AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **37** (2007), Pages 11–20

Chromaticity of the complements of some sparse graphs*

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Abstract

For a graph G, let $\overline{G}$ be its complement and $h(G,x)$ its adjoint polynomial. Let $\mathcal{L} = \{P_i|i \geq 2\} \cup \{C_j|j \geq 4\} \cup \{D_k|k \geq 4\} \cup \{F_s|s \geq 6\} \cup \{K_4^-, K_4\}$, where P_i denotes the path with i vertices, C_j denotes the cycle with j vertices, D_k denotes the graph obtained from K_3 and P_{k-2} by identifying a vertex of K_3 with an end-vertex of P_{k-2}, F_s denotes the graph obtained from K_3 and D_{s-2} by identifying a vertex of K_3 with the vertex of degree 1 of D_{s-2}, and K_4^- denotes the graph obtained from complete graph K_4 by deleting an edge. In this paper, we obtain a necessary and sufficient condition for each graph of form $\overline{aK_3 \cup \bigcup_i G_i}$ to be chromatically unique when $h(K_3,x) \not| h(G_i,x)$ and $G_i \in \mathcal{L}$ for each i. Moreover many known results are generalized.

* Research supported by National Science Foundation of China and the Science Foundation of the State Education Ministry of China.

1 Introduction

All graphs considered here are finite and simple. Undefined notation and terminology will conform to those in [1, 2]. Let $V(G)$, $E(G)$, $p(G)$, $q(G)$, $\delta(G)$ and $\overline{G}$ denote the set of vertices, the set of edges, the number of vertices, the number of edges, the minimum degree of vertices and the complement of a graph G, respectively.

For a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$ is called an *r-independent partition* of a graph G if every A_i is a nonempty independent set of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of $V(G)$. Then, the chromatic polynomial of G is given by $P(G, \lambda) = \sum_{r\geq 1} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - r + 1)$ for all $r \geq 1$, see [3,4] for more details. Two graphs G and H are called *chromatically equivalent*, denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is called *chromatically unique* (or simply *χ-unique*) if $H \cong G$ whenever $H \sim G$.

For a graph G with p vertices. If H is a spanning subgraph of G and each component of H is complete, then H is called *an ideal subgraph* of G [10]. Let $b_i(G)$ denote the number of ideal subgraphs H in G with $p - i$ components. It is clear that $b_0(G) = 1$, $b_1(G) = q(G)$ and $b_i(G) = \alpha(\overline{G}, p - i)$ for each i. The polynomial

$$h(G, x) = \sum_{i=0}^{p-1} b_i(G)x^{p-i}$$

is called the *adjoint polynomial* of the graph G.

Two graphs G and H are said to be *adjointly equivalent*, denoted by $G \sim_h H$, if $h(G, x) = h(H, x)$. Clearly, "$\sim_h$" is an equivalence relation on the family of all graphs. Let $[G]_h = \{H | H \sim_h G\}$. A graph G is said to be *adjointly unique* if $H \cong G$ whenever $H \sim_h G$. For a set $\mathcal{G}$ of graphs, if $[G]_h \subset \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is called *adjointly closed*. More details on $h(G, x)$ can be found in [3,4,10–15].

From the above definitions, we have

Theorem 1.1

(i) $G \sim H$ *if and only if* $\overline{G} \sim_h \overline{H}$*;*

(ii) $[G] = \{H | \overline{H} \in [\overline{G}]_h\}$;

(iii) G *is adjointly unique if and only if* $\overline{G}$ *is* χ*-unique.*

□

Let G be a graph and $h(G, x) = x^{\alpha(G)} h_1(G, x)$, where $h_1(G, x)$ is a polynomial with a nonzero constant term. If $h_1(G, x)$ is an irreducible polynomial over the rational number field, then G is called an *irreducible graph*.

