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# Bipartite probe interval graphs, circular arc graphs, and interval point bigraphs

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**Abstract**

An intersection graph is a graph whose vertices are in bijective correspondence to a collection of sets so that vertices are adjacent if and only if their corresponding sets intersect. A graph $G$ is a probe interval graph if it has a vertex partition $V(G) = (P, N)$ and an interval of $\mathbb{R}$ assigned to each vertex such that vertices are adjacent if and only if their corresponding intervals intersect and at least one of the vertices belongs to $P$. The sets $P$ and $N$ are called the probes and nonprobes, respectively. A circular arc graph is the intersection graph of arcs of a circle. An interval point bigraph is a bipartite intersection graph of points and intervals, that is, a graph $G$ with bipartition $V(G) = X \cup Y$ in which one of the partite sets corresponds to a collection of points of $\mathbb{R}$ and the other to intervals with vertices adjacent if and only if the point for one is contained in the interval for the other. We show that the complements of a class of 2-clique circular arc graphs, and a class of bipartite probe interval graphs are each equivalent to interval point bigraphs. Specifically, we characterize the bipartite probe interval graphs in which the probe/nonprobe partition can correspond to the bipartition. We also give a characterization for the aforementioned via a consecutively orderable edge partition into stars, and a new characterization for probe interval graphs by a consecutively orderable collection of quasi cliques.

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## 1 Background and Introduction

We will denote a bipartite graph $G$ whose vertices are partitioned into $X$ and $Y$ by $G = (X, Y, E)$. An *interval bigraph* $G = (X, Y, E)$ is the bipartite graph with vertices in $X$ in correspondence with a collection of intervals $\mathcal{I}_X$ of $\mathbb{R}$ and $Y$ in correspondence with another collection of intervals $\mathcal{I}_Y$ of $\mathbb{R}$ such that vertices are adjacent if and only if their corresponding intervals intersect. We call the collection $\mathcal{I}_X \cup \mathcal{I}_Y$ the *representation* of interval bigraph $G$. If one of the collections $\mathcal{I}_X$ or $\mathcal{I}_Y$ can be made to correspond to points of $\mathbb{R}$, then the interval bigraph is an *interval point bigraph*. In an interval point bigraph the partite set that corresponds to points will be called the *point partition*. Interval bigraphs were introduced in [10] as "bi-interval graphs", but the research waned until Das et al. introduced interval digraphs in [6], a class of directed graphs with the same model as interval bigraphs. An interval digraph is a directed graph with an ordered pair of intervals $(S_u, T_u)$ corresponding to each vertex $u$ with $u \to v \Leftrightarrow S_u \cap T_v \neq \emptyset$. Like the authors of this paper, Das et al. were apparently unaware of the bi-interval graphs of [10], as both introduced their respective classes of graphs as an analogue of interval graphs. But the work in [6] has provided the foundation for the continuing research on interval bigraphs. For a summary of this research see [2]. In [15] Müller gives a recognition algorithm for both interval digraphs and interval bigraphs and gives forbidden substructures for the bipartite model. Brown et al. study interval bigraphs' relationship with bipartite probe interval graphs, defined below, and also give characterizations and forbidden substructures in [3].

A *circular arc graph* is a graph with an arc of a circle $\mathcal{C}$ corresponding to each vertex such that vertices are adjacent if and only if their corresponding arcs intersect. The circle $\mathcal{C}$ from which arcs are taken will be called the *host circle*. Circular arc graphs have been studied extensively. There are around 50 papers in the literature giving various properties and characterizations for circular arc graphs. The seminal papers are perhaps those by Tucker: [21, 19]. A *2-clique graph* is a graph whose vertices can be partitioned into two sets, where each set induces a complete graph. For example, the complement of any bipartite graph is a 2-clique graph. The 2-clique circular arc graphs are characterized by the bipartite graphs forbidden as induced subgraphs in their complements in [14]. Feder, Hell, and Huang in [7] simplify this characterization by describing the substructure that forbids each of the bipartite graphs in the list of [14]. This new characterization is related to the work of Müller in [15] and leads to Theorem 1.1 proved in [11]. Since this result inspired the investigations that led to Theorem 5.2 proved herein, we record it here for perspective.

**Theorem 1.1** (Hell, Huang, [11]) *A bipartite graph $G$ is an interval bigraph if and only if $\overline{G}$ is a 2-clique circular arc graph with a representation in which no two arcs cover the host circle.*

Interval point bigraphs, like interval bigraphs, have a directed graph analogue called interval point digraphs. Both have been studied and characterized via their adjacency matrices (for the directed graphs) and by their reduced adjacency matrices (for the bigraph model) in [6] and [3], respectively. This matrix characterization will serve us well here and it is recorded as Theorem 3.1. Figure 1 gives examples of interval point bigraphs.

