyclic decomposition of a permutation $\pi \in S_n$ equals $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, and the average number of components for an element $f \in \mathcal{T}_n$ equals $\sum_{k=1}^{n}\frac{n!}{k(n-k)!n^k}$, see [12, Lemma 6.1.12] or [16].

3 Chains and Orbits

Let L_n denote the total number of chains in the chain decompositions of all elements in $\mathcal{IS}_n$.

Proposition 5. $L_n = \sum_{k=0}^{n}(n-k)R_{n,k}$.

Proof. Each element of rank k has defect $n-k$ and thus contains $n-k$ chains. □

Comparing the last formula with Proposition 4 we get

Corollary 2.

$$\sum_{k=0}^{n-1}(n-k)\binom{n}{k}^2 k! = \sum_{k=1}^{n}[n]_k|\mathcal{IS}_{n-k}|.$$

One more recursive relation for the cardinalities of $\mathcal{IS}_n$ is given by

Proposition 6.

$$\frac{1}{n}\sum_{k=1}^{n}\left(k\cdot R_{n,k}+[n]_k|\mathcal{IS}_{n-k}|\right) = |\mathcal{IS}_n|.$$

Proof. We have $\mathrm{rank}(\alpha)+\mathrm{def}(\alpha) = n$ for every $\alpha \in \mathcal{IS}_n$. Hence the sum of the average rank and the average defect of all elements in $\mathcal{IS}_n$ must be equal to n as well. Therefore

$$\frac{1}{|\mathcal{IS}_n|}\sum_{k=1}^{n}k\cdot R_{n,k}+\frac{1}{|\mathcal{IS}_n|}\sum_{k=1}^{n}[n]_k|\mathcal{IS}_{n-k}| = n,$$

which completes the proof. □

Theorem 1.

$$L_n = \sum_{\alpha\in\mathcal{IS}_n}\mathrm{st.rank}(\alpha).$$

Proof. Consider the sets

$$A = \{(\alpha, c, x) : \alpha \in \mathcal{IS}_n, c \text{ is a cycle of } \alpha, x \text{ is a point of } c\},$$
$$B = \{(\beta, l) : \beta \in \mathcal{IS}_n, l \text{ is a chain from the chain decomposition of } \beta\}.$$

The statement of the theorem is equivalent to the equality $|A| = |B|$.

Consider the map $f : A \to B$, which is defined as follows: $f((\alpha, (x, a, \dots, b), x)) = (\beta, [x, a, \dots, b])$, where β is obtained from α substituting the cycle $(x, a, \dots, b)$ with the chain $[x, a, \dots, b]$. Consider also the map $g : B \to A$, which is defined as follows: $g((\beta, [x, a, \dots, b])) = (\alpha, (x, a, \dots, b), x)$, where α is obtained from β substituting the chain $[x, a, \dots, b]$ with the cycle $(x, a, \dots, b)$. Obviously f and g are inverse to each other and thus $|A| = |B|$. □

Remark 3. It is obvious that $\sum_{\alpha \in \mathcal{IS}_n} \operatorname{st.rank}(\alpha)$ is equal to the total sum of lengths of all cycles of all elements in $\mathcal{IS}_n$.

Let P_n denote the total number of fixed points for all elements in $\mathcal{IS}_n$. From Burnside's lemma it follows that the average number of fixed points for permutations in S_n equals 1. An analogue of this statement for $\mathcal{IS}_n$ is the following

Theorem 2.

$$P_n + \frac{1}{n} L_n = |\mathcal{IS}_n|. \qquad (2)$$

Proof. Consider the following sets:

$$A = \{(\alpha, x) : \alpha \in \mathcal{IS}_n, x \in N\},$$
$$B = \{(\beta, l) : \beta \in \mathcal{IS}_n, l \text{ is a chain for the chain decomposition of } \beta\},$$
$$C = \{(\gamma, y, z) : \gamma \in \mathcal{IS}_n, y \text{ is a fixed point of } \gamma, z \in N\}.$$

The equality (2) is equivalent to the equality $|A| = |B| + |C|$. To prove the latter we decompose A into a disjoint union $A = A_1 \cup A_2$, where

$$A_1 = \{(\alpha, x) \in A : x \text{ belongs to some chain of } \alpha\},$$
$$A_2 = \{(\alpha, x) \in A : x \text{ belongs to some cycle of } \alpha\}.$$

Consider the transformation, which maps the cycle $(x, a, \dots, b)$ with a base point x to the chain $[x, a, \dots, b]$. Obviously, this transformation induces a bijection $A_2 \to B$. Hence $|A_2| = |B|$.

To prove $|A_1| = |C|$ we construct mutually inverse bijections $f : A_1 \to C$ and $g : C \to A_1$. Consider any element $(\alpha, x) \in A_1$. If x is the source of some chain $[x, a, \dots, b]$ of length at least 2 from the chain decomposition of α, we define $f((\alpha, x)) = (\gamma, x, a)$, where γ is obtained from α substituting the chain $[x, a, \dots, b]$ with the cycle (x) and the cycle $(a, \dots, b)$. If x is the only point of the chain $[x]$, we define $f((\alpha, x)) = (\gamma, x, x)$, where γ is obtained from α substituting the chain $[x]$ with the cycle (x). Finally, if x is contained in some chain $[a, \dots, b, x, c, \dots, d]$ and is different from the source of this chain, we define $f((\alpha,x))=(\gamma,x,b)$, where γ is obtained from α substituting the chain $[a,\dots,b,x,c,\dots,d]$ with the cycle (x) and the chain $[a,\dots,b,c,\dots,d]$.

Let now $(\gamma,y,z)\in C$. If $y=z$, we define $g((\gamma,y,z))=(\alpha,z)$, where α is obtained from γ substituting the cycle (y) with the chain $[y]$. If z is a point of some chain $[a_1,\dots,a_s,z,b_1,\dots,b_t]$ in the chain decomposition of γ, we set $g((\gamma,y,z))=(\alpha,z)$, where α is obtained from γ substituting the cycle (y) and the chain $[a_1,\dots,a_s,z,b_1,\dots,b_t]$ with the chain $[a_1,\dots,a_s,z,y,b_1,\dots,b_t]$. Finally, if z is a point of some cycle $(a_1,\dots,a_s,z)$ of γ, we set $g((\gamma,y,z))=(\alpha,z)$, where α is obtained from γ substituting the cycles (y) and $(z,a_1,\dots,a_s)$ with the chain $[y,z,a_1,\dots,a_s]$.

Obviously, f and g are inverse to each other implying $|A_1|=|C|$, and the theorem follows. □

If $x\in\mathrm{dom}(\alpha)$, the set $\{x,\alpha(x),\alpha^2(x),\dots\}$ is called the *orbit* of x under α and the cardinality of this set is called the *length* of the orbit. If $x\notin\mathrm{dom}(\alpha)$, we say that the orbit is empty and consequently the length of the orbit is 0. Since for every transposition $(x,y)\in S_n$ the conjugation $\alpha\mapsto(x,y)\alpha(x,y)$ maps orbits of x to orbits of y and vice versa, it is enough to study the orbits of the element 1.

It is easy to calculate that the average length of the orbit of 1 under the action of the symmetric group S_n equals $(n+1)/2$, and the number of orbits of 1 of length i does not depend on i and equals $(n-1)!$. The corresponding situation in the semigroup $\mathcal{T}_n$ is much more interesting. For example, it is shown in [9] that the random function $X_n(\alpha)$, whose value is the cardinality of the permutational part of $\alpha\in\mathcal{T}_n$, and the random function $Y_n(\alpha)$, whose value is the length of the orbit of 1 for $\alpha\in\mathcal{T}_n$, have the same distribution. Later on an elementary proof of this fact was found in [1] (see also the historical review of this fact in [11]). For $\mathcal{IS}_n$ the corresponding statement does not hold, however, one has the following

Theorem 3. *The sum of lengths of the orbits of* 1 *over all elements* $\alpha\in\mathcal{IS}_n$ *equals the total number of chains in all elements in* $\mathcal{IS}_n$*.*

Proof. Let

$$A=\{(\alpha,x):\alpha\in\mathcal{IS}_n, x \text{ is a member of the orbit of 1 for } \alpha\},$$
$$B=\{(\beta,l):\beta\in\mathcal{IS}_n, l \text{ is a chain from the chain decomposition of } \beta\}.$$

The statement of the theorem is equivalent to the equality $|A|=|B|$. To prove the latter let us construct mutually inverse bijections $f:A\to B$ and $g:B\to A$.