For convenience, we simply denote $h(G, x)$ by $h(G)$ and $h_1(G, x)$ by $h_1(G)$. Next we introduce some notation: For a graph G and $v \in V(G)$, by $N_G(v)$ we denote the set of all vertices of G adjacent to v. For $e = v_1v_2 \in E(G)$, set $N_G(e) = N_G(v_1) \cup N_G(v_2) - \{v_1, v_2\}$ and $d(e) = d_G(e) = |N_G(e)|$. Let G and H be two graphs, $G \cup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies of H. For $g(x), f(x) \in Q[x]$, let $(g(x), f(x))$ denote the greatest common factor of $g(x)$ and $f(x)$, $g(x)|f(x)$ (respectively, $g(x) \not| f(x)$) mean that $g(x)$ divides $f(x)$ (respectively, $g(x)$ does not divide $f(x)$).

By C_n (respectively, P_n) we denote the cycle (respectively, the path) with n vertices. By K_4^- we denote the graph obtained by deleting an edge from K_4. The graphs shown in Figure 1 are frequently used throughout the paper. In Figure 1, a dotted line denotes a path whose number of vertices is at least 2, and n denotes the number of vertices in each graph. We write $\mathcal{L} = \{P_n, C_{n+2}, D_{n+2}, F_{n+4} | n \geq 2\} \cup \{K_4^-, K_4\}$.

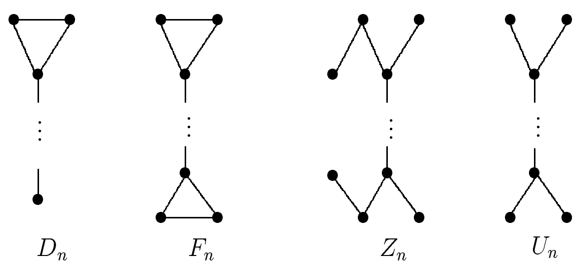

Figure 1. Graphs D_n, F_n, Z_n and U_n.

For the study of chromatic uniqueness of graphs, in addition to the chromatic polynomial, the following polynomials have been employed: the σ-polynomial (see [5, 6]) and the adjoint polynomial of graphs (see[3–4] and [8–15]). In [8–12], when P_n, C_n, D_n and F_n are irreducible graphs, the chromatic uniqueness of $\overline{\cup P_{n_i}}$, $\overline{\cup C_{m_i}}$, $\overline{(\cup P_{n_i}) \cup (\cup C_{m_j})}$, $\overline{(\cup C_{n_i}) \cup (\cup D_{m_j})}$ and $\overline{F_n}$ were studied. In [5, 6], Du discussed the chromatic uniqueness of $\overline{lK_3 \cup (\cup_i P_{n_i})}$ and $\overline{\cup_j C_{m_j}}$, and obtained that $\overline{lK_3 \cup (\cup_i P_{n_i})}$ and $\overline{\cup_j C_{m_j}}$ are chromatically unique if $n_i \not\equiv 4 (mod\ 10)$ and n_i is even, $m_j \geq 3$ and $m_j \neq 4$. Very recently, Dong et al. in [4] investigated chromaticity of complements of $H = aK_3 \cup bD_4 \cup \bigcup_{1 \leq i \leq s} P_{u_i} \cup \bigcup_{1 \leq j \leq t} C_{v_j}$, where $a, b \geq 0, u_i \geq 3, u_i \not\equiv 4 (mod\ 5), v_j \geq 4$, and obtained a necessary and sufficient condition for $\overline{H}$ to be chromatically unique.

In this paper, we first show that $\mathcal{F}_a$ is adjointly closed, where

$$\mathcal{F}_a = \{aK_3 \cup \bigcup_i G_i \mid G_i \in \mathcal{L}, h(K_3) \not| h(G_i)\}.$$

We then investigate the chromaticity of $\overline{G} \in \mathcal{F}_a$ and give a necessary and sufficient for $\overline{G}$ to be chromatically unique. Many of the results in [9–12] are generalized.