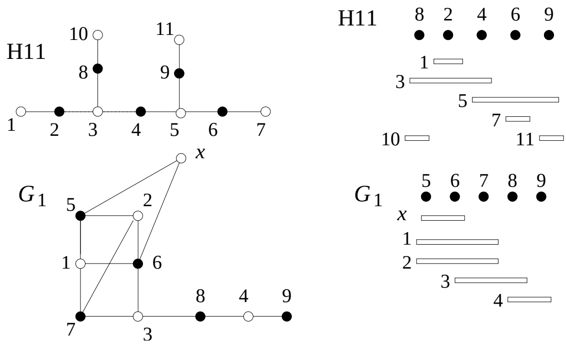


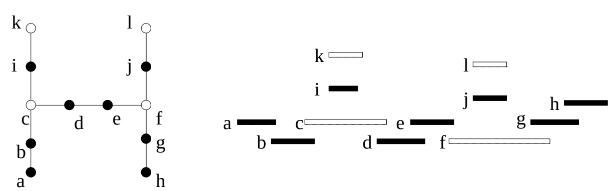

Figure 1: Two interval point bigraphs, solid vertices belong to the point partition.

The vertex set of graph $G$ will be denoted $V(G)$, and its edge set $E(G)$. A *probe interval graph* $G$ has a vertex partition $V(G) = (P, N)$ so that an interval of $\mathbb{R}$ can be assigned to each vertex with vertices adjacent if and only if their corresponding intervals intersect and at least one of the vertices belongs to $P$. The sets $P$ and $N$ are called *probes* and *nonprobes*, respectively. [One of the authors likes to note that if "at least" is replaced with "exactly" in the definition for probe interval graphs, then we obtain a definition for interval bigraphs.] Probe interval graphs were introduced to model a problem in the human genome project called the physical mapping of DNA, see [22, 23]. Because of complexity issues and the amount of data, only small fragments of DNA can be considered at a time, and then the original DNA must be reconstructed. The small fragments are typically called clones and the reconstruction is based on whether information is shared between clones. If a pair of clones contain the same information, then they may come from the same segment on the original DNA. Furthermore, interval graphs do not suffice for modeling this problem since it is often desirable to pay attention to only certain overlap information among a restricted collection of clones. The probe interval graph model allows for this: certain fragments can be labeled as nonprobes and their overlap can be ignored.

We have focused on investigating the structure of probe interval graphs, and in this paper we report on results about probe interval graphs that are bipartite. The problem of recognizing whether a given $n$-vertex graph $G$ with $m$ edges and partition $V(G) = (P, N)$ is a probe interval graph (with probes being $P$) is solvable in time $\mathcal{O}(n^2)$ via a method involving PQ-trees, see [12]. Another method given by [16] uses modular decomposition and has complexity $\mathcal{O}(n + m \log n)$. However the problem of recognizing whether a given graph with no partition specified is a probe interval graph remains open, as does the problem of determining a list of forbidden induced subgraphs in the general case. The trees, however, that are probe interval graphs have been characterized by two forbidden induced subgraphs in [18]. Also, the trees that are unit probe interval graphs (probe interval graphs with all intervals the same length) have been characterized by forbidden induced subgraphs in [1].

The paper [5] indicates that characterization of probe interval graphs by forbidden induced subgraphs in general will be difficult; specifically, the paper is devoted to developing a large list of forbidden induced subgraphs for the 2-trees that are probe interval graphs. Two of the results in this paper indicate the difficulty of the problem in the bipartite case. A collection of induced subgraphs $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$ of a graph $G$ is *consecutively ordered* if for each vertex $v$ of $G$, if $v \in G_i \cap G_k$, then $v \in G_j$ holds for $i < j < k$. Interval graphs, [8], probe interval graphs, [13], and interval bigraphs, [6, 4], all have characterizations by consecutively orderable collections of particular subgraphs. Theorem 4.1 shows that bipartite probe interval graphs also have such a characterization, but it also indicates a subtlety complicating the issue for bipartite probe interval graphs. Also, Conjecture 6.1 presents a lengthy list of forbidden induced subgraphs even for a restricted subclass of bipartite probe interval graphs.

Among our results in this paper is one that characterizes the bipartite probe interval graphs whose vertex partition can be a bipartition. For a simple example illustrating that the probe/nonprobe partition does not always correspond to a bipartition, consider H12 of Figure 2. In a probe interval representation, vertices c and f must be nonprobes, but they belong to different partite sets in the bipartition of H12. As foreshadowing, we note that H12 is not an interval point bigraph.




H12

Figure 2: Example of a bipartite probe interval graph with $P, N$ partition not a bipartition. Darkened vertices and intervals represent probes.

## 2 A Note on Probe Interval Graphs

In this section we record, for perspective, some results on various interval intersection graphs with consecutive order characterizations, and revise a result of Zhang, [22].

A graph is an *interval graph* if its vertices can correspond to a collection of intervals of $R$ with vertices adjacent if and only if their corresponding intervals intersect. It could be said that motivation for the recent work on graphs that have interval or arc intersection models stems from the abundance of nice results discovered for interval graphs, see [9] and [17] for excellent introductions to interval graphs and for discussions of these nice properties and other references. We record Theorem 2.1

because it supports Theorem 2.2 and Theorem 2.3, and all of these give context to Theorem 3.2. As indicated above, the characterization given in Theorem 2.1 is of a type common to graphs with interval intersection models.