Let $(\alpha,x)\in A$. If x is a point of the cycle $(x,\dots,1,\dots,y)$, we define $f((\alpha,x))=(\beta,[x,\dots,1,\dots,y])$, where β is obtained from α substituting the cycle $(x,\dots,1,\dots,y)$ with the chain $[x,\dots,1,\dots,y]$. If x is a point of the chain $[a,\dots,1,\dots,b,x,\dots,c]$ and $x\neq 1$, we define $f((\alpha,x))=(\beta,[x,\dots,c])$, where β is obtained from α substituting the chain $[a,\dots,1,\dots,b,x,\dots,c]$ with two chains, $[a,\dots,1,\dots,b]$ and $[x,\dots,c]$. Finally, if $x=1$ and it is a point of the chain $[a,\dots,1,b,\dots,c]$, we define $f((\alpha,x))=(\beta,[b,\dots,c])$, where β is obtained from α substituting the chain $[a,\dots,1,b,\dots,c]$ with the cycle $(a,\dots,1)$ and the chain $[b,\dots,c]$.

Let $(\beta, l) \in B$. If l contains 1 and has the form $l = [x, \dots, 1, \dots, y]$, we set $g((\beta, l)) = (\alpha, x)$, where α is obtained from β substituting the chain $[x, \dots, 1, \dots, y]$ with the cycle $(x, \dots, 1, \dots, y)$. If $l = [x, \dots, c]$ does not contain 1 and 1 belongs to another chain, $[a, \dots, 1, \dots, b]$ say, we set $g((\beta, l)) = (\alpha, x)$, where α is obtained from β substituting the chains $[a, \dots, 1, \dots, b]$ and $[x, \dots, c]$ with the chain $[a, \dots, 1, \dots, b, x, \dots, c]$. If $l = [x, \dots, c]$ does not contain 1 and 1 belongs to a cycle, $(a, \dots, 1)$ say, we set $g((\beta, l)) = (\alpha, 1)$, where α is obtained from β substituting the chain $[x, \dots, c]$ and the cycle $(a, \dots, 1)$ with the chain $[a, \dots, 1, x, \dots, c]$. It is obvious that under the definition of g the point x in the pair (α, x) always belongs to the orbit of 1 under the action of α.

It is easy to check that f and g are mutually inverse bijections, which completes the proof. □

Theorem 4. *Let $l_{n,k}$ denote the total number of orbits of 1, having length k, in all elements from $\mathcal{IS}_n$. Then*

(i) $l_{n,0} = |T_n|$,

(ii) $l_{n,1} = |\mathcal{IS}_{n-1}|$,

(iii) $l_{n,k} = [n-1]_{k-1}(L_{n-k} + 2|\mathcal{IS}_{n-k}|)$ *for* $1 < k \leq n$.

Proof. (i). According to the definition, $l_{n,0}$ is the cardinality of the set

$$E(n, 0) = \{\alpha \in \mathcal{IS}_n : 1 \notin \mathrm{dom}(\alpha)\}.$$

Consider the following decomposition of $E(n, 0)$ into a disjoint union of subsets:

$$E(n, 0) = \bigcup_{A \subset \{2,3,\dots,n\}} E_A,$$

where $E_A = \{\alpha \in E(n, 0) : A = \cap_{k>0} \mathrm{dom}(\alpha^k)\}$. In other words, E_A contains all those elements from $E(n, 0)$, for which A is the domain of the permutational part.

Consider also the following decomposition of T_n into a disjoint union of subsets:

$$T_n = \bigcup_{A \subset \{2,3,\dots,n\}} T_A,$$

where $T_A = \{\beta \in T_n : \beta$ contains the chain $[\dots, 1, a_1, \dots, a_k]$ and $\{a_1, \dots, a_k\} = A\}$. Set $\overline{A} = N \setminus A$. If we substitute the chain $[b_1, \dots, b_m, 1, a_1, \dots, a_k]$ with its initial subchain $[b_1, \dots, b_m]$, then every $\beta \in T_A$ can be transformed into the element $\overline{\beta}$ from the set $\tilde{T}_A$ of all those nilpotent elements from $\mathcal{IS}_{\overline{A}}$, which are not defined in the point 1. Moreover, every such nilpotent will be obtained exactly $|A|!$ times. Hence $|T_A| = |A|! \cdot |\tilde{T}_A|$.

On the other hand, the set $\overline{A}$ is α-invariant for every $\alpha \in E_A$, moreover, the restriction $\alpha|_{\overline{A}}$ is a nilpotent element from $\tilde{T}_A$. Since the restriction $\alpha|_{\overline{A}}$ does not depend on the permutational part of α, we get $|E_A| = |A|! \cdot |\tilde{T}_A|$.

Therefore $|T_A| = |E_A|$ for all $A \subset \{2, 3, \dots, n\}$ and hence $l(n, 0) = |E(n, 0)| = |T_n|$.

Remark 4. The equality $|T_A| = |E_A|$ can also be proved purely combinatorially, using a bijection, analogous to that, constructed in Remark 1.

(ii). The orbit of 1 under the action of α has length 1 if and only if 1 is a fixed point of α. The elements from $\mathcal{IS}_n$, for which 1 is a fixed point, are identified with $\mathcal{IS}_{\{2,3,\dots,n\}} \simeq \mathcal{IS}_{n-1}$ in a natural way.

(iii). If the orbit of 1 under the action of α has length $k > 1$, the element α has one of the following three types:

(I) $\alpha = (1, a_2, \dots, a_k) \dots$. We have $(n-1)\dots(n-k)|\mathcal{IS}_{n-k}| = [n-1]_{k-1}|\mathcal{IS}_{n-k}|$ elements of this type.

(II) $\alpha = [1, a_2, \dots, a_k] \dots$. We again have $[n-1]_{k-1}|\mathcal{IS}_{n-k}|$ elements of this type.

(III) $\alpha = [b_1, \dots, b_m, 1, a_2, \dots, a_k] \dots$.

With every α of type (III) we associate the pair $(\beta, [b_1, \dots, b_m])$, where $\beta \in \mathcal{IS}_{N\setminus\{1,a_2,\dots,a_k\}} \simeq \mathcal{IS}_{n-k}$ is obtained from α substituting the chain $[b_1, \dots, b_m, 1, a_2, \dots, a_k]$ with the chain $[b_1, \dots, b_m]$. It is obvious that this map is a bijection to the set

$$\{(\beta, l) : \beta \in \mathcal{IS}_{N\setminus\{1,a_2,\dots,a_k\}}, l \text{ is a chain from the chain decomposition of } \beta\}.$$

The elements $a_2, \dots, a_k$ can be chosen in $[n-1]_{k-1}$ different ways, and the pair (β, l) in L_{n-k} different ways. Hence the number of elements of type (III) equals $[n-1]_{k-1} \cdot L_{n-k}$. Adding up the last three numbers we obtained, we complete the proof of the theorem. □

Corollary 3. *$|T_n|$ equals the total number of partial injections from the set $\{2, 3, \dots, n\}$ to the set $\{1, 2, \dots, n\}$ (or from $\{1, 2, \dots, n\}$ to $\{2, 3, \dots, n\}$).*

Proof. Follows from Theorem 4(i) and natural identification of $E(n, 0)$ with partial injections from $\{2, 3, \dots, n\}$ to $\{1, 2, \dots, n\}$. Inverses for the later partial injections are exactly partial injections from $\{1, 2, \dots, n\}$ to $\{2, 3, \dots, n\}$. □

4 Nilpotent elements

Some aspects of combinatorial properties of nilpotent elements in $\mathcal{IS}_n$ were studied in [8, 2, 3, 6], however, the main objects in these papers were not the elements from T_n but rather certain nilpotent subsemigroups in $\mathcal{IS}_n$, that is some special subsets of T_n. The problem of calculating the cardinalities of such subsemigroups lead to interesting combinatorial schemes, involving such classical combinatorial objects as Bell numbers, Catalan numbers, Stirling numbers of the 2nd kind and others. An overview of the results in this direction can be found in [5].

In this section we will investigate the combinatorial properties of the set T_n itself. A striking phenomenon we discover is a kind of duality between the cardinalities of certain combinatorial sets, associated with $\mathcal{IS}_n$ and T_n. This duality will also appear in the next section and in the present section it will be visible in most statements.