2 Preliminaries

Lemma 2.1 ([10]) *Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

$$h(G) = \prod_{i=1}^{k} h(G_i).$$

Let G be a graph and $e = v_1v_2 \in E(G)$. A new graph $G * e$ is defined as follows: the set of vertices of $G * e$ is $V(G)\backslash\{v_1, v_2\} \cup \{v\}$, where $v \notin V(G)$, and the set of edges of $G * e$ is $\{e'|e' \in E(G), e'$ is not incident with v_1 or $v_2\} \cup \{uv|u \in N_G(v_1) \cap N_G(v_2)\}$. For example, let C_4 be the cycle on 4 vertices with an edge uv, and let $H = C_4 + e$ be the graph obtained from C_4 by adding a chord e. Then $C_4 * uv = K_1 \cup P_2$ and $H * e = P_3$.

Lemma 2.2 ([3,8]) *Let G be a graph and $e \in E(G)$. Then*

$$h(G, x) = h(G - e, x) + h(G * e, x).$$

Lemma 2.3 ([11])

(i) For all $n \geq 2$, $h(P_n) = \sum_{k \leq n} \binom{k}{n-k} x^k$;

(ii) For all $n \geq 4$, $h(C_n) = \sum_{k \leq n} \frac{n}{k} \binom{k}{n-k} x^k$;

(iii) For all $n \geq 4$, $h(D_n) = \sum_{k \leq n} \left(\frac{n}{k} \binom{k}{n-k} + \binom{k-2}{n-k-3} \right) x^k$.

Lemma 2.4 ([11]) *(i) For $n \geq 3$, $h(P_n, x) = x(h(P_{n-1}, x) + h(P_{n-2}, x))$.*

(ii) For $n \geq 6$, $h(C_n, x) = x(h(C_{n-1}, x) + h(C_{n-2}, x))$.

(iii) For $n \geq 6$, $h(D_n, x) = x(h(D_{n-1}, x) + h(D_{n-2}, x))$.

(iv) For $n \geq 8$, $h(F_n, x) = x(h(F_{n-1}, x) + h(F_{n-2}, x))$.

Lemma 2.5 *(i)* ([11]) *For all $n, m \geq 2$, $h(P_n)|h(P_m)$ if only and if $(n+1)|(m+1)$.*

(ii) ([12]) *For all $m \geq 4$, $h(P_4) \not| h(C_m)$.*

Lemma 2.6([15]) *Let $\{g_i(x)|i \geq 0\}$ be a polynomial sequence with integer coefficients and $g_n(x) = x(g_{n-1}(x) + g_{n-2}(x))$. Then $h_1(P_n)|g_{k(n+1)+i}(x)$ if and only if $h_1(P_n)|g_i(x)$, where $0 \leq i \leq n$ and $n \geq 2$.*

Lemma 2.7 *(i) For $n \geq 4$, $h_1(P_4)|h(D_n)$ if and only if $n = 5k + 3$, where $k \geq 1$;*

(ii) For $n \geq 6$, $h_1(P_4)|h(F_n)$ if and only if $n = 5k + 2$, where $k \geq 1$.

Proof. (i) Let $m \geq 0$ and $g_m(x) = h(D_{m+4})$. By Lemma 2.4 we have

$$g_m(x) = x(g_{m-1}(x) + g_{m-2}(x)).$$

Without loss of generality, let $m = 5k + i$, where $0 \leq i \leq 4$. By Lemma 2.6 we have $h_1(P_4)|g_{5k+i}(x)$ if and only if $h_1(P_4)|g_i(x)$, where $0 \leq i \leq 4$. By Lemma 2.3 we obtain the following: $h_1(P_4) = x^2 + 3x + 1$, $g_0(x) = h(D_4) = x^2(x^2 + 4x + 2)$, $g_1(x) = h(D_5) = x^2(x+1)(x^2+4x+1)$, $g_2(x) = h(D_6) = x^3(x^3+6x^2+9x+3)$, $g_3(x) = h(D_7) = x^3(x^4+7x^3+14x^2+8x+1)$ and $g_4(x) = h(D_8) = x^4(x+1)(x+4)(x^2+3x+1)$. When $i = 0, 1, 2, 3, 4$, it is easy to verify that $h_1(P_4)|g_i(x)$ if and only if $h_1(P_4)|g_4(x)$. So, by Lemma 2.6 it follows that $h_1(P_4)|h(D_n)$ if and only if $n = 5k+3$, where $k \geq 1$.