**Theorem 2.1** (Fulkerson, Gross, 1965, [8]) *A graph is an interval graph if and only if its maximal cliques can be consecutively ordered.*

Let $\mathcal{G} = \{G_i\}$ be a collection of subgraphs of a graph $G$. If each edge of $G$ is contained in some $G_i \in \mathcal{G}$, then $\mathcal{G}$ is an *edge cover* for $G$; if there is exactly one $G_i$ containing each edge of $G$, then $\mathcal{G}$ is an *edge partition* for $G$. We have a characterization for interval bigraphs via a cover of consecutively orderable bicliques, that follows as a corollary to a result in [6] for interval digraphs, but also from a more general result on interval $k$-graphs in [4]. We will refer to this result later, so we record it here.

**Corollary 2.1** (Das et al., Brown et al. [6, 4]) *A bipartite graph $G$ is an interval bigraph if and only if it has a consecutively orderable edge cover of bicliques.*

We also have a characterization for probe interval graphs via a consecutively orderable cover of maximal quasi cliques, given by Theorem 2.3, below. But let us first define some terms. If $G$ is a graph with vertices partitioned into $(U, W)$, then we may write $G = (U, W, E)$; for example if $G$ is a probe interval graph, and the partition of vertices into probes and nonprobes is known, we write $G = (P, N, E)$. If $S$ is a subset of vertices of a graph $G$, then the graph induced on $S$ is denoted $G(S)$. A subset of vertices $S$ of a graph $G$ is an *independent set* if $G(S)$ has no edges. A *quasi clique* in a graph $G = (U, W, E)$, where $G(W)$ is an independent set, is a set $Q$ of vertices with $G(Q \cap P)$ a clique, and any vertex of $Q \cap N$ adjacent to all vertices of $Q \cap P$. A *maximal quasi clique* of $G$ is a quasi clique that is not contained in any larger quasi clique. A *complete set of maximal quasi cliques* of $G$ is a collection of maximal quasi cliques in which each maximal clique of $G$ is in exactly one maximal quasi clique of the set. A collection of sets is said to have the *Helly property* if whenever a subcollection $S_1, \ldots, S_k$ of them intersect pairwise, then $\bigcap_{i=1}^{k} S_i$ is nonempty. Any collection of intervals has the Helly property. Theorem 2.2 is purportedly a consequence of Theorem 2.1 and because intervals have the Helly property, see [13]. Aside from this claim, there is no published proof of Theorem 2.2, so we attempted to give one here, but the result we proved turned out to be a result with fewer conditions in the hypothesis; it is Theorem 2.3. An *interval split graph* is a graph $G = (U, W, E)$ with $G(U_1)$ an interval graph and $G(U_2)$ an independent set.

**Theorem 2.2** (Zhang, 1994, [22]) *An interval split graph $G = (U_1, U_2, E)$, $G(U_2)$ an independent set, is a probe interval graph with respect to the same partition $U_1 = P$, $U_2 = N$ if and only if there is a complete set of maximal quasi cliques that can be consecutively ordered.*

The following is simply a revision of the above theorem with the hypotheses simplified, and with the condition that the collection of maximal quasi cliques be complete relegated.

**Theorem 2.3** *Let $G = (U_1, U_2, E)$ be a graph with $G(U_2)$ an independent set. $G$ is a probe interval graph with $P = U_1, N = U_2$ if and only if $G$ has an edge cover of quasi cliques that can be consecutively ordered.*

**Proof.** Let $G = (U_1, U_2, E)$ be a graph with $G(U_2)$ an independent set.

Suppose $\mathcal{Q} = \{Q_1, \ldots, Q_s\}$ is an edge cover of quasi cliques that is consecutively ordered. For each $v \in V(G)$ define an interval $I(v) = [l(v), r(v)]$, where $l(v) = \min\{i : v \in Q_i\}$, and $r(v) = \max\{j : v \in Q_j\}$, and $N = U_2, P = U_1$. We claim $\{I(v) : v \in V(G)\}$ together with the partition is a probe interval representation for $G$. By definition of $\mathcal{Q}$, for vertices $u, v$, we have $I(u) \cap I(v) \neq \emptyset$ if and only if they belong to the same quasi clique, in which case $uv \in E$ unless $u, v \in N = U_2$; but if $u, v \in N$, then the intersection of their intervals does not induce an edge in $G$. Thus, $G$ is a probe interval graph with $U_1 = P$ and $U_2 = N$.

Now suppose $G$ has a probe interval representation $\mathcal{I} = \{I(v)\}_{v \in V}$, and let $r_1 < r_2 < \cdots < r_m$ be the distinct right endpoints among intervals of $\mathcal{I}$. Define $Q_i$ to be the subgraph of $G$ induced on $\bigcup_{\{v \,:\, r_i \in I(v)\}} v$; this subgraph is a quasi clique, and the collection $\mathcal{Q} = \{Q_i\}_{i=1}^{m}$ is a consecutively ordered collection of quasi cliques that covers the edges of $G$. ■

Theorem 4.1 in section 4 will sharpen this result, and indicate a subtlety, with respect to bipartite probe interval graphs.

## 3 A Consecutive Order Characterization For Interval Point Bigraphs

Our next result gives a characterization for interval point bigraphs in terms of a consecutive ordering of stars that form a partition of the edges, and hence, because of a theorem in [4], characterizes how interval point bigraphs and interval bigraphs differ in their biclique structure. We call a $K_{1,n}$, for $n \geq 0$, a *star* and the *center* of a star is the partite set of size 1 (for a $K_{1,1}$ either vertex may be thought of as the center). We will use the Theorem 3.1, a characterization for interval point bigraphs via their reduced adjacency matrices given in [3] for the bipartite graph model and in [6] for the directed graph model.