However, we do not have any satisfactory explanation for its existence. We start with the theorem, which is in some sense dual to Theorem 4. We denote by $L^{(n)}$ the total number of chains in the chain decompositions of elements in T_n, and by $l^{n,k}$ the total number of orbits of 1 of length k for the elements in T_n.

Theorem 5. *(i)* $l^{n,0} = |\mathcal{IS}_{n-1}|$.

(ii) $|\{\alpha \in T_n :$ *the chain decomposition of* α *contains the chain* $[1]\}| = |T_{n-1}|$.

(iii) $l^{n,k} = [n-1]_{k-1} \cdot (L^{(n-k)} + |T_{n-k}|)$ *for* $1 < k \leq n$.

Proof. To prove (i) we note that, by definition, $l^{n,0}$ is the cardinality of the set $B = \{\alpha \in T_n : 1 \notin \mathrm{dom}(\alpha)\}$. The chain decomposition of every element from the set B has the form $\alpha = [a_1, \dots, a_k, 1] \dots$, where $k \geq 0$. Let us order the elements in $\{a_1, \dots, a_k\}$ in the increasing order: $a_{i_1} < a_{i_2} < \dots < a_{i_k}$. Note that the set $A = N \setminus \{a_1, \dots, a_k, 1\}$ is α-invariant, and define $\overline{\alpha} \in \mathcal{IS}_{\{2,3,\dots,n\}}$ in the following way: $\overline{\alpha}|_A = \alpha|_A$, $\overline{\alpha}|_{\{a_1,\dots,a_k\}} = \begin{pmatrix} a_{i_1} & \dots & a_{i_k} \\ a_1 & \dots & a_k \end{pmatrix}$. The map $\alpha \mapsto \overline{\alpha}$ is obviously a bijection from B to $\mathcal{IS}_{\{2,3,\dots,n\}}$ and the statement follows.

(ii) is obvious.

To prove (iii) we partition the elements of the set

$$\{\alpha \in T_n : \text{ the orbit of 1 under the action of } \alpha \text{ has length } k\}$$

into two classes, with respect to whether 1 is a starting point of some chain in the chain decomposition of α or not. The chain decomposition of every element α from the first class has the form $\alpha = [1, a_1, \dots, a_{k-1}] \dots$, where $a_1, \dots, a_{k-1}$ can be chosen in $[n-1]_{k-1}$ different ways, and all the other chains of α define some nilpotent element from $\mathcal{IS}_{N \setminus \{1, a_1, \dots, a_{k-1}\}}$. Hence the first class contains $[n-1]_{k-1} \cdot |T_{n-k}|$ elements.

The chain decomposition of every element from the second class has the form $\alpha = [b_1, \dots, b_m, 1, a_1, \dots, a_{k-1}] \dots$, where $m > 0$. The elements $a_1, \dots, a_{k-1}$ again can be chosen in $[n-1]_{k-1}$ different ways. If we now fix $a_1, \dots, a_{k-1}$, we can associate the corresponding elements α to the pair $(\beta, [b_1, \dots, b_m])$, where $\beta \in \mathcal{IS}_{N \setminus \{1, a_1, \dots, a_{k-1}\}}$ is obtained from α substituting the chain $[b_1, \dots, b_m, 1, a_1, \dots, a_{k-1}]$ with the chain $[b_1, \dots, b_m]$. This defines, for fixed $a_1, \dots, a_{k-1}$, a bijection from the set of all corresponding α to the set

$$\{(\beta, l) : \beta \text{ is a nilpotent element from } \mathcal{IS}_{N \setminus \{1, a_1, \dots, a_{k-1}\}}, l \text{ is a chain of } \beta\}.$$

Hence the second class contains $[n-1]_{k-1} \cdot L^{(n-k)}$ elements. □

Remark 5. The first parts of Theorems 4 and 5 are completely dual to each other. The last parts of these theorems are almost dual, however, one could not expect a perfect duality for this statement as there are no orbits of length 1 for nilpotent elements.

Theorem 6. *1.* $|T_n| = \frac{1}{n} L_n$.

2. $|\mathcal{IS}_n| = \frac{1}{n+1}L^{(n+1)}$.

Proof. The element $\alpha \in \mathcal{IS}_n$ can have some fixed points only in the case, when the permutational part of α is not trivial, that is if α is not nilpotent. For every $\alpha \notin T_n$ let $A_\alpha = \mathrm{dom}(\alpha^n)$ and $\overline{A_\alpha} = N \setminus A_\alpha$. Consider the set

$$M_\alpha = \{\beta \in \mathcal{IS}_n : \mathrm{dom}(\beta^n) = A_\alpha \text{ and } \alpha|_{\overline{A_\alpha}} = \beta|_{\overline{A_\alpha}}\}.$$

Since the permutational parts of the elements from M_α correspond to all permutations in S_{A_α}, it follows that the average number of fixed points for elements in M_α equals 1. Since $M_{\alpha_1} = M_{\alpha_2}$ or $M_{\alpha_1} \cap M_{\alpha_2} = \varnothing$ for arbitrary M_{α_1} and M_{α_2}, the sets M_α form a partition of $\mathcal{IS}_n \setminus T_n$ into a disjoint union of subsets. Hence the total number P_n of the fixed points equals $|\mathcal{IS}_n \setminus T_n|$. Theorem 2 now implies $\frac{1}{n}L_n = |\mathcal{IS}_n| - |\mathcal{IS}_n \setminus T_n| = |T_n|$. This proves (1).

To prove (2) it is enough to show that the cardinalities of the sets

$$A = \{(x, \alpha) : x \in \{1, 2, \dots, n+1\}, \alpha \in \mathcal{IS}_{\{1,2,\dots,n+1\}\setminus\{x\}}\} \qquad \text{and}$$
$$B = \{(\beta, l) : \beta \in T_{n+1}, l \text{ is a chain of } \beta\}$$

are the same. For this we define the map $f : A \to B$ in the following way. Let $(x, \alpha) \in A$ and $\begin{pmatrix} a_1 & \dots & a_k \\ a_{i_1} & \dots & a_{i_k} \end{pmatrix}$ be the permutational part of α. Assume that $a_1 < a_2 < \dots < a_k$ and set $f((x, \alpha)) = (\beta, l) \in B$, where $l = [a_{i_1}, \dots, a_{i_k}, x]$ and β is obtained from α substituting the permutational part with l.

We define the map $g : B \to A$, $g : (\beta, l) \mapsto (x, \alpha)$ in the following way: if $l = [a_1, \dots, a_k, a_{k+1}]$, we set $x = a_{k+1}$ and α is obtained from β substituting l with the permutational part $\begin{pmatrix} a_{i_1} & \dots & a_{i_k} \\ a_1 & \dots & a_k \end{pmatrix}$, where $a_{i_1} < a_{i_2} < \dots < a_{i_k}$ are elements $a_1, \dots, a_k$, written with respect to the natural increasing order.

It is easy to check that f and g are mutually inverse to each other, which implies $|A| = |B|$ and completes the proof. $\square$

Theorem 7. *1.* $|T_n| = |\mathcal{IS}_{n-1}| + L_{n-1}$.

2. $|\mathcal{IS}_n| = |T_n| + L^{(n)}$.

Proof. We start with (1). According to the first part of Theorem 4, we have $|T_n| = |B|$, where $B = \{\alpha \in \mathcal{IS}_n : 1 \notin \mathrm{dom}(\alpha)\}$. We partition B into two disjoint subsets $B_1 = \{\alpha \in B : 1 \notin \mathrm{im}(\alpha)\}$ and $B_2 = \{\alpha \in B : 1 \in \mathrm{im}(\alpha)\}$. The elements of B_1 are identified with the elements of $\mathcal{IS}_{\{2,3,\dots,n\}} \simeq \mathcal{IS}_{n-1}$ in a natural way. Hence $|B_1| = |\mathcal{IS}_{n-1}|$.

The elements from B_2 have chain decomposition of the form $\alpha = [b_1, \dots, b_k, 1] \dots$, where $k > 0$. Sending every such α to the pair $(\beta, [b_1, \dots, b_k])$, where $\beta \in \mathcal{IS}_{\{2,3,\dots,n\}}$ is obtained from α substituting the chain $[b_1, \dots, b_k, 1]$ with the chain $[b_1, \dots, b_k]$, we get a bijection from B_2 to the set $\{(\beta, l) : \beta \in \mathcal{IS}_{\{2,3,\dots,n\}}, l \text{ is a chain of } \beta\}$. Hence $|B_2| = L_{n-1}$.