(ii) By Lemma 2.2 we have $h(F_n) = h(D_n) + h(P_2)h(D_{n-3})$. By Lemma 2.3 we have $h(F_6) = x^2(x^4 + 7x^3 + 13x^2 + 7x + 1)$, $h(F_7) = x^3(x^2 + 3x + 1)(x^2 + 5x + 3)$, $h(F_8) = x^3(x+1)(x^4+8x^3+18x^2+9x+1)$, $h(F_9) = x^4(x^2+4x+2)(x^3+6x^2+8x+2)$ and $h(F_{10}) = x^4(x^6+11x^5+43x^4+72x^3+51x^2+14x+1)$. It is not difficult to verify that when $6 \leq n \leq 10, h(P_4)|h(F_n)$ if and only if $n = 7$. Similar to the proof of (i), we can show that (ii) holds. □

3 Invariants for Adjointly Equivalent Graphs

Let G be a graph. Liu [11] introduced an invariant $R_1(G)$ for adjointly equivalent graphs as follows:

$$R_1(G) = \begin{cases} 0 & \text{if } q(G) = 0, \\ b_2(G) - \binom{b_1(G) - 1}{2} + 1 & \text{if } q(G) > 0. \end{cases}$$

For the invariant $R_1(G)$, the following results can be found in [3–6] and [10–14].

Lemma 3.1 ([11]) *Let G and H be two graphs. If $h(G, x) = h(H, x)$, then*

$$R_1(G) = R_1(H).$$

Lemma 3.2 ([11]) *Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

$$R_1(G) = \sum_{i=1}^{k} R_1(G_i).$$

Lemma 3.3 ([11]) *Let G be a connected graph and $e \in E(G)$. Then*

$$R_1(G) = R_1(G - e) - d_G(e) + 1.$$

Very recently Dong et al. introduced an graph invariant, denoted by $R_2(G)$ (see [4]), for adjointly equivalent graphs. In [16], Zhao introduced a parameter $R_3(G)$ of a graph as follows:

$$R_3(G) = R_1(G) + q(G) - p(G).$$

Evidently, $p(G)$ and $q(G)$ are invariant. Thus $R_3(G)$ is an invariant for adjointly equivalent graphs. The following theorem follows from Lemma 3.2.

Theorem 3.1 *Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then*

$$R_3(G) = \sum_{i=1}^{k} R_3(G_i).$$

□

By Lemma 3.3 we obtain

Theorem 3.2 *Let G be a connected graph and $e \in E(G)$. Then*

$$R_3(G) = R_3(G-e) - d_G(e) + 2.$$

□

The following result can be found in [3].

Theorem 3.3 ([3]) *For any connected graph G with $G \notin \{K_3, K_4\}$,*

(i) if $-1 \le R_1(G) \le 1$, then $R_1(G) \le p(G) - q(G)$ with equality if and only if

$$G \in \{P_n, C_{n+2}, D_{n+2}, F_{n+4} | n \ge 2\} \cup \{K_4^-\}.$$

(ii) if $R_1(G) \le -2$, then $R_1(G) \le p(G) - q(G) - 1$.

It is not hard to see that the above theorem is equivalent to the following theorem.

Theorem 3.4 ([3, 16]) *Let G be a connected graph. Then*

(i) $R_3(G) \le 1$, and the equality holds if and only if $G \cong K_3$.

(ii) $R_3(G) = 0$ if and only if $G \in \mathcal{L}$.

Theorem 3.5 *Let $\mathcal{F}_a = \{aK_3 \cup \bigcup_i G_i | G_i \in \mathcal{L}$ and $h(K_3) \not| h(G_i)\}$. Then $\mathcal{F}_a$ is adjointly closed.*

Proof. Suppose that $G \in \mathcal{F}_a$ and $H \sim_h G$. It is sufficient to prove that $H \in \mathcal{F}_a$. So, we shall show that H contains exactly a components K_3 and each of the other components of H belongs to $\mathcal{L}$.