**Theorem 3.1** *A bipartite graph $G$ is an interval point bigraph if and only if its reduced adjacency matrix has the consecutive 1's property for rows or the consecutive 1's property for columns.*

**Theorem 3.2** *A bipartite graph $G = (X, Y, E)$ is an interval point bigraph if and only if it has a consecutively orderable edge partition of stars with all centers in the same partite set.*

**Proof.** Let $G = (X, Y, E)$ be an interval point bigraph. By Theorem 3.1, $A(G) = [a_{i,j}]$ exhibits the consecutive 1's property for rows or for columns. Let it have the consecutive 1's property for rows, since the argument for $A(G)$ having the consecutive

1's property for columns is similar. Let $\alpha_i$ be the first column in which a 1 appears in row $i$. Permute rows of $A(G)$ so that $\{\alpha_i\}$ is a nondecreasing sequence. Now, with $y_j \in Y$ and $x_i \in X$ corresponding to column $j$ and row $i$, respectively, take each $y_j$ as the center of star $S_j$ and put $S_j = \{y_j\} \cup \{x_i : x_i y_j \in E\}$. That is, take the star given by each column of the matrix $A(G)$ for which $\{\alpha_i\}$ forms a nondecreasing sequence; the order of the columns gives the order of the stars. To see that this ordering is consecutive, it suffices to check that $x_i \in S_a \cap S_c \implies x_i \in S_b$ for any $a < b < c$. If $x_i \in S_a \cap S_c$, but $x_i \notin S_b$, then $a_{i,a} = 1 = a_{a,c}$, but $a_{i,b} = 0$, contradicting the fact that $A(G)$ has consecutive 1's in the rows.

Conversely, let $\mathcal{S} = \{S_1, \ldots, S_r\}$ be a partition of $G = (X, Y, E)$ consisting of stars that are consecutively ordered with order given by their indexing, and so that each star has its center in $X$. We will show that $G$ is an interval point bigraph with $X$ the point partition. Note that if the centers all belonged to $Y$, then we would obtain an interval point bigraph with $Y$ the point partition from the appropriate analogue of the following construction. Make a collection of points $\mathcal{P} = p(x_1) < p(x_2) < \cdots < p(x_r)$, where $x_i$ is the center of star $S_i$. Now, for each $y \in Y$, make $I(y) = [l(y), r(y)]$, where $l(y) = \min\{i : y \in S_i\}$ and $r(y) = \max\{i : y \in S_i\}$. We have $p(x_i) \in I(y)$ if and only if $x_i$ and $y$ are both contained in some star together which happens only if $x_i y \in E$. Thus, the collection of intervals and points is an interval point representation for $G$. ■

## 4 Bipartite Probe Interval Graphs and Interval Point Bigraphs

Our first goal for this section is to develop a result that gives more precision to the relationships among bipartite probe interval graphs, interval point bigraphs, and interval bigraphs. We will begin with a lemma showing that the way in which the edge cover of quasi cliques was chosen in Theorem 2.3 results in a collection in which each maximal clique appears in exactly one quasi clique in the cover.

**Lemma 4.1** *If $G$ is a probe interval graph, then the consecutive cover of quasi cliques $\mathcal{Q}$ given by Theorem 2.3 can be made so that each maximal clique of $G$ is contained in exactly one quasi clique of $\mathcal{Q}$.*

**Proof.** Let $G$ be a PIG with $\mathcal{Q}$ a consecutively ordered edge cover of quasi cliques defined by the distinct right endpoints of the probe interval representation for $G$ as in the proof of Theorem 2.3. That is, let $Q_i = G\left(\bigcup_{\{v \,:\, r_i \in I(v)\}} v\right)$, where $r_1 < \cdots < r_m$ are the distinct right endpoints. Note that a maximal clique $C$ of $G$ is a quasi clique containing at most one nonprobe. We will show that each maximal clique is contained in at most one $Q_i$ and at least one $Q_i$. No $C$ is contained in more than one $Q_i$ because each vertex $v$ with interval $[l(v), r_i]$ will not be contained in $Q_{i+1}$. Now, suppose $C$ consists of $P_0 \subset P$ and $n \in N$. We must have $\bigcap_{p \,:\, p \in P_0} I(p) \cap I(n)$ containing some common point $q$ by the Helly property. Either $q = r_i$ or $q \in (r_i, r_{i+1})$, for some $i$. If $q = r_i$, then $C$ is contained in $Q_i$. If $q \in (r_i, r_{i+1})$, then $C$ is contained in $Q_{i+1}$, since $r(v) \geq r_{i+1}$ for all $v \in C$. Thus, $C$ is in at least one and no more than one $Q_i$, for some $i$. ∎

We will use the following definition in our next result. Let $G$ be a bipartite graph with vertex partition $V = (U, N)$, not necessarily a bipartition, where $G(N)$ is an independent set. A *$U$-star* of $G$ is either a star with center in $U$ and all other vertices in $N$, or a $K_2$ contained in $U$. If a collection of $U$-stars of $G$ is consecutively ordered, and forms an edge partition for $G$, we call it a *consecutive $U$-star partition.*