Now we prove (2). Using the first part of Theorem 5, we can substitute $\mathcal{IS}_n$ by $B = \{\alpha \in T_{n+1} : n+1 \notin \mathrm{dom}(\alpha)\}$. The chain decomposition of every $\beta \in B$

contains the chain of the form $[a_1, \dots, a_k, n+1]$, where $0 \le k \le n$. For $k = 0$ the corresponding elements are identified with T_n in a natural way, hence the number of such elements in $|T_n|$. If $k > 0$, we map the element β to the pair $(\overline{\beta}, [a_1, \dots, a_k])$, where $\overline{\beta} \in \mathcal{IS}_n$ is obtained from β by substituting the chain $[a_1, \dots, a_k, n+1]$ by the chain $[a_1, \dots, a_k]$. It is easy to see that this map is a bijection to the set $\{(\alpha, l) : \alpha \in T_n, l$ is a chain of $\alpha\}$. Hence the number of such pairs equals $L^{(n)}$, which completes the proof. □

Remark 6. The first part of Theorem 7 implies that nilpotent elements form a substantial part of $|\mathcal{IS}_n|$, in particular, the inequality $|T_n| > |\mathcal{IS}_{n-1}|$ is very rough.

The following statement provides a precise connection between the cardinalities of $\mathcal{IS}_n$ and T_n:

Proposition 7.

$$|\mathcal{IS}_n| = \sum_{k=0}^{n} [n]_k |T_{n-k}| = \sum_{k=1}^{n} [n-1]_{k-1} (n+k) |T_{n-k}|.$$

Proof. The first equality follows from the fact that for a fixed $k > 0$ the number of elements in $\mathcal{IS}_n$, which have stable rank k equals $\binom{n}{k} \cdot k! \cdot |T_{n-k}| = [n]_k \cdot |T_{n-k}|$.

To prove the second equality one shows, analogously to the proof of Proposition 6, that the average number of fixed points in elements of stable rank $k > 0$ is 1. Moreover, the total number of points in the domains of the permutational parts of these elements equals $k \cdot \binom{n}{k} \cdot k! \cdot |T_{n-k}|$. Using Theorem 2 and Theorem 1 we now get

$$|\mathcal{IS}_n| = \sum_{k=1}^{n} \binom{n}{k} \cdot k! \cdot |T_{n-k}| + \frac{1}{n} \sum_{k=1}^{n} k \binom{n}{k} \cdot k! \cdot |T_{n-k}| = \sum_{k=1}^{n} [n-1]_{k-1} (n+k) |T_{n-k}|.$$

□

Corollary 4.

$$|T_n| = \sum_{k=1}^{n} k [n-1]_{k-1} |T_{n-k}|.$$

Proof. Follows from the right equality of Proposition 7. □

5 Various asymptotics

Lemma 1. *For every $n > 1$ the following holds*

(1) $2n - 1 \ge |T_n|/|T_{n-1}| \ge n + 1$, *moreover, both inequalities are strict for* $n > 2$,

(2) $2n > |\mathcal{IS}_n|/|\mathcal{IS}_{n-1}| > n + 1$.

Proof. To prove (1) we consider a chain $[a_1, \dots, a_k]$ from the chain decomposition of some $\alpha \in T_{n-1}$. Inserting the point n on different places into this chain we obtain $k+1$ different chains $[n, a_1, \dots, a_k]$, $[a_1, n, a_2, \dots, a_k]$, $\dots$, $[a_1, \dots, a_k, n]$. If we now perform this for every chain from the chain decomposition of α, we will get $(n-1) + \mathrm{def}(\alpha)$ different nilpotent elements in T_n. One more nilpotent element is obtained by adding the chain $[n]$ to α. Since $1 \le \mathrm{def}(\alpha) \le n-1$, we get

$$2n - 1 \ge (n-1) + \mathrm{def}(\alpha) + 1 \ge n + 1. \tag{3}$$

Therefore for every $\alpha \in T_{n-1}$ we get at least $n+1$ and at most $2n-1$ different elements from T_n. Certainly, performing this construction for all elements from T_{n-1} we will obtain all elements from T_n, moreover, each element will be obtained only once. Hence

$$(2n-1) \cdot |T_{n-1}| \ge |T_n| \ge (n+1) \cdot |T_{n-1}|. \tag{4}$$

If $n > 2$, then the left inequality in (3) is strict for all $\alpha \in T_{n-1}$ such that $\mathrm{def}(\alpha) = 1$, and the right inequality in (3) is strict for all $\alpha \in T_{n-1}$ such that $\mathrm{def}(\alpha) = n-1$. Hence both inequalities in (4) are strict in this case.

The proof of (2) is analogous with the following differences: one can insert the point n in a cycle of length k in k different ways, one can add both the cycle (a) and the chain $[a]$ to the chain decomposition of $\alpha \in \mathcal{IS}_{n-1}$. □

If we recall that T_n contains exactly $L'(n,k)$ and $\mathcal{IS}_n$ contains exactly $R_{n,n-k}$ elements of defect k, the proof of Lemma 1 immediately implies

Lemma 2. *1.* $|T_{n+1}| = \sum_{k=1}^{n} (n+k+1) L'(n,k)$,

2. $|\mathcal{IS}_{n+1}| = \sum_{k=0}^{n} (n+k+2) R_{n,n-k} = \sum_{k=0}^{n} (2n-k+2) R_{n,k}$.

Lemma 3. *If* $1 \le k < \sqrt{n+1} - 1$ *then* $L'(n,k+1) > L'(n,k)$, *and if* $\sqrt{n+1} - 1 < k < n$ *then* $L'(n,k+1) < L'(n,k)$.

Proof. We have $\frac{L'(n,k+1)}{L'(n,k)} = \frac{n-k}{k(k+1)}$ and we have

$$\frac{n-k}{k(k+1)} > 1 \Leftrightarrow k^2 + 2k - n > 0 \Leftrightarrow 1 \le k < \sqrt{n+1} - 1.$$

□

Using analogous arguments we can see that

Lemma 4. *If* $1 \le k < n + \frac{1}{2} - \sqrt{n + 5/4}$ *then* $R_{n,k+1} > R_{n,k}$, *and if* $n + \frac{1}{2} - \sqrt{n + 5/4} < k < n$ *then* $R_{n,k+1} < R_{n,k}$.

Lemma 5.

$$\lim_{n\to\infty} \frac{n \cdot L'(n, 3[\sqrt{n}])}{\sqrt{n} \cdot L'(n, 2[\sqrt{n}])} = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{n \cdot R_{n,n-3[\sqrt{n}]}}{\sqrt{n} \cdot R_{n,n-2[\sqrt{n}]}} = 0.$$

Proof. To prove the first formula we set $m = [\sqrt{n}]$. Using the Stirling formula for $n!$ we get

$$\frac{n \cdot L'(n,3m)}{\sqrt{n} \cdot L'(n,2m)} = \frac{n \cdot \frac{(n-1)!}{(n-3m)!(3m-1)!} \cdot \frac{n!}{(3m)!}}{\sqrt{n} \cdot \frac{(n-1)!}{(n-2m)!(2m-1)!} \cdot \frac{n!}{(2m)!}}$$
$$= \sqrt{n} \cdot \frac{\sqrt{2 \cdot (2m-1)(n-2m)}}{\sqrt{3 \cdot (3m-1)(n-3m)}} \cdot \exp(m) \cdot \left(\frac{n-2m}{n-3m}\right)^{n-3m} \cdot \left(\frac{2m-1}{3m-1}\right)^{2m-1}$$
$$\times \left(\frac{n-2m}{3m-1}\right)^{m} \cdot \left(\frac{4}{27}\right)^{m} \cdot \frac{1}{m^m} \cdot (1+o(1)).$$

But we have

$$\frac{\sqrt{2 \cdot (2m-1)(n-2m)}}{\sqrt{3 \cdot (3m-1)(n-3m)}} = \frac{2}{3}(1+o(1)), \qquad \left(\frac{n-2m}{n-3m}\right)^{n-3m} = \exp(m)(1+o(1)),$$
$$\left(\frac{2m-1}{3m-1}\right)^{2m-1} = \left(\frac{2}{3}\right)^{2m-1} \cdot \exp(-1/3) \cdot (1+o(1)),$$
$$\left(\frac{n-2m}{3m-1}\right)^{m} \le \left(\frac{m^2}{3m-1}\right)^{m} = \left(\frac{m}{3}\right)^{m} \cdot \exp(1/3) \cdot (1+o(1)).$$