Clearly, $h(H) = h(G)$. Denote by N_A the number of the components K_3 in H. By Theorems 3.1 and 3.4, we have $R_3(G) = R_3(H) = a$ and $N_A \geq a$.
Since $[h(K_3)]^{a+1} \not| h(G)$, we have $[h_1(K_3)]^{a+1} \not| h(H)$, and so $N_A \leq a$. Thus $N_A = a$, which implies that H has exactly a components K_3 and $R_3(H_i) = 0$ for every component H_i of H except K_3. By Theorem 3.4, $H_i \in \mathcal{L}$ except K_3. Hence $H \in \mathcal{F}_a$.
□

4 Chromatic uniqueness of graphs

In this section, we denote by $A, A_i, B, B_i, C, M_i, E$ and E_i the multisets of some positive integer numbers, where $i = 1, 2$. For a graph G, let $f(G, x)$ denote the characteristic polynomial of an adjacency matrix of G. We denote by $\gamma(G)$ and $\beta(G)$, respectively, the maximum real root of $f(G, x)$ and the minimum real root of $h(G, x)$. The following lemmas can be found in [13, 14].

Lemma 4.1([13]) *For $n \geq 6$, $h(F_n \cup 2K_1) = h(Z_{n+2})$.*

Lemma 4.2 ([13]) *For a tree T, $\beta(T) = -(\gamma(T))^2$.*

Lemma 4.3 ([14]) *(i) For $n \geq 2$, $-4 < \beta(P_n) < \beta(P_{n-1})$;*

(ii) For $n \geq 4$, $-4 < \beta(C_{n+1}) < \beta(C_n) < -3$ and $\beta(D_{n+1}) < \beta(D_n)$;

(iii) For $n \geq 4$, $\beta(D_n) < \beta(C_n) < \beta(P_n)$;

(iv) For $n \geq 9$, $\beta(D_n) < -4$.

Lemma 4.4 ([13]) *Let $f_1(x), f_2(x)$ and $f_3(x)$ be polynomials in x with real positive coefficients. If (i) $f_3(x) = f_2(x) + f_1(x)$ and $\partial f_3(x) - \partial f_1(x) \equiv 1 (mod\ 2)$, where β_i (or $\partial f_i(x)$) denotes the minimum real root (or the degree) of $f_i(x)$ $(i = 1, 2, 3)$, (ii) both of $f_1(x)$ and $f_2(x)$ have real roots, and $\beta_2 < \beta_1$, then $f_3(x)$ has at least one real root β_3 such that $\beta_3 < \beta_2$.*

Recently, the authors of [14] determined all connected graphs G with $-4 \leq \beta(G) \leq 0$ and proved the following result:

Theorem 4.1 ([14]) *Let G be a graph with p vertices and $\delta(G) \geq p - 3$; then G is χ-unique if and only if $\overline{G}$ is one of the following graphs:*

(i) $rK_1 \cup (\cup P_i)$ for $r = 0$, $i \equiv 0 (mod\ 2)$ and $i \neq 4$; or $r = 0$ and $i = 3, 5$; or $r \neq 0$, $i \equiv 0 (mod\ 2)$ and $i \neq 4$; or $r \neq 0$ and $i = 3$;

(ii) $t_1 P_2 \cup t_2 P_3 \cup t_3 P_5 \cup (\cup_j P_j) \cup (\cup_k C_k) \cup lC_3$ for $t_1 = 0, l \geq 0, k \neq j + 1$ and j is even; or $t_1 \neq 0, l \geq 0, k \neq j + 1$, $k \neq 6, 9, 15$ and j is even, where $j \geq 6, k \geq 5$.

An *internal x_1—x_k path* of a graph G is a sequence $x_1, x_2, x_3, \cdots, x_k$ such that all x_i are distinct (except possibly $x_1 = x_k$), the vertex degrees $d(x_i)$ satisfy $d(x_1) \geq 3$,

$d(x_2) = d(x_3) = \cdots = d(x_{k-1}) = 2$ (unless $k = 2$), $d(x_k) \geq 3$ and x_i is adjacent to x_{i-1}, where $i = 1, 2, \cdots, k-1$.