**Theorem 4.1** *Let $G$ be a bipartite graph. Then $G$ is a probe interval graph if and only if there is a vertex partition $V = U \cup N$, with $G(N)$ an independent set, and there is a consecutive $U$-star partition with respect to this partition.*

**Proof.** Let $G = (V, E)$ be a bipartite probe interval graph with $V = P \cup N$, where $P$ and $N$ are the sets of probes and nonprobes, respectively. Let $\mathcal{Q}$ be the consecutively ordered collection of quasi cliques that cover the edges of $G$ as guaranteed by Theorem 2.3. By definition of a quasi clique, each member of $\mathcal{Q}$ is either a $K_2 \subset P$, or a star with center in $P$ and all other vertices in $N$. By Lemma 4.1 we can choose members of $\mathcal{Q}$ so that each maximal clique of $G$ is contained in exactly one element of $\mathcal{Q}$. Since $G$ is bipartite, a maximal clique is isomorphic to a $K_2$, and so the intersection of any two elements of $\mathcal{Q}$ cannot contain an edge. Thus, $\mathcal{Q}$ can be made to form an edge partition of stars, and by the third sentence in this proof, each star is a $U$-star, with $P = U$.

Now assume that $G$ has a consecutive $U$-star partition, where $N$ is an independent set and $U = V \setminus N$. Let $S_1, \ldots, S_r$ be the collection of $U$-stars that partition the edges of $G$. We construct a probe interval representation for $G$, that is, we create a collection of intervals and decide on a partition of $V$ into probes $P$ and nonprobes $N$ that represents $G$. For each vertex of $G$, put $I(v) = [l(v), r(v)]$, with $l(v) = \min\{i : v \in S_i\}$, and $r(v) = \max\{j : v \in S_j\}$. For any two vertices $u, v$, $I(u) \cap I(v) \neq \emptyset$ if and only if $u$ and $v$ belong to the same $S_i$. If $u, v$ both belong to $S_i$ and $uv \in E(G)$, then either (wolog) $u \in U$, $v \in N$, or $u, v \in U$ and $S_i \cong K_2$. So if we put $P = U$, and let nonprobes be $N$, then we have a probe/nonprobe partition with $uv \in E(G)$ if and only if $I(u) \cap I(v) \neq \emptyset$ and at least one of $u, v$ belongs to $P$. Therefore, $G$ is a bipartite probe interval graph. ∎

Let us illustrate the subtlety Theorem 4.1 points out. Consider H10 in Figure 3. H10 is not a probe interval graph by a result of Sheng in [18], but it has a consecutive edge partition into stars. However, and it is left as an exercise for the reader to check this, there is no consecutive $U$-star partition.

Some remarks: If $G$ is an interval point bigraph, then by Theorem 3.2 there is a partition of stars with all centers in one partite set. By defining $P$ and $N$ to be the centers and non-centers of the stars, respectively, we obtain a complete set of maximal quasi cliques of $G$ that are consecutively ordered. H12 of Figure 2 is a bipartite probe interval graph, but not an interval point bigraph, which will be shown below. If $G$ is a bipartite probe interval graph, then the consecutive $U$-star partition

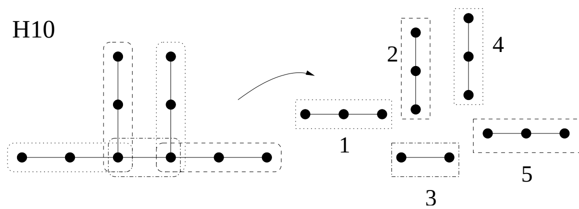



Figure 3: H10 has a consecutive partition into stars, does not have a consecutive $U$-star partition, and is not a probe interval graph.

given by Theorem 4.1 is a consecutively ordered biclique cover and so $G$ is an interval bigraph by Corollary 2.1. It is easy to check that H10 is an interval bigraph. We have the following corollary.

**Corollary 4.1** *An interval point bigraph is a bipartite probe interval graph, but not always conversely, and a bipartite probe interval graph is an interval bigraph, but not always conversely.*

Next, we characterize those bipartite probe interval graphs in which the partition of vertices into probes and nonprobes can correspond to a bipartition.

**Theorem 4.2** *$G = (X, Y, E)$ is a bipartite probe interval graph in which the probe/nonprobe partition can correspond to the bipartition if and only if $G$ is an interval point bigraph.*

**Proof.** Let $G = (X, Y, E)$ be a interval point bigraph with $X$ the point partition and interval point representation $\mathcal{I} \cup \mathcal{P}$. If we make all vertices in $X$ probes, and all other vertices nonprobes, we get $G = (P, N, E)$ is a probe interval graph with $\mathcal{I} \cup \mathcal{P}$ its probe interval representation. Similarly, if $G$ is an interval point bigraph with $Y$ the point partition, then put $N = X, P = Y$, and use the interval point representation as the probe interval representation.