Hence

$$\frac{n \cdot L'(n,3m)}{\sqrt{n} \cdot L'(n,2m)} \le \sqrt{n} \cdot \left(\frac{16\exp(2)}{3^6}\right)^{m} \cdot (1+o(1)).$$

As $\sqrt{n} = m(1+o(1))$ and $16\exp(2)/3^6 < 1$, we obtain

$$\lim_{n\to\infty} \frac{n \cdot L'(n,3[\sqrt{n}])}{\sqrt{n} \cdot L'(n,2[\sqrt{n}])} = 0.$$

The proof of the second formula is analogous, using $R_{n,k} = \binom{n}{k}^2 \cdot k!$. □

Theorem 8.

$$\lim_{n\to\infty} \frac{|T_{n+1}|}{(n+2)|T_n|} = \lim_{n\to\infty} \frac{|\mathcal{IS}_{n+1}|}{(n+2)|\mathcal{IS}_n|} = 1.$$

Proof. From Lemma 1 it follows that for all n we have

$$\frac{|T_{n+1}|}{(n+2)|T_n|} \ge \frac{(n+2)|T_n|}{(n+2)|T_n|} = 1, \quad \text{and} \quad \frac{|\mathcal{IS}_{n+1}|}{(n+2)|\mathcal{IS}_n|} > \frac{(n+2)|\mathcal{IS}_n|}{(n+2)|\mathcal{IS}_n|} = 1.$$

Hence to prove the theorem it is enough to show that both sequences, $\frac{|T_{n+1}|}{(n+2)|T_n|}$ and $\frac{|\mathcal{IS}_{n+1}|}{(n+2)|\mathcal{IS}_n|}$, are majorized by a sequence, which converges to 1. For the sequence

$\frac{|T_{n+1}|}{(n+2)|T_n|}$ we have, using Lemma 2(1), the following:

$$\frac{|T_{n+1}|}{(n+2)|T_n|} = \frac{\sum_{k=1}^{n}(n+k+1)L'(n,k)}{(n+2)\sum_{k=1}^{n}L'(n,k)}$$
$$< \frac{\sum_{k<3[\sqrt{n}]}(n+3[\sqrt{n}]+1)L'(n,k) + \sum_{k\geq 3[\sqrt{n}]}(2n+1)L'(n,k)}{(n+2)\sum_{k=1}^{n}L'(n,k)}$$
$$< \frac{n+3[\sqrt{n}]+1}{n+2} + \frac{2n+1}{n+2}\cdot\frac{\sum_{k\geq 3[\sqrt{n}]}L'(n,k)}{\sum_{k=1}^{n}L'(n,k)}. \quad (5)$$

By Lemma 3 we have

$$\sum_{k\geq 3[\sqrt{n}]} L'(n,k) < \sum_{k\geq 3[\sqrt{n}]} L'(n,3[\sqrt{n}]) < n\cdot L'(n,3[\sqrt{n}])$$

and

$$\sum_{k=1}^{n} L'(n,k) > \sum_{[\sqrt{n}]\leq k\leq 2[\sqrt{n}]} L'(n,k) > [\sqrt{n}]\cdot L'(n,2[\sqrt{n}]).$$

Applying the first part of Lemma 5 we get that the second summand of (5) converges to 0. It is obvious that the first summand converges to 1, which completes the proof for the sequence $\frac{|T_{n+1}|}{(n+2)|T_n|}$.

For the sequence $\frac{|\mathcal{IS}_{n+1}|}{(n+2)|\mathcal{IS}_n|}$ we have, using Lemma 2(2), the following:

$$\frac{|\mathcal{IS}_{n+1}|}{(n+2)|\mathcal{IS}_n|} = \frac{\sum_{k=0}^{n}(2n-k+2)R_{n,k}}{(n+2)\sum_{k=0}^{n}R_{n,k}}$$
$$< \frac{\sum_{k<n-3[\sqrt{n}]}(2n+2)R_{n,k} + \sum_{k\geq n-3[\sqrt{n}]}(n+3[\sqrt{n}]+2)R_{n,k}}{(n+2)\sum_{k=1}^{n}R_{n,k}}$$
$$< \frac{2n+2}{n+2}\cdot\frac{\sum_{k<n-3[\sqrt{n}]}R_{n,k}}{\sum_{k=1}^{n}R_{n,k}} + \frac{n+3[\sqrt{n}]+2}{n+2}. \quad (6)$$

By Lemma 4 we have

$$\sum_{k<n-3[\sqrt{n}]} R_{n,k} < \sum_{k<n-3[\sqrt{n}]}^{n} R_{n,n-3[\sqrt{n}]} < n \cdot R_{n,n-3[\sqrt{n}]}$$

and

$$\sum_{k=1}^{n} R_{n,k} > \sum_{n-3[\sqrt{n}]\leq k\leq n-2[\sqrt{n}]} R_{n,k} > [\sqrt{n}] \cdot R_{n,n-2[\sqrt{n}]}.$$

Applying the second part of Lemma 5 we get that the first summand of (5) converges to 0. It is obvious that the second summand converges to 1, which completes the proof. □

Theorem 9.

$$\lim_{n\to\infty} \frac{|T_n|}{|\mathcal{IS}_n|} = 0.$$

Proof. Using the first statement of Theorem 7, Proposition 5 and Lemma 2(2) we have

$$\frac{|T_n|}{|\mathcal{IS}_n|} = \frac{|\mathcal{IS}_{n-1}| + L_{n-1}}{|\mathcal{IS}_n|} = \frac{|\mathcal{IS}_{n-1}|}{|\mathcal{IS}_n|} + \frac{\sum_{k=0}^{n-1}(n-k-1)R_{n-1,k}}{|\mathcal{IS}_n|} = \frac{|\mathcal{IS}_{n-1}|}{|\mathcal{IS}_n|} + \frac{\sum_{k=0}^{n-1}(n-k-1)R_{n-1,k}}{\sum_{k=0}^{n-1}(2n-k)R_{n-1,k}}.$$

By Lemma 1(2) the first summand of the last sum converges to 0. Let us study the second summand in more detail:

$$\frac{\sum_{k=0}^{n-1}(n-k-1)R_{n-1,k}}{\sum_{k=0}^{n-1}(2n-k)R_{n-1,k}} < \frac{\sum_{k<n-3[\sqrt{n-1}]}(n-k-1)R_{n-1,k} + \sum_{k\geq n-3[\sqrt{n-1}]}(n-k-1)R_{n-1,k}}{\sum_{k\geq n-3[\sqrt{n-1}]}(2n-k)R_{n-1,k}}$$

$$< \frac{n\cdot\sum_{k<n-3[\sqrt{n-1}]} R_{n-1,k}}{n\cdot\sum_{k\geq n-3[\sqrt{n-1}]} R_{n-1,k}} + \frac{3[\sqrt{n-1}]\cdot\sum_{k\geq n-3[\sqrt{n-1}]} R_{n-1,k}}{n\cdot\sum_{k\geq n-3[\sqrt{n-1}]} R_{n-1,k}}.$$

As in the proof of Theorem 8, Lemma 4 and the second part of Lemma 5 guarantee that the first summand in the last sum converges to 0. It is obvious that the second summand $3[\sqrt{n-1}]/n$ converges to 0 as well. This completes the proof. □

Theorem 10. *Let $m \in \mathbb{N}$ be fixed. Then the distribution of the ranks of the elements of $\mathcal{IS}_n$ modulo m is asymptotically uniform, that is for all $p \in \mathbb{Z}$ we have*

$$\lim_{n\to\infty} \frac{\sum\limits_{k\in A_{n,p}} R_{n,k}}{|\mathcal{IS}_n|} = \frac{1}{m},$$

where $A_{n,p} = \{x \in \mathbb{Z} : 0 \leq x \leq n,\ x \equiv p \mod m\}$.