Lemma 4.5 ([2]) *Let G_{xy} be the graph obtained from G by introducing a new vertex on the edge xy of G. If xy is an edge on an internal path of G and $G \ncong U_n$ for any $n \geq 6$(see Figure 1), then $\gamma(G_{xy}) < \gamma(G)$.*

Theorem 4.2 *(i) For $n \geq 6$, $\beta(F_{n-1}) < \beta(F_n) < \beta(D_n) < \beta(D_{n-1})$;*

(ii) $\beta(K_4) < \beta(F_6) < -4$;

(iii) If G is connected and $G \in \mathcal{L}$, then $\beta(G) = -4$ if and only if $G \cong K_4^-$ or $G \cong D_8$.

Proof. (i) By Lemma 4.1, $h_1(Z_{n+2}) = h_1(F_n)$. From Lemma 4.2, we have $\beta(Z_n) = -\gamma^2(Z_n)$. By Lemma 4.5, $\gamma(Z_{n+2}) < \gamma(Z_{n+1})$. So, $\beta(Z_{n+1}) < \beta(Z_{n+2})$. This implies $\beta(F_{n-1}) < \beta(F_n)$.

By Lemma 2.2, one can obtain that $h(F_n) = h(D_n) + h(P_2)h(D_{n-3})$. By Lemma 4.3, $\beta(D_n) < \beta(h(P_2)h(D_{n-3}))$. By Lemma 4.4, $\beta(F_n) < \beta(D_n)$.

(ii) Since $h_1(K_4) = x^3 + 6x^2 + 7x + 1$ and $h_1(F_6) = x^4 + 7x^3 + 13x^2 + 7x + 1$, it follows immediately that $\beta(K_4) < \beta(F_6) < -4$, by direct calculation.

(iii) As $h_1(D_8) = (x^2 + 3x + 1)(x^2 + 5x + 4) = h_1(K_3)h_1(K_4^-)$, we have $\beta(D_8) = \beta(K_4^-) = -4$. By (i) and (ii) of the theorem, for $n \geq 7$ and $m \geq 10$ we have

$$\beta(K_4) < \beta(F_{n-1}) < \beta(F_n) < \beta(D_m) < \beta(D_{m-1}) < \cdots < \beta(D_9) < \beta(D_8) = -4.$$

From Lemma 4.3, for $i \geq 2$, $j \geq 3$ and $4 \leq k \leq 7$ we have

$$\beta(P_i) > -4, \beta(C_j) > -4, \beta(D_k) > -4.$$

So, (iii) holds. □

Theorem 4.3 *Let a, t, r be nonnegative integers and*

$$G = (\bigcup_{i \in A} P_i) \cup (\bigcup_{j \in B} C_j) \cup (\bigcup_{k \in M} D_k) \cup (\bigcup_{s \in E} F_s) \cup aK_3 \cup tK_4^- \cup rK_4,$$

where $A = \{i \mid i \geq 2,\ i \equiv 0 \pmod 2$ and $i \not\equiv 4 \pmod{10}\}$, $B = \{j \mid j \geq 5\}$, $M = \{k \mid k \geq 9, k \not\equiv 3 \pmod 5\}$, $E = \{s \mid s \geq 6, s \not\equiv 2 \pmod 5\}$. Then $\overline{G}$ is χ-unique if and only if $\{i+1 \mid i \in A\} \cap B = \emptyset$ if $2 \notin A$, or $\{i+1 \mid i \in A\} \cap B = \emptyset$ and $\{5, 6, 7\} \cap B = \emptyset$ if $2 \in A$.

Proof. It is not difficult to see that we need only to prove that G is adjointly unique if and only if $\{i + 1 | i \in A\} \cap B = \emptyset$ if $2 \notin A$, or $\{i + 1 | i \in A\} \cap B = \emptyset$ and $\{5, 6, 7\} \cap B = \emptyset$ if $2 \in A$.