For the converse, suppose $G = (P, N, E)$ is a bipartite probe interval graph with $P, N$ each corresponding to a partite set. Let $\{p_i\}_{i=1}^m = P$, $\{I_{p_i}\}$ be the set of intervals corresponding to $P$, and $\{I_{n_j}\}$ be the set of intervals corresponding to $N = \{n_j\}$. Since $P$ is independent, for any $i \neq j$, $I_{p_i} \cap I_{p_j} = \emptyset$. Label $\{p_i\}_{i=1}^m$ so that $l(p_i) < l(p_j)$ if and only if $i < j$. Now, for each $n_j$, extend $I_{n_j}$ so that $l(n_j) = l(p_i)$ for the smallest $i$ such that $p_i \in N(n_j)$. Now shrink each $I_{p_i}$ to its left-endpoint and get $G$ is an interval point bigraph in which $P$ becomes the point partition. ■

## 5 Circular Arc Graphs and Interval Point Bigraphs

Recall that a circular arc graph is the intersection graph of arcs of a circle, and that the circle from which the arcs are obtained is called the host circle. Theorem 1.1

and the next theorem we record, both due to P. Hell and J. Huang, give context to and motivate the main result of this section. A *proper circular arc graph* is one in which no arc contains another properly in some representation. A *proper interval bigraph* is an interval bigraph in which no interval properly contains another among both sets of intervals that represent it.

**Theorem 5.1** (Hell, Huang, [11]) *Let $G$ be a bipartite graph. Then $G$ is a proper interval bigraph if and only if $\overline{G}$ is a proper circular arc graph.*

For the most part, the work in this paper is a consequence of trying to complement Theorem 1.1 and Theorem 5.1 by exploring which circular arc graphs correspond to the complements of interval point bigraphs. Clearly, the circular arc graphs we seek are 2-clique graphs. A 2-clique graph will sometimes be denoted $G = (U, W, E)$, in which case it is meant to indicate that $G(U)$ and $G(W)$ are two cliques that render $G$ a 2-clique graph. Although this is the same notation for a bipartite graph, the context will make clear the intention of the notation.

Now, we introduce some terms used for the main result of this section. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a subset of a set $X$. The indexing of elements of $S$ is a *modular consecutive order* with respect to a binary relation $R$ on $X$ if, for some $i < j$, the image of each $x \in X \setminus S$ under $R$ is either $\{s_i, s_{i+1}, \ldots, s_j\}$ or $\{s_j, s_{j+1}, \ldots, s_n, s_1, s_2, \ldots, s_i\}$. If $G = (U, W, E)$ is a 2-clique graph in which $U$ or $W$ has a modular consecutive indexing with respect to adjacency and $X = U \cup W$, then $G$ is a *modular consecutive 2-clique graph*. To relate this concept to bipartite graphs, realize that another way to state Theorem 3.1 is as follows. A bipartite graph $G = (X, Y, E)$ is an interval point bigraph if $X$ or $Y$ can be ordered with $<$ such that (if $X$ is ordered) $ux, uz \in E \implies uy \in E$ for $u \in Y, x, y, z \in X$ and $x < y < z$ (switch roles of $X$ and $Y$ if $Y$ is ordered). If bipartite $G = (X, Y, E)$ has $X$ ordered as above, we say it is *$X$-consecutive*; if $Y$ is, then $G$ is *$Y$-consecutive*. These terms are the ones Tucker used in [20]. Also, following Tucker in [19], we say that a (0,1)-matrix $M$ has the *circular 1's property for columns* if the rows can be permuted so that the 1's in each column are circular, that is, if the matrix were wrapped around a cylinder the 1's would appear consecutively. The *circular 1's property for rows* is defined similarly. Suppose $G = (U, W, E)$ is a 2-clique graph with $|U| = m$ and $|W| = n$, and $U$ modularly ordered. Then the adjacency matrix for $G$, $M(G)$ has the form

$$\begin{bmatrix} J_{m\times m} - I_{m\times m} & J_{m\times n} - A(\overline{G}) \\ J_{n\times m} - A(\overline{G})^T & J_{n\times n} - I_{n\times n} \end{bmatrix},$$

where $J$ is a matrix of all 1's, $I$ the identity matrix, and $A(\overline{G})$ is the $m \times n$ reduced adjacency matrix of the bipartite graph $\overline{G}$. Since $U$ is modularly indexed, $J_{m\times n} - A(\overline{G})$ has the circular 1's property for rows, and hence, $J_{n\times m} - A(\overline{G})^T$ (the transpose of $J_{m\times n} - A(\overline{G})$) has the circular 1's property for columns. For convenience, we use $\overline{I} = J - I$, and more generally, $\overline{A} = J - A$, where the sizes of the matrices make sense, and $A$ is any (0,1)-matrix. Also, for any (0,1)-matrix $A = [a_{i,j}]$, there corresponds a bipartite graph $G = (X, Y, E)$ with $x_i y_j \in E$ if and only if $a_{i,j} = 1$; we denote this graph by $G(A)$.