Proof. Denote $F_p = \sum_{k\in A_{n,p}} R_{n,k}$ and let $k_0 = \lceil n + 1/2 - \sqrt{n+5/4}\rceil$. For $p \in \mathbb{Z}$ let $A^0_{n,p}$ denote the set of all $x \in A_{n,p}$ satisfying $x \leq k_0$, and $A^1_{n,p} = A_{n,p} \setminus A^0_{n,p}$. From Lemma 4 it follows that $R_{n,k}$ is increasing for $0 \leq k \leq k_0$ and decreasing for $k_0 \leq k \leq n$. For k_0 the value R_{n,k_0} is the maximal one (for a fixed n). Hence for all p, q we have

$$|F_p - F_q| \leq \left|\sum_{k\in A^0_{n,p}} R_{n,k} - \sum_{k\in A^0_{n,q}} R_{n,k}\right| + \left|\sum_{k\in A^1_{n,p}} R_{n,k} - \sum_{k\in A^1_{n,q}} R_{n,k}\right| < 2R_{n,k_0}.$$

Lemma 6. *Let $n > 10000$ and $|k - k_0| < \frac{\sqrt[4]{n}}{6} - 1$. Then $\left|\frac{R_{n,k+1}}{R_{n,k}} - 1\right| < \frac{1}{\sqrt[4]{n}}$.*

Proof. For $s = k - (n + 1/2 - \sqrt{n+5/4})$ we obviously have $|s| < \frac{\sqrt[4]{n}}{6}$. By direct calculation we get

$$\left|\frac{R_{n,k+1}}{R_{n,k}} - 1\right| = \left|\frac{n^2 - 2nk + k^2 - k - 1}{k+1}\right| = \left|\frac{s^2 - 2s\sqrt{n+5/4}}{n + 3/2 - \sqrt{n+5/4} + s}\right|.$$

Again by direct calculation it is easy to show that for $|s| \leq 1$ we have

$$\left|\frac{s^2 - 2s\sqrt{n+5/4}}{n + 3/2 - \sqrt{n+5/4} + s}\right| < \left|\frac{4\sqrt{n+5/4}}{n - \sqrt{n+5/4}}\right| < \frac{6\sqrt{n}}{n} < \frac{1}{\sqrt[4]{n}};$$

and that for $|s| > 1$ we have

$$\left|\frac{s^2 - 2s\sqrt{n+5/4}}{n + 3/2 - \sqrt{n+5/4} + s}\right| < \left|\frac{4s\sqrt{n+5/4}}{n - 2\sqrt{n}}\right| < \frac{6|s|}{\sqrt{n}} < \frac{1}{\sqrt[4]{n}}.$$

□

Lemma 7. *For all n big enough the inequality $k_0 - \left[\frac{\sqrt[4]{n}}{6}\right] + 1 \leq k \leq k_0 + \left[\frac{\sqrt[4]{n}}{6}\right] - 1$ implies the inequality $\frac{R_{n,k_0}}{R_{n,k}} < 2$.*

Proof. From Lemma 6 it follows that for all such k we have

$$\frac{R_{n,k_0}}{R_{n,k}} < \left(1 + \frac{1}{\sqrt[4]{n}}\right)^{\sqrt[4]{n}/6} = e^{1/6}(1 + o(1)).$$

The remark that $e^{1/6} < 2$ completes the proof. □

From Lemma 7 it follows that for all n big enough and for all p and q we have

$$\frac{|F_p - F_q|}{|\mathcal{IS}_n|} < \frac{2R_{n,k_0}}{|\mathcal{IS}_n|} < \frac{2R_{n,k_0}}{\sum\limits_{k \in B_n} R_{n,k}} < \frac{2R_{n,k_0}}{2\left(\left[\frac{\sqrt[4]{n}}{6}\right] - 1\right) \cdot \frac{R_{n,k_0}}{2}} = \frac{2}{\left[\frac{\sqrt[4]{n}}{6}\right] - 1},$$

where $B_n = \left\{k \in \mathbb{Z} : k_0 - \left[\frac{\sqrt[4]{n}}{6}\right] + 1 \leq k \leq k_0 + \left[\frac{\sqrt[4]{n}}{6}\right] - 1\right\}$. Hence

$$\lim_{n\to\infty} \frac{F_p}{|\mathcal{IS}_n|} = \lim_{n\to\infty} \frac{F_q}{|\mathcal{IS}_n|}.$$

As $F_0 + F_1 + \cdots + F_{m-1} = |\mathcal{IS}_n|$, we finally get $\lim\limits_{n\to\infty} \frac{F_p}{|\mathcal{IS}_n|} = \frac{1}{m}$. □

6 Random products

We consider the products $x_1x_2\dots x_k$ of elements from $\mathcal{IS}_n$ of length k. We assume that the elements $x_1, x_2, \dots, x_k$ are chosen randomly and independently, with the uniform distribution of probabilities, that is the probability to choose the element $\alpha \in \mathcal{IS}_n$ does not depend on α and equals $\frac{1}{|\mathcal{IS}_n|}$.

Lemma 8. *Given $\alpha \in \mathcal{IS}_n$, the probability of the following random event "the random product $x_1x_2\dots x_k$ of elements from $\mathcal{IS}_n$ of length k equals α" depends only on* $\mathrm{rank}(\alpha)$*.*

Proof. Let $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta) = m$ and $\alpha = \begin{pmatrix} a_1 & \dots & a_m \\ b_1 & \dots & b_m \end{pmatrix}$, $\beta = \begin{pmatrix} c_1 & \dots & c_m \\ d_1 & \dots & d_m \end{pmatrix}$. Let $\mu, \tau \in S_n$ be such that $\mu = \begin{pmatrix} c_1 & \dots & c_m & \dots \\ a_1 & \dots & a_m & \dots \end{pmatrix}$, $\tau = \begin{pmatrix} b_1 & \dots & b_m & \dots \\ d_1 & \dots & d_m & \dots \end{pmatrix}$. Then $\mu\alpha\tau = \beta$ and the map

$$\{(x_1, \dots, x_k) : x_1 \dots x_k = \alpha\} \to \{(y_1, \dots, y_k) : y_1 \dots y_k = \beta\},$$
$$(x_1, \dots, x_k) \mapsto (\mu x_1, x_2, \dots, x_{k-1}, x_k\tau)$$

is injective. Hence $\Pr(\alpha = x_1 \dots x_k) \leq \Pr(\beta = y_1 \dots y_k)$. The opposite inequality follows by switching α and β. □

Let $P_{k,n}^{(i)}$ denote the probability of the random event "the random product $x_1x_2\dots x_k$ of elements from $\mathcal{IS}_n$ of length k is equal to a fixed element of rank i". From Lemma 8 it follows that $P_{k,n}^{(i)}$ is well-defined, that it does not depend on the choice of the element of rank i.

Corollary 5. *Let $M \subset \mathcal{IS}_n$ and m_i, $i = 0, \dots n$, denote the number of elements in M of rank i. Then the probability of the following random event: "the random product $x_1x_2\dots x_k$ of elements from $\mathcal{IS}_n$ of length k belongs to M" equals $\sum_{i=0}^{n} m_i P_{k,n}^{(i)}$. In particular, the probability of the random event "the random product $x_1x_2\dots x_k$ of elements from $\mathcal{IS}_n$ of length k belongs to T_n" equals $\sum_{i=0}^{n-1} L'(n,n-i)P_{k,n}^{(i)}$.*

Proposition 8.

$$P_{k,n}^{(i)} = \left(\frac{|\mathcal{IS}_{n-i}|}{|\mathcal{IS}_n|}\right)^k \cdot ([n]_i)^{k-1} \cdot P_{k,n-i}^{(0)}.$$

Proof. We fix $\alpha \in \mathcal{IS}_n$ such that $\mathrm{rank}(\alpha) = i$ and have $P_{k,n}^{(i)} = \Pr(x_1 \dots x_k = \alpha)$.

Take any random product $x_1 \dots x_k$ and set $A_0 = \mathrm{dom}(x_1 \dots x_k)$, $A_1 = x_1(A_0)$, $A_2 = x_2(A_1)$,..., $A_k = x_k(A_{k-1}) = \mathrm{im}(x_1 \dots x_k)$. Set $B_j = N \setminus A_j$, $j = 1, \dots, k$. Then with every x_j we associate two maps: the bijection $y_j = x_j|_{A_{j-1}} : A_{j-1} \to A_j$ and the partial injection $z_j = x_j|_{B_{j-1}} : B_{j-1} \to B_j$. Moreover, the equality $x_1 \dots x_k = \alpha$ becomes equivalent to the following pair of equalities: $y_1 \dots y_k = \alpha$, $z_1 \dots z_k = 0$. The sets $A_1, A_2, \dots, A_{k-1}$ and bijections $y_1, y_2 \dots, y_{k-1}$ can be chosen arbitrarily and this can be done in $([n]_i)^{k-1}$ different ways. After this choice the factor y_k is uniquely determined.