Let H be a graph such that $h(H) = h(G)$. Since $h_1(K_3) = h_1(P_4)$, by Lemmas 2.5 and 2.7 one can see that $h_1(K_3) \not| h_1(Y)$ for each $Y \in \{P_i \mid i \geq 2, i \equiv 0 \pmod 2), i \not\equiv 4 \pmod{10}\} \cup \{C_j \mid j \geq 4\} \cup \{D_k \mid k \geq 4, k \not\equiv 3 \pmod 5)\} \cup \{F_s | s \geq 6, s \not\equiv 2 \pmod 5)\}$.

So, by Theorem 3.5, $H \in \mathcal{F}_a$. Assume $H = aK_3 \cup H_1$ and $G = aK_3 \cup G_1$. Then $h(G_1) = h(H_1)$. Without lost of generality, we assume

$$G_1 = (\bigcup_{i\in A} P_i) \cup (\bigcup_{j\in B} C_j) \cup (\bigcup_{k\in M} D_k) \cup (\bigcup_{s\in E} F_s) \cup tK_4^- \cup rK_4$$

and

$$H_1 = (\bigcup_{i_1\in A_1} P_{i_1}) \cup (\bigcup_{j_1\in B_1} C_{j_1}) \cup (\bigcup_{k_1\in M_1} D_{k_1}) \cup (\bigcup_{s_1\in E_1} F_{s_1}) \cup t_1K_4^- \cup r_1K_4,$$

where i, j, k and s satisfy the condition of the theorem.

It is enough to prove that $H_1 \cong G_1$. Since $h_1(D_8) = (x^2 + 5x + 4)h_1(K_3)$, H_1 does not contain the component D_8. By Theorem 4.2, we have

$$\begin{aligned}&\beta(K_4) < \beta(F_6) < \beta(F_7) < \cdots < \beta(F_{n-1}) < \beta(F_n) < \beta(D_m)\\ &< \beta(D_{m-1}) < \cdots < \beta(D_9) < \beta(D_8) = \beta(K_4^-) = -4.\end{aligned}$$

By comparing the minimum real root of $h_1(G_1)$ with that of $h_1(H_1)$, we know that $r = r_1, t = t_1, |E| = |E_1|, M \subseteq M_1$. Eliminating all components G^* with $\beta(G^*) \leq -4$ from G_1 and H_1, we obtain

$$h((\bigcup_{i\in A} P_i) \cup (\bigcup_{j\in B} C_j)) = h((\bigcup_{i\in A_1} P_i) \cup (\bigcup_{j\in B_1} C_j) \cup (\bigcup_{k\in M_2} D_k)),$$

where $M_2 = M_1 - M$.

By Theorem 4.1, we know that $\overline{(\cup_{i\in A} P_i) \cup (\cup_{j\in B} C_j)}$ is χ-unique if and only if $\{i+1 \mid i \in A\} \cap B = \emptyset$ if $2 \notin A$, or $\{i+1 \mid i \in A\} \cap B = \emptyset$ and $\{5,6,7\} \cap B = \emptyset$ if $2 \in A$ when i and j satisfy the condition of the theorem. Hence $M_2 = \emptyset$ and $M = M_1$, which implies that $H_1 \cong G_1$ and $H \cong G$. □

It is not difficult to see that all the chromatically unique graphs given in [9–12] are special cases of our Theorems 4.3. In particular, from Theorem 4.3 we have

Corollary 4.1 *Let* $k \not\equiv 3 \pmod 5$ *and* $k \geq 9$, $s \not\equiv 2 \pmod 5$ *and* $s \geq 6$. *Then* $\overline{(\cup_s F_s) \cup (\cup_k D_k)}$ *is* χ*-unique.* □

Corollary 4.2 *Let* $K(n_1, n_2, \cdots, n_t)$ *be the complete* t*-partite graph. Then* $K(2, \cdots, 2, 3, \cdots, 3, 4, \cdots, 4)$ *is* χ*-unique.* □

Acknowledgements

The authors are greatly indebted to the referees for their valuable comments and suggestions, which are very helpful for improving the presentation of the paper.

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(Received 1 Nov 2003; revised 21 Apr 2004 and 13 July 2006)