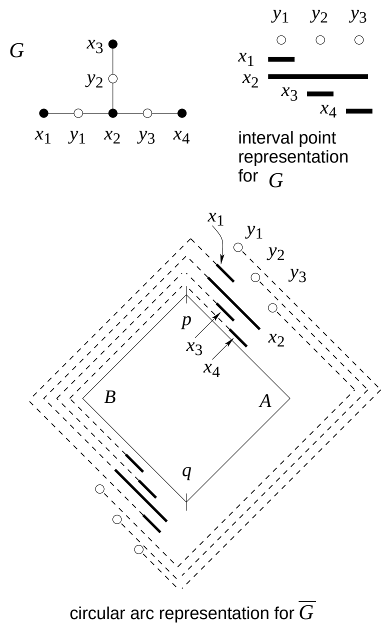



Figure 4: A circular arc representation of the complement of an interval point bigraph $G$ using the method in the proof of Theorem 5.2. The dashed lines are meant to indicate the arcs for $\overline{G}$. Our host circle is a square because it made producing an accurate figure a little easier.

**Theorem 5.2** *A bipartite graph $G = (X, Y, E)$ is an interval point bigraph if and only if $\overline{G}$ is a modular consecutive 2-clique circular arc graph.*

**Proof.** Suppose $G = (X, Y, E)$ is an interval point bigraph with $Y$ the point partition and $|Y| = n$; we will prove that $\overline{G}$ is a 2-clique circular arc graph in which $Y$ has a modular consecutive order. If $X$ were the point partition, then $\overline{G}$ would turn out to be a 2-clique circular arc graph in which $X$ has a modular consecutive order.

Suppose $G$ is an interval point bigraph that is $Y$-consecutive. So we may assume that an arbitrarily chosen $x \in X$ is adjacent to $y_i, \ldots y_k \in Y$ in $G$, for $1 \le i \le k \le n$, and hence, $x$ is adjacent to $y_{k+1}, \ldots, y_n, y_1, \ldots, y_{i-1}$ in $\overline{G}$. We construct a circular arc representation for $\overline{G}$. Let $\mathcal{C}$ be a circle with two specified, diametrically opposed points $p$ and $q$, with $A$ (respectively $B$) the segment of $\mathcal{C}$ extending clockwise from $p$ to $q$ (respectively $q$ to $p$). Let $\mathcal{I}$ be the interval point representation for $G$. The structure of $G$ dictates that $p(y_1) < p(y_2) < \cdots < p(y_n)$, that we may use $I(x) = [p(y_i), p(y_k)]$, and we may assume that the points are spaced equidistantly by some constant, say $\epsilon$. We may also assume that the total width of $\mathcal{I}$ is $p(y_n) - p(y_1)$; that is, the leftmost interval has left endpoint equal to $p(y_1)$ and the rightmost interval has right endpoint equal to $p(y_n)$. Place a copy of $\mathcal{I}$ in $A$ with $p(y_1) = p$ and place a copy of $\mathcal{I}$ in $B$

with $p(y_1) = q$. Of course, we assume $A$ and $B$ are each large enough to contain $\mathcal{I}$. Construct open $R(v) = (cc(v), cl(v))$ for each $v \in V(\overline{G})$ as follows. Put $(cc(y_i)) = p(y_i) \in A$ and $cl(y_i) = p(y_i) \in B$. Put $cc(x) = p(y_k) \in B$ and $cl(x) = p(y_i) \in A$. In this representation $R(x) \cap R(y_j) \neq \emptyset$ whenever $j \in \{k+1, \ldots, n, 1, \ldots, i-1\}$, so $Y$ is circularly indexed. Since $x$ was arbitrary, this construction applied to each $x \in X$ gives a circular arc representation of $\overline{G}$, a 2-clique graph with $Y$ modularly ordered.

Let $G = (X, Y, E)$ be a 2-clique graph, and $M(G) = [m_{i,j}]$ the adjacency matrix for $G$. If $X$ can be modularly indexed, then we can permute corresponding rows and columns so that

$$M(G) = \begin{bmatrix} \overline{I}_{|X|\times|X|} & A \\ A^T & \overline{I}_{|Y|\times|Y|} \end{bmatrix},$$

$A = M(G)[X;Y]$, that is, $A$ is the matrix induced on the rows corresponding to vertices in $X$ versus the columns corresponding to the vertices in $Y$. Since $X$ is circularly indexed, $A$ has the circular 1's property for columns, and $A^T$ has the circular 1's property for rows. Thus,

$$M(\overline{G}) = \begin{bmatrix} I_{|X|\times|X|} & \overline{A} \\ \overline{A^T} & I_{|Y|\times|Y|} \end{bmatrix}.$$

Clearly, $\overline{A}$ has the consecutive 1's property for columns. Hence, $G(\overline{A})$ is an interval point bigraph with $X$ the point partition, by Theorem 3.1. Taking complements, and disregarding which clique can be modularly indexed, we see that if $\overline{G}$ is a modular consecutive 2-clique graph, then $G$ is an interval point bigraph. ■

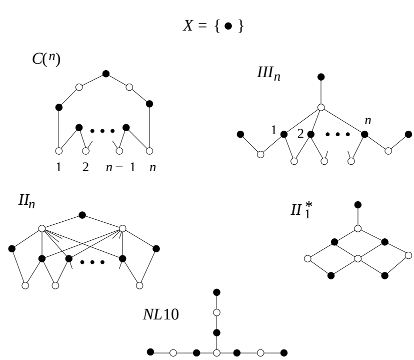



Figure 5: The forbidden subgraphs for the consecutive 1's property for columns.