For every j, $j = 0, 1, \dots, k$, we fix a bijection $B_j \to \{1, 2, \dots, n-i\}$. Then every $z_j : B_{j-1} \to B_j$ is associated in a natural way with a partial injection, $\hat{z}_j \in \mathcal{IS}_{n-i}$. Moreover, the condition $z_1 \dots z_k = 0$ becomes equivalent to the condition $\hat{z}_1 \dots \hat{z}_k = 0$. Since for the last equation the factors can be chosen in $|\mathcal{IS}_{n-i}|^k \cdot P_{k,n-i}^{(0)}$ different ways, we get

$$P_{k,n}^{(i)} = \left(\frac{|\mathcal{IS}_{n-i}|}{|\mathcal{IS}_n|}\right)^k \cdot ([n]_i)^{k-1} \cdot P_{k,n-i}^{(0)},$$

which completes the proof. □

Corollary 6.

$$\frac{|\mathcal{IS}_{n-i}|^k}{|\mathcal{IS}_n|^k} \cdot ([n]_i)^{k-1} \geq P_{k,n}^{(i)} \geq \frac{|\mathcal{IS}_{n-i}|^{k-1}}{|\mathcal{IS}_n|^k} \cdot ([n]_i)^{k-1}.$$

Proof. This follows from Proposition 8 and the obvious inequality $1 \geq P_{k,n-i}^{(0)} \geq \frac{1}{|\mathcal{IS}_{n-i}|}$. □

Corollary 7. *Let n and $i > 0$ be fixed. Then $\lim_{k\to\infty} P_{k,n}^{(i)} = 0$.*

Proof. Using Corollary 6 and Lemma 1(2) we get

$$\begin{aligned} P_{k,n}^{(i)} &\le \frac{1}{[n]_i}\left(\frac{|\mathcal{IS}_{n-i}|}{|\mathcal{IS}_n|}\cdot [n]_i\right)^k \\ &= \frac{1}{[n]_i}\left(\frac{(n-i+1)|\mathcal{IS}_{n-i}|}{|\mathcal{IS}_{n-i+1}|}\cdot\dots\cdot\frac{(n-1)|\mathcal{IS}_{n-2}|}{|\mathcal{IS}_{n-1}|}\cdot\frac{n\cdot|\mathcal{IS}_{n-1}|}{|\mathcal{IS}_n|}\right)^k \\ &< \frac{1}{[n]_i}\left(\frac{(n-i+1)}{(n-i+2)}\cdot\dots\cdot\frac{n-1}{n}\cdot\frac{n}{n+1}\right)^k = \frac{1}{[n]_i}\left(\frac{n-i+1}{n+1}\right)^k. \end{aligned}$$

But $\lim\limits_{k\to\infty}\frac{1}{[n]_i}\left(\frac{n-i+1}{n+1}\right)^k = 0$, which completes the proof. □

Corollary 8. *Let n be fixed. Then $\lim\limits_{k\to\infty} P_{k,n}^{(0)} = 1$.*

Proof. Since $x_1\dots x_k \in \mathcal{IS}_n$ we get $\sum\limits_{i=0}^{n} P_{k,n}^{(i)}\cdot R_{n,i} = 1$. Since $R_{n,0}=1$, we obtain the equality $P_{k,n}^{(0)} = 1-\sum\limits_{i=1}^{n} P_{k,n}^{(i)}\cdot R_{n,i}$. From Corollary 7 it follows that $\sum_{i=1}^n P_{k,n}^{(i)}\cdot R_{n,i}\to 0$ if $k\to\infty$, and hence $P_{k,n}^{(0)}\to 1$. □

Remark 7. Corollary 8 implies that the semigroup $\mathcal{IS}_n$ is "almost nilpotent" in the sense that for all k big enough almost all products $x_1\dots x_k$ of elements from $\mathcal{IS}_n$ equal 0.

Corollary 9. $P_{k,n}^{(n)} = \frac{(n!)^{k-1}}{|\mathcal{IS}_n|^k}$.

Proof. Follows from Proposition 8 and the fact that $P_{k,0}^{(0)} = 1$ as $\mathcal{IS}_0 = \{0\}$. □

Corollary 10. *For fixed n and i we have*

$$P_{k,n}^{(i)} = \left(\frac{|\mathcal{IS}_{n-i}|}{|\mathcal{IS}_n|}\right)^k\cdot([n]_i)^{k-1}\cdot(1+o(1)).$$

Proof. Follows from Proposition 8 and Corollary 8. □

For $i,j\in\mathbb{N}$ we denote by $I(i,j)$ the number of partial injections from $\{1,2,\dots,i\}$ to $\{1,2,\dots,j\}$. It is obvious that $I(i,i)=|\mathcal{IS}_i|$, $I(i,j)=I(j,i)$, and

$$I(i,j) = \sum_{k=0}^{\min(i,j)}\binom{i}{k}\binom{j}{k}k!.$$

Consider the $(n+1)\times(n+1)$-matrix $\mathcal{A}=(A_{i,j})$, $i,j=0,1,2,\dots,n$, where

$$A_{i,j} = \begin{cases}\binom{n-i}{j-i}\binom{n}{j}\cdot j!\cdot I(n-i,n-j), & \text{if } i\le j,\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 11. *For all positive integers n and k we have the following equality of vectors*

$$\left(P_{k,n}^{(0)}, P_{k,n}^{(1)}, \dots, P_{k,n}^{(n)}\right)^t = \frac{1}{|\mathcal{IS}_n|^k}\mathcal{A}^{k-1} \cdot (1, 1, \dots, 1)^t.$$

Proof. We use induction in k and note that the statement is obvious for $k = 1$. Let us now calculate $P_{k+1,n}^{(i)} = \Pr(x_1 \dots x_k x_{k+1}) = \alpha$, where $\alpha \in \mathcal{IS}_n$ is a fixed element of rank i. It is obvious that

$$P_{k+1,n}^{(i)} = \sum_{j=i}^{n} \Pr(x_1 \dots x_k x_{k+1} = \alpha \text{ and } \operatorname{rank}(x_1 \dots x_k) = j).$$

The product $x_1 \dots x_k$ can be arbitrary, satisfying $\operatorname{dom}(x_1 \dots x_k) \supset \operatorname{dom}(\alpha)$. Under the additional assumption $\operatorname{rank}(x_1 \dots x_k) = j$, we get that the product $x_1 \dots x_k$ can have exactly $\binom{n-i}{j-i}\binom{n}{j}j!$ different values, where $\binom{n-i}{j-i}$ is the total number of extensions of $\operatorname{dom}(\alpha)$ up to $\operatorname{dom}(x_1 \dots x_k)$, $\binom{n}{j}$ is the number of ways to choose $\operatorname{im}(x_1 \dots x_k)$ and $j!$ is the number of ways to construct a bijection from $\operatorname{dom}(x_1 \dots x_k)$ to $\operatorname{im}(x_1 \dots x_k)$.

For a fixed $x_1 \dots x_k$ the action of x_{k+1} on $\operatorname{im}(x_1 \dots x_k)$ is uniquely defined, and the action of x_{k+1} on $N \setminus \operatorname{im}(x_1 \dots x_k)$ can be arbitrary with the only restriction $x_{k+1}(N \setminus \operatorname{im}(x_1 \dots x_k)) \subset N \setminus \operatorname{im}(\alpha)$. Hence for fixed $x_1 \dots x_k$ we have exactly $I(n-j, n-i)$ possibilities to choose x_{k+1}. This implies that

$$\Pr(x_1 \dots x_k x_{k+1} = \alpha \text{ and } \operatorname{rank}(x_1 \dots x_k) = j) = P_{k,n}^{(j)} \cdot \binom{n-i}{j-i}\binom{n}{j} j! \cdot \frac{I(n-j, n-i)}{|\mathcal{IS}_n|},$$

where $P_{k,n}^{(j)} \cdot \binom{n-i}{j-i}\binom{n}{j}j!$ is the probability of the occurrence of the necessary factor $x_1 \dots x_k$, and $\frac{I(n-j,n-i)}{|\mathcal{IS}_n|}$ is the probability of the occurrence of the independent necessary factor x_{k+1}.