In Figure 4 we have illustrated the idea behind the proof of Theorem 5.2 by constructing a circular arc representation for $\overline{G}$ using the construction in the proof. The figure drawing environment at the disposal of the authors is not conducive to drawing arcs and circles, so we have used a square as our "host circle" — nothing is lost of course, since, topologically, circles and squares are the same.

## 6 Summary and Forbidden Induced Subgraph Conjecture

We will summarize with a list of equivalences that incorporates all of the results developed herein and some results from other sources to give the most complete picture we can. First, we note a result of Tucker from [20] that gives a structure theorem for the consecutive 1's property for columns in (0,1)-matrices via properties of their corresponding bipartite graphs. Recall that we use the convention that the rows of a (0,1)-matrix $A$ correspond to the partite set $X$ in $G(A)$. An *asteroidal triple* in a graph is a set of three vertices with a path between any two that avoids the neighborhood of the third.

**Theorem 6.1** (Tucker, [20]) *A (0,1)-matrix A has the consecutive 1's property for columns if and only if the corresponding bipartite graph $B(A) = (X, Y, E)$ contains no asteroidal triple in $X$.*

The above theorem gives the following corollary, which we incorporate into Theorem 6.2.

**Corollary 6.1** *A bipartite graph $G = (X, Y, E)$ is an interval point bigraph if and only if $X$ contains no asteroidal triple of $G$, or $Y$ contains no asteroidal triple of $G$.*

**Theorem 6.2** *Let $G = (X, Y, E)$ be a bipartite graph. The following are equivalent:*
*(1.) G is a probe interval graph in which the probe/nonprobe partition can correspond to the bipartition;*
*(2.) $\overline{G}$ is a modular consecutive 2-clique circular arc graph;*
*(3.) G is an interval point bigraph;*
*(4.) There is no asteroidal triple of G contained in $X$, or there is no asteroidal triple of G contained in $Y$;*
*(5.) The reduced adjacency matrix for G has the consecutive 1's property for rows or for columns;*
*(6.) G has an edge partition of stars with all centers in the same partite set that can be consecutively ordered.*

Before we conjecture the list of forbidden induced subgraphs for interval point bigraphs, and the other classes of structured graphs to which they are equivalent, we present another result of Tucker from [20].

**Theorem 6.3** (Tucker, 1972, [20]) *A (0,1)-matrix has the consecutive 1's property for columns if and only if its corresponding bipartite graph $G = (X, Y, E)$ has no induced subgraph isomorphic to NL10, $II_1^*$, $C(n)$, $II_n$, or $III_n$ of Figure 5.*

Some of the graphs in Figure 5 are interval point bigraphs, since we do not require that $X$ be the point partition. Namely, $II_1, III_1, III_2, III_3$ are interval point bigraphs. It is tedious but straightforward to check that the condition in the following conjecture is necessary.

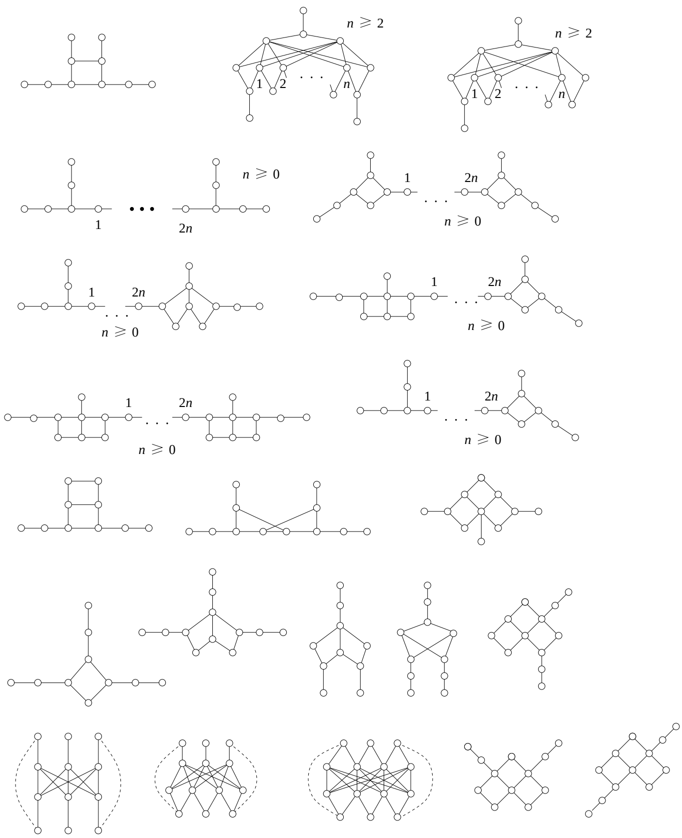



Figure 6: Forbidden interval point bigraphs. Include NL10, $C(n)$, $II_1^*$, $II_n$ $(n \geq 2)$, and $III_n$ $(n \geq 4)$ of Figure 5.

**Conjecture 6.1** *A bipartite graph $G$ is an interval point bigraph if and only if it has no induced NL10, $C(n)$, $II_1^*$, $II_n$ $(n \geq 2)$, $III_n$ $(n \geq 4)$, of Figure 5, or any of the graphs in Figure 6 as an induced subgraph.*

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