Therefore

$$P_{k+1,n}^{(i)} = \sum_{j=i}^{n} P_{k,n}^{(j)} \cdot \binom{n-i}{j-i}\binom{n}{j} j! \cdot \frac{I(n-j, n-i)}{|\mathcal{IS}_n|} = \frac{1}{|\mathcal{IS}_n|}\sum_{j=i}^{n} P_{k,n}^{(i)} A_{i,j},$$

and hence

$$\left(P_{k+1,n}^{(0)}, \dots, P_{k+1,n}^{(n)}\right)^t = \frac{1}{|\mathcal{IS}_n|} \cdot \mathcal{A} \cdot \left(P_{k,n}^{(0)}, \dots, P_{k,n}^{(n)}\right)^t.$$

Taking into account the inductive assumption we complete the proof. □

We remark that the matrix $\mathcal{A}$ is upper triangular with the positive integers $A_{i,i} = [n]_i |\mathcal{IS}_{n-i}|$ on the diagonal. Hence these numbers are the eigenvalues of $\mathcal{A}$. Furthermore, according to Lemma 1(2), we have

$$\frac{A_{i,i}}{A_{i+1,i+1}} = \frac{[n]_i |\mathcal{IS}_{n-i}|}{[n]_{i+1} |\mathcal{IS}_{n-i-1}|} = \frac{|\mathcal{IS}_{n-i}|}{(n-i)|\mathcal{IS}_{n-i-1}|} > \frac{n-i+1}{n-i} > 1,$$

and thus all eigenvalues of $\mathcal{A}$ are different. Hence $\mathcal{A}$ has $n+1$ linearly independent eigenvectors.

Proposition 9. *The vectors*

$$\begin{aligned}
f_0 &= (R_{n,0}, 0, \dots, 0)^t,\\
f_1 &= (-R_{n,1}, R_{n-1,0}, 0, \dots, 0)^t,\\
&\dots\\
f_k &= ((-1)^k R_{n,k}, (-1)^{k-1} R_{n-1,k-1}, \dots, R_{n-k,0}, 0, \dots, 0)^t,\\
&\dots\\
f_n &= ((-1)^n R_{n,n}, (-1)^{n-1} R_{n-1,n-1}, \dots, -R_{1,1}, R_{0,0})^t
\end{aligned}$$

are the eigenvectors of $\mathcal{A}$ with eigenvalues $A_{0,0}$, $A_{1,1}, \dots$, $A_{n,n}$ respectively.

Proof. We are going to prove the statement using induction in n. For this we have to denote the matrix $\mathcal{A}$ of order $n+1$ by $\mathcal{A}_n$ and the corresponding vectors $f_0, \dots, f_n$ by $f_0^{(n)}, \dots, f_n^{(n)}$ respectively. Under this notation we have

$$\mathcal{A}_n = \left(\begin{array}{c|ccc} I(n,n) & R_{n,1}I(n,n-1) & \dots & R_{n,n}I(n,0) \\ \hline 0 & & n \cdot \mathcal{A}_{n-1} & \end{array}\right)$$

and $f_k^{(n)} = ((-1)^k R_{n,k} | f_{k-1}^{(n-1)})^t$.

For $n = 0$ we have $\mathcal{A}_0 = (1)$ and $f_0^{(0)} = (1)$ and the statement is obvious.

Let us now assume that the statement is true for $\mathcal{A}_{n-1}$. Then

$$\mathcal{A}_n \cdot f_k^{(n)} = \left(\begin{array}{c|ccc} I(n,n) & R_{n,1}I(n,n-1) & \dots & R_{n,n}I(n,0) \\ \hline 0 & & n \cdot \mathcal{A}_{n-1} & \end{array}\right) \cdot \left(\begin{array}{c} (-1)^k R_{n,k} \\ \hline f_{k-1}^{(n-1)} \end{array}\right)$$

$$= \left(\begin{array}{c} I(n,n) \cdot (-1)^k R_{n,k} + (R_{n,1}I(n,n-1), \dots, R_{n,n}I(n,0)) \cdot f_{k-1}^{(n-1)} \\ \hline n\mathcal{A}_{n-1} \cdot f_{k-1}^{(n-1)} \end{array}\right).$$

From the inductive assumption we get $\mathcal{A}_{n-1} \cdot f_{k-1}^{(n-1)} = [n-1]_{k-1} |\mathcal{IS}_{n-k}| f_{k-1}^{(n-1)}$ and hence $n\mathcal{A}_{n-1} \cdot f_{k-1}^{(n-1)} = [n]_k |\mathcal{IS}_{n-k}| f_{k-1}^{(n-1)}$.

The only thing, which is left to complete the proof, is to show the following equality for the first coordinate:

$$\sum_{i=0}^{k} R_{n,i} I(n, n-i) \cdot (-1)^{k-i} \cdot R_{n-i,k-i} = [n]_k I(n-k, n-k) \cdot (-1)^k R_{n,k}. \tag{7}$$

But we have $R_{n,i} \cdot R_{n-i,k-i} = R_{n,k}\binom{k}{i}$, and hence, canceling $R_{n,k} \cdot (-1)^k$, we reduce (7) to the following equality:

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} I(n, n-i) = [n]_k I(n-k, n-k). \tag{8}$$

To prove (8) we count the number F of those $\alpha \in \mathcal{IS}_n$, for which $\mathrm{dom}(\alpha) \supset \{1, 2, \dots, k\}$, in two different ways. The number of those $\alpha \in \mathcal{IS}_n$, which are not defined in $a_1, \dots, a_i$, equals $I(n-i, n)$. Therefore, using the principle of inclusion and exclusion, we get

$$F = \sum_{i=0}^{k} (-1)^i \binom{k}{i} I(n, n-i).$$

On the other hand, if $\alpha \in \mathcal{IS}_n$ satisfies $\{1, 2, \dots, k\} \subset \mathrm{dom}(\alpha)$, we can choose the values for α on the elements from $\{1, 2, \dots, k\}$ in $\binom{n}{k} \cdot k! = [n]_k$ different ways. If the action of α on $\{1, 2, \dots, k\}$ is already defined, the extension to N is naturally identified with a partial injection on $N \setminus \{1, 2, \dots, k\}$, and thus can be performed in $I(n-k, n-k)$ different ways. Hence $F = [n]_k \cdot I(n-k, n-k)$, which completes the proof of (8) and of the proposition. □

Proposition 10.

$$\sum_{k=0}^{n} (-1)^k |\mathcal{IS}_{n-k}| \cdot R_{n,k} = 1.$$

Proof. As we have seen in the proof of Proposition 9, the number of those $\alpha \in \mathcal{IS}_n$, which are defined in the given k points, equals $[n]_k \cdot I(n-k, n-k) = [n]_k \cdot |\mathcal{IS}_{n-k}|$. Hence, by the principle of inclusion and exclusion, the number of those elements in $\mathcal{IS}_n$, which are not defined in any point, equals

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} [n]_k |\mathcal{IS}_{n-k}| = \sum_{k=0}^{n} (-1)^k R_{n,k} |\mathcal{IS}_{n-k}|.$$

On the other hand, $\mathcal{IS}_n$ contains exactly one element, 0, which is not defined in any point. □

Corollary 11. *The vector* $(1, 1, \dots, 1)^t$ *has coordinates* $(|\mathcal{IS}_n|, |\mathcal{IS}_{n-1}|, \dots, |\mathcal{IS}_1|, |\mathcal{IS}_0|)$ *in the basis, formed by vectors* $f_0, f_1, \dots, f_n$ *(see Proposition 9).*

Proof. The vectors $f_0, f_1, \dots, f_n$ form a basis as eigenvectors, which correspond to different eigenvalues for a linear operator with simple spectrum. Let $T = (t_{i,j})$ be the transformation matrix to the basis $f_0, f_1, \dots, f_n$. For the entries of this matrix we have:

$$t_{i,j} = \begin{cases} (-1)^{j-i} R_{n-i,j-i}, & \text{if } i \le j, \\ 0, & \text{otherwise.} \end{cases}$$

The necessary statement is now equivalent to the equality

$$T \cdot (|\mathcal{IS}_n|, |\mathcal{IS}_{n-1}|, \dots, |\mathcal{IS}_1|, |\mathcal{IS}_0|)^t = (1, 1, \dots, 1)^t,$$

which follows immediately from Proposition 10. □

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Note added in proofs. An alternative approach to some questions studied in the present paper and some further results in this direction can be found in: S. Janson, V. Mazorchuk, Some remarks on the combinatorics of $\mathcal{IS}_n$, *Semigroup Forum* **70** (2005), no. 3, 391–405.